Harish-Chandra's Tempered Representations and Geometry IV

Tempered homogeneous spaces — Interaction with topology and geometry

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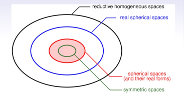
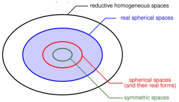
18th Discussion Meeting in Harmonic Analysis
(In honour of centenary year of Harish Chandra)
Indian Institute of Technology Guwahati, India, 16 December 2023

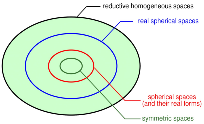
Plan of Lectures





- Talk 1: Is rep theory useful for global analysis?
  —Multiplicity: Approach from PDEs

- Talk 2: Tempered homogeneous spaces
  —Dynamical approach



- Talk 3: Classification theory of tempered G/H
  —Combinatorics of convex polyhedra

- Talk 4:

Plan

	Method	Topic
Lecture 1	PDEs	Multiplicity in $C^\infty(G/H)$
Lecture 2	Dynamical approach	L^q-estimate of $L^2(G/H)$
Lecture 3	Combinatorics	Classification of non-tempered G/H

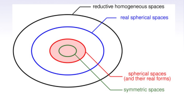
- Plan for Today (Lecture 4)

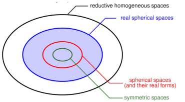
0. Temperedness criterion (generalization)

Explore yet another relation of tempered homogeneous spaces with other disciplines.

1. Topology: Deforming Lie algebras
2. Geometry: Geometric quantization

Plan of Lectures





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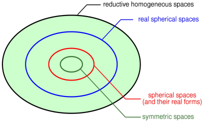
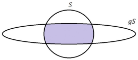


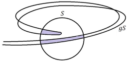
- Talk 3: Classification theory of tempered G/H
  —Combinatorics of convex polyhedra

- Talk 4: Tempered homogeneous spaces
  —Interaction with topology and geometry

Temperedness criterion (generalization)

Lecture 2





						Method
(Theorem E)	Case 1	G semisimple	$\curvearrowright$	V linear		Dynamical approach
(Theorem F′)	Case 2	G semisimple	$\supset$	H reductive		Global geometry + Case 1
((Theorem H)	Case 3	G semisimple	$\supset$	H any		Domination of G-spaces)

Today

(Theorem O)	Case 4	G any	$\supset$	H any		"Limit algebras"

Reminder from Lecture 2

$\mathfrak{a}$: max split abelian subspace of a Lie algebra $\mathfrak{h}$

p_V is defined for a linear action $\mathfrak{h} \curvearrowright V$ by

$$p_V = \max_{Y \in \mathfrak{h} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} = \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\sum |\text{eigenvalues of } Y \curvearrowright \mathfrak{h}|}{\sum |\text{eigenvalues of } Y \curvearrowright V|}.$$

Levi decomposition

- (Hulanicki–Reiter) For solvable Lie groups, all unitary reps are tempered.

- Levi decomposition

$$\underset{\text{semisimple}}{\mathfrak{g} = \mathfrak{g}_s} \oplus \underset{\text{solvable}}{\mathfrak{u}} \qquad \text{(Levi decomposition)}$$

$$G \supset G_s \qquad \text{(semisimple part)}$$

- For a unitary representation π of a Lie group G, we shall discuss temperedness of π as a representation of the semisimple part G_s.

Temperedness criterion in the general case

Setting $H \subset G$ real algebraic Lie groups.

We allow G and H to be non-reductive.

Take maximal semisimple subgroups H_s and G_s of H and G, respectively, such that $H_s \subset G_s$. Consider

$$G_s \subset G \curvearrowright L^2(G/H)$$

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

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Theorem O* $L^2(G/H)$ is G_s-tempered $\Longleftrightarrow$?

