Harish-Chandra's Tempered Representations and Geometry II

Tempered homogeneous spaces and tempered subgroups — Dynamical approach

Toshiyuki Kobayashi

The Graduate School of Mathematical Sciences
The University of Tokyo

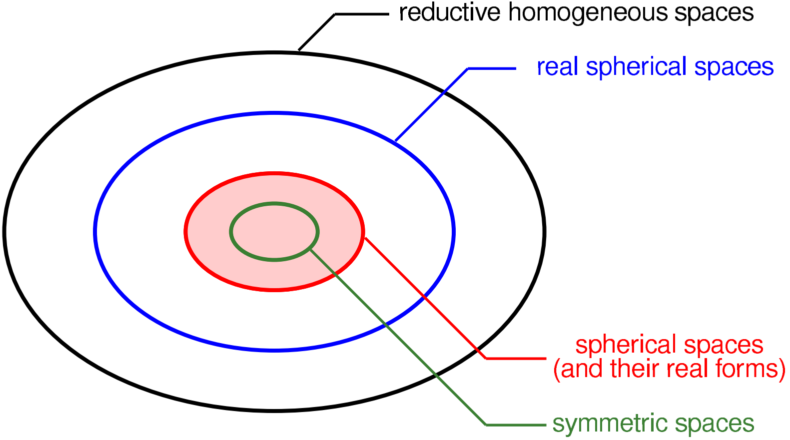
http://www.ms.u-tokyo.ac.jp/~toshi/

18th Discussion Meeting in Harmonic Analysis
(In honour of centenary year of Harish Chandra)
Indian Institute of Technology Guwahati, India, 13 December 2023

Reductive homogeneous space G/H

G: real reductive groups

H: reductive subgroup

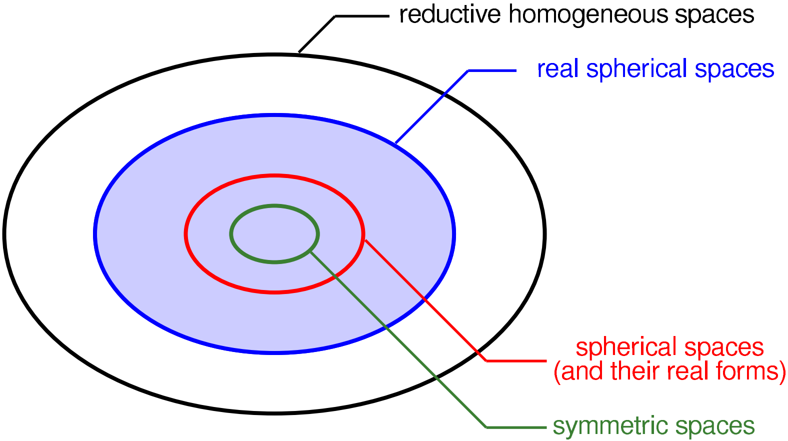


We shall also discuss when G and H are not nesssarily reductive.

Reductive homogeneous space G/H

G: real reductive groups

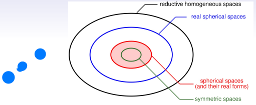
H: reductive subgroup

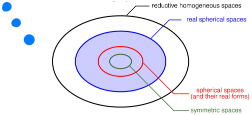


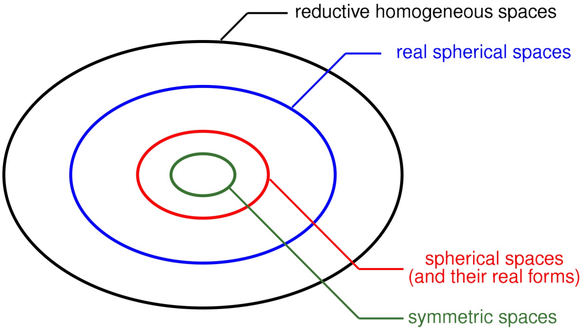
We shall also discuss when G and H are not nesssarily reductive.

Plan of Lectures

- Talk 1: Is rep theory useful for global analysis?
  —Multiplicity: Approach from PDEs







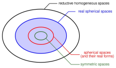
Topic of Yesterday (Lecture 1)

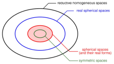
$$G \curvearrowright X = G/H \rightsquigarrow G \curvearrowright C^\infty(X),\ L^2(X)$$

Geometry　　　　Functions

What is a geometric condition for $G \curvearrowright X$ that assures a "strong grip" of G on $C^\infty(X)$ in the sense of "multiplicities"?

	Multiplicity	Geometry
Thm A :	"finite"	$\cdots G/H$ is real spherical.
Thm B :	"uniformly bounded"	$\cdots G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.

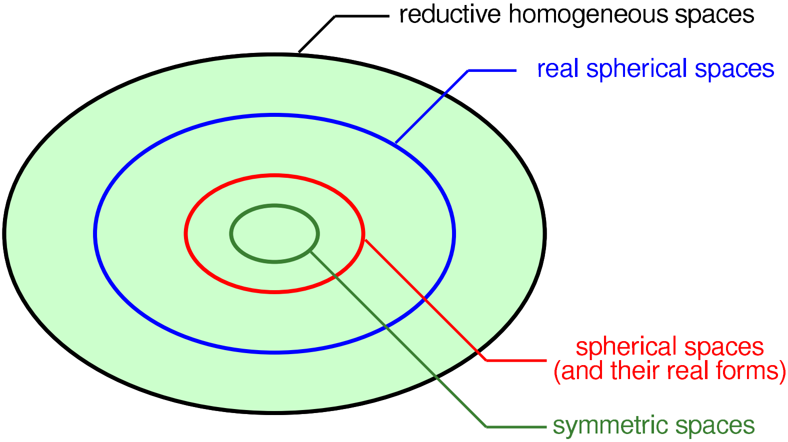




(Thms C and D $\cdots$ counterpart for the restriction $G \downarrow H$.)

Reductive homogeneous space G/H

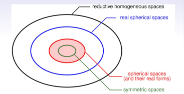
G: real reductive groups
H: reductive subgroup

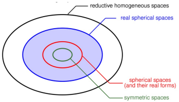