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Temperedness criterion in the general case

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$$G_s \subset G \curvearrowright L^2(G/H)$$

We set $V := \mathfrak{g}/\mathfrak{h} + \mathfrak{g}/\mathfrak{g}_s \cdots H_s$-module

<u>Theorem O</u>* $L^2(G/H)$ is G_s-tempered $\iff p_V \leq 1$.

$$\iff \rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}_s$$

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Temperedness criterion in the general case

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$$G_s \subset G \curvearrowright L^2(G/H)$$

We set $V := \mathfrak{g}/\mathfrak{h} + \mathfrak{g}/\mathfrak{g}_s \cdots H_s$-module

Theorem O* $L^2(G/H)$ is G_s-tempered $\iff p_V \leq 1$.

When G is semisimple, *i.e.*, $G = G_s$, Theorem O implies:

Theorem H (Lecture 2: G semisimple, H reductive case)
$L^2(G/H)$ is G-tempered $\iff p_{\mathfrak{g}/\mathfrak{h}} \leq 1$.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Plan of Lecture 4

0. Temperedness criterion (generalization)

Explore yet another relation of tempered homogeneous spaces with other disciplines.

1. Topology: Deforming Lie algebras
2. Geometry: Geometric quantization

Deformation of space forms S^n, $\mathbb{R}^n$, and H^n

$$K = SO(n+1) \curvearrowright S^n \qquad \mathfrak{k} = \mathfrak{so}(n+1)$$

$$\downarrow \qquad \downarrow \text{ limit algebra in } \mathfrak{g}$$

$$MN = SO(n) \ltimes \mathbb{R}^n \curvearrowright \mathbb{R}^n \qquad \mathfrak{m} + \mathfrak{n} = \mathfrak{so}(n) \ltimes \mathbb{R}^n$$

$$\uparrow \qquad \downarrow \text{ limit algebra in } \mathfrak{g}$$

$$H = SO(n,1) \curvearrowright H^n \qquad \mathfrak{h} = \mathfrak{so}(n,1)$$

View point from transformation groups

$G = SO(n+1,1)$ contains K, MN, and H.

Deformation of space forms S^n, $\mathbb{R}^n$, and H^n

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View point from transformation groups

$G = SO(n+1,1)$ contains K, MN, and H.

Deforming Lie algebras (1) — Example

Consider two equi-dimensional subalgebras of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$:

$$\mathfrak{k} = \mathfrak{so}(n), \qquad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

reductive   nilpotent

Observation $\exists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\,\mathfrak{k} = \mathfrak{n}$

Proof. $(n = 2)$ Take $g_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{-j} \end{pmatrix}$. Then

$$\mathrm{Ad}(g_j)\,\mathfrak{k} = \mathrm{Ad}(g_j)\mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbb{R}\begin{pmatrix} 0 & -2^{2j} \\ 2^{-2j} & 0 \end{pmatrix} \xrightarrow{j\to\infty} \mathbb{R}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \mathfrak{n}.$$

Remark $\nexists$ sequence $g_j \in SL(n, \mathbb{R})$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\,\mathfrak{n} = \mathfrak{k}$.

Limit algebras (2) — Formulation

By forgetting the Lie algebra structure of $\mathfrak{g}$, one considers

$$G \overset{\mathrm{Ad}}{\curvearrowright} \mathrm{Gr}(\mathfrak{g}) := \coprod_{m=0}^{\dim \mathfrak{g}} \mathrm{Gr}_m(\mathfrak{g}), \quad \text{(Grassmann variety).}$$

$\mathfrak{h}$: a subalgebra of $\mathfrak{g}$, with dimension m.

$\rightsquigarrow$ $\mathfrak{h}$ may be regarded as a point of $\mathrm{Gr}_m(\mathfrak{g})$.

$\mathrm{Gr}(\mathfrak{g}) \underset{\text{submanifold}}{\supset} \mathrm{Ad}(G)\mathfrak{h}$, which may or may not be closed.

$$\mathrm{Gr}(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty \quad \text{(limit algebra)}$$

Definition (limit algebra) $\mathfrak{h}_\infty$ $(\subset \mathfrak{g})$ is a limit algebra of $\mathfrak{h}$ in $\mathfrak{g}$ if $\exists$ sequence $g_j \in G$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h} = \mathfrak{h}_\infty$ in $\mathrm{Gr}(\mathfrak{g})$.

Limit algebras (3) — Properties

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $\mathrm{Gr}(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

Remark Limit algebra is not unique.

Basic properties

0) $\mathfrak{h}$ itself is a limit algebra of $\mathfrak{h}$.

1) Any limit algebra $\mathfrak{h}_\infty$ is an equi-dimensional Lie algebra.

2) If $\mathfrak{h}$ is $\begin{cases} \text{abelian} \\ \text{nilpotent} \\ \text{solvable} \end{cases}$ then any limit algebra $\mathfrak{h}_\infty$ is also $\begin{cases} \text{abelian} \\ \text{nilpotent} \\ \text{solvable} \end{cases}$.

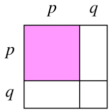
"Semisimple" $\mathfrak{h}$ may collapse to " solvable " $\mathfrak{h}_\infty$, but not vice versa.

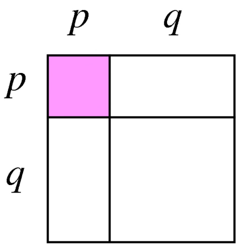
Limit algebras (4) — Example

$\mathfrak{g} \supset \mathfrak{h}$ subalgebra $\rightsquigarrow$ $Gr(\mathfrak{g}) \supset \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni \mathfrak{h}_\infty$ (limit algebra)

Remark $\mathfrak{h}_\infty$ is determined not only by $\mathfrak{h}$ itself but by how $\mathfrak{h}$ is embedded in $\mathfrak{g}$.

Exercise Fix p, and consider $\mathfrak{h} = \mathfrak{sl}_p \hookrightarrow \mathfrak{g} = \mathfrak{sl}_{p+q}$
Find a necessary and sufficient condition on (p, q) such that $\overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}$ solvable $\mathfrak{h}_\infty$.

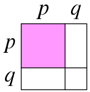




Deforming Lie algebras to solvable ones

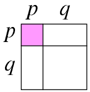
Example $\mathfrak{h} = \mathfrak{sl}_p \hookrightarrow \mathfrak{g} = \mathfrak{sl}_{p+q}$

$q \leq p$



does not have a solvable limit.

$q \geq p + 1$



has a solvable limit.

Definition (solvable limit algebra) $\mathfrak{h} \subset \mathfrak{g}$ Lie algebras

We say $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$ if

$^{\exists}\{g_j\} \in G$ such that $\lim_{j \to \infty} \mathrm{Ad}(g_j)\mathfrak{h}$ is a solvable Lie algebra.

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

Formulation: Consider the variety of all subalgebras in $\mathfrak{g}$.

$$Gr(\mathfrak{g}) \equiv \coprod_{N=0}^{\dim \mathfrak{g}} \mathrm{Gr}_N(\mathfrak{g}) \qquad \cdots \text{ algebraic variety}$$

$$\cup$$

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\} \qquad \cdots \text{ algebraic variety}$$

$$\cup$$

$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \text{ solvable}\}$$

$$\cup$$

$$\{\text{solvable subalgs}\} \qquad \cdots \text{ algebraic variety}$$

Question What does $\mathcal{S}$ look like in $\mathcal{L}$?

Variety of all Lie algebras $\mathcal{L}$ and its subset $\mathcal{S}$

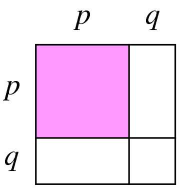
$\mathfrak{g}$: Lie algebra.

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$

$$\cup$$

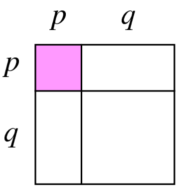
$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \ \text{solvable}\}$$

Question What does $\mathcal{S}$ look like in $\mathcal{L}$?



$p \geq q$

$\notin \mathcal{S} \ni$



$p \leq q - 1$

Topology of $\mathcal{S} = \{\mathfrak{h} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_{\infty}$ solvable$\}$

Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

<u>Open Problem P</u> Is $\mathcal{S}$ open in $\mathcal{L}$?

Topology of $\mathcal{S} = \{\mathfrak{h} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable$\}$

Suppose $\mathfrak{g}$ is an algebraic Lie algebra $/\mathbb{C}$.

Open Problem P Is $\mathcal{S}$ open in $\mathcal{L}$?

Theorem Q*

(1) $\mathcal{S}$ is closed in $\mathcal{L}$.

(2) $\mathcal{S}$ is open and closed in $\mathcal{L}$ if $\mathfrak{g}$ is semisimple.

Recall

$$\mathcal{L} := \{\text{subalgebras of } \mathfrak{g}\}$$
$$\cup$$
$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty \text{ solvable}\}$$

Our proof for Theorem Q uses unitary representation theory.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

$\mathcal{S}$ and temperedness of $L^2(G/H)$

G : complex algebraic Lie group,

H : algebraic subgroup.

We recall

$\mathcal{L}$:= {subalgebras of $\mathfrak{g}$}

$\cup$

$\mathcal{S}$:= $\{\mathfrak{h} \in \mathcal{L} : \overline{\mathrm{Ad}(G)\mathfrak{h}} \ni {}^{\exists}\mathfrak{h}_\infty$ solvable $\}$

<u>Theorem R*</u>

$L^2(G/H)$ is G_s-tempered $\iff$ $\mathfrak{h} \in \mathcal{S}$.

Since temperedness criterion $\rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}}$ in Theorem O is a closed condition, $\mathcal{S}$ is closed in $\mathcal{L}$, showing Theorem Q (1).

Sketch of Proof of Theorem R (easier part)

We explain an easier part of the inplication in Theorem R.

$$L^2(G/H) \text{ is } G_s\text{-tempered} \Longrightarrow \mathfrak{h} \in \mathcal{S}.$$

Take $\mathfrak{h}_\infty \in \overline{\mathrm{Ad}(G)\mathfrak{h}}$ such that $\mathrm{Ad}(G)\mathfrak{h}_\infty$ is closed. We show

$$\rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}_s \Longrightarrow \mathfrak{h}_\infty \text{ is solvable.}$$

- Can assume $\mathfrak{h} = \mathfrak{h}_\infty$.
- Can find a parabolic $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is an ideal of $\mathfrak{q}$

$\rightsquigarrow$

$\rho_{\mathfrak{g}_s} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{h}_s$ implies $\mathfrak{h}_s = 0$ after some elementary computation. Hence, $\mathfrak{h}$ is solvable.

Plan of Lecture 4

0. Temperedness criterion (generalization)

Explore yet another relation of tempered homogeneous spaces with other disciplines.

1. Topology: Deforming Lie algebras

2. Geometry: Geometric quantization

Geometric quantization and temperedness

$\mathrm{Ad}\colon G \to GL_{\mathbb{R}}(\mathfrak{g})$ adjoint representation.

$\mathrm{Ad}^*\colon G \to GL_{\mathbb{R}}(\mathfrak{g}^*)$ coadjoint representation.

Coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$ for $\lambda \in \mathfrak{g}^*$.

Lemma (Kostant–Kirillov–Souriau)
Every coadjoint orbit $\mathcal{O}_\lambda$ carries a natural symplectic structure.

"Geometric quantization":

$$\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \overset{?}{\rightsquigarrow} \pi_\lambda \in \widehat{G}$$

symplectic mfd unitary rep

Expect

$$\mathfrak{g}^* / \mathrm{Ad}^*(G) \fallingdotseq \widehat{G}$$

From orbit philosophy by Kirillov–Kostant

We assume now G is a complex reductive Lie group.

$$\mathfrak{g}^* \supset \mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\lambda \text{ is of maximal dimension}\},$$
$$\mathfrak{g}^* \supset \mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}.$$

Orbit philosophy by Kirillov–Kostant

$$\begin{array}{ccc}
\mathrm{Ad}^*(G)\mathfrak{h}^\perp / \mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\
\cap & & \cap \\
\mathfrak{g}^* / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\
\cup & & \cup \\
\mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}}
\end{array}$$

Remark $\mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset \iff \mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \underset{\text{dense}}{\subset} \mathfrak{h}^\perp$

Geometric quantization and temperedness

"Geometric quantization": $\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \overset{?}{\rightsquigarrow} \pi_\lambda \in \widehat{G}$

symplectic mfd — unitary rep

$$\begin{array}{ccc} \mathrm{Ad}^*(G)\mathfrak{h}^\perp / \mathrm{Ad}^*(G) & \fallingdotseq & \mathrm{Supp}(L^2(G/H)) \\ \cap & & \cap \\ \mathfrak{g}^* / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G} \\ \cup & & \cup \\ \mathfrak{g}^*_{\mathrm{reg}} / \mathrm{Ad}^*(G) & \fallingdotseq & \widehat{G}_{\mathrm{temp}} \end{array}$$

Theorem S*

Suppose G is a complex reductive Lie group, and H a connected closed subgroup. Then (i) $\Leftrightarrow$ (ii).

(i) $G \curvearrowright L^2(G/H)$ is tempered.

(ii) $\mathfrak{g}^*_{\mathrm{reg}} \cap \mathfrak{h}^\perp \neq \emptyset$.

$\mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G) \cdot \lambda$ is of maximal dimension$\}$

$\mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, 28 pages, 2022.

Further interactions for "tempered spaces"

Theorem T Let $\mathfrak{g}$ be a complex reductive Lie algebra.
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (Analysis) $L^2(G/H)$ is tempered .

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

(iii) (Geometric quantization) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (Topology) $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.

Application Representation theory $\Longrightarrow$ Topology

Corollary U (Topology) The property "having solvable limit" is an open and closed condition for subalgebras in a complex reductive Lie algebra $\mathfrak{g}$, namely, $\mathcal{S}$ is open and closed in $\mathcal{L}$.

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Sketch of Proof for Theorem S: Tempered homogeneous spaces

Thm T Let $\mathfrak{g}$ be a complex reductive Lie algebra.

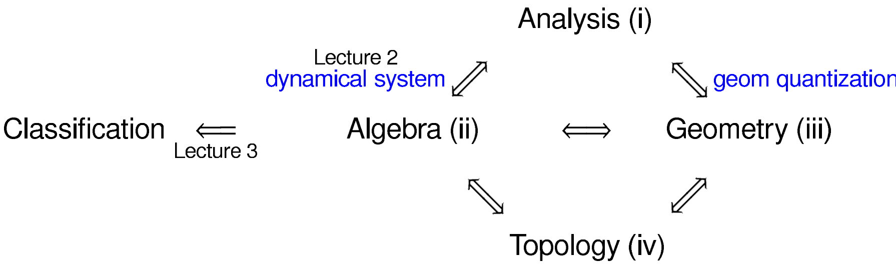
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (unitary rep) $L^2(G/H)$ is tempered .

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

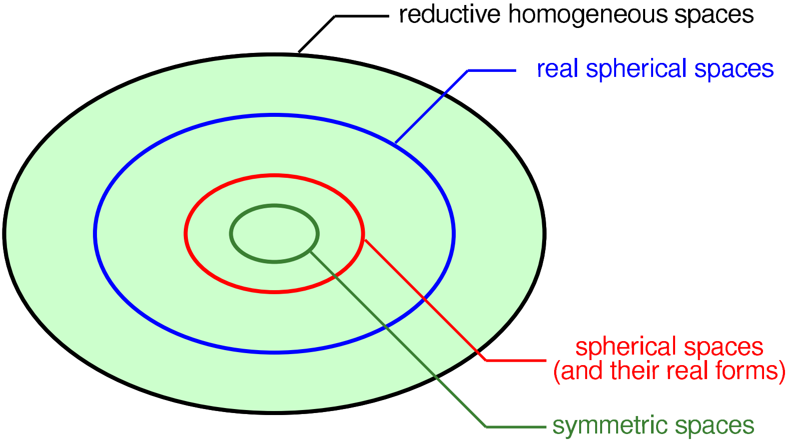
(iii) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

(iv) (limit algebra) $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.



Reductive homogeneous space G/H

G: real reductive groups
H: reductive subgroup



We shall also discuss when G and H are not nesssarily reductive.

Basic Questions
in Group-Theoretic Analysis on Manifolds

$$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X),\ L^2(X),\ \cdots$$

Geometry — Function Space

Basic Question 1 (Lecture 1)

Does the group G "control well" $C^\infty(X)$?

Use a system of PDEs.

Formulation Consider the dimension of

$$\mathrm{Hom}_G(\pi, C^\infty(X)) \quad \text{for } \pi \in \mathrm{Irr}(G).$$

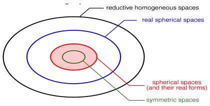
infinite, finite, bounded, 0 or 1

control better

Basic Questions
in Group-Theoretic Analysis on Manifolds

$G \curvearrowright X$ $\rightsquigarrow$ $G \curvearrowright C^\infty(X), L^2(X), \cdots$

Geometry — Function Space



<u>Basic Question 1</u> (Lecture 1)

Does the group G "control well" $C^\infty(X)$?

Use a system of PDEs.

Lecture 1

<u>Theorem B</u> *The following conditions are all equivalent:

(i) (Analysis & rep theory) There exists $C > 0$ s.t.

$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \le C$ for all $\pi \in \mathrm{Irr}(G)$.

(ii) (Complex geometry) $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical.

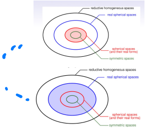
(ii)′ (Algebra) The ring $\mathbb{D}_G(X)$ is commutative.

(ii)″ (Algebra) The ring $\mathbb{D}_G(X)$ is a polynomial ring.

Basic Questions
in Group-Theoretic Analysis on Manifolds

$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X),\ L^2(X),\ \cdots$

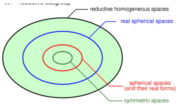
Geometry — Function Space



Basic Question 1 (Lecture 1)

Does the group G "control well" $C^\infty(X)$?

Use a system of PDEs.



Basic Question 2 (Lectures 2–4)

What is the spectrum of $L^2(X)$?

Can we decompose $L^2(X)$ by irreducible tempered reps?

Use ideas of dynamical system, combinatorics, and deformation.

Thank you very much!

Main References

Lecture 1.

T. Kobayashi–T. Oshima, Finite multiplicity theorems for induction and restriction, Adv. Math. (2013).

T. Kobayashi, Conjectures on Reductive Homogeneous Spaces, Lecture Notes in Mathematics **2313**, Springer, (2023).

Lectures 2–4.

Y. Benoist–T. Kobayashi, Tempered homogeneous spaces I, J. Eur. Math. Soc. (2015); II, Univ. Chicago Press (2022); III, J. Lie Theory (2021); IV, J. Inst. Math. Jussieu (2023).

Short Expository Articles for Lectures 1–4.

T. Kobayashi, Topics on global analysis of manifolds and representation theory of reductive groups, PROMS, Springer (2020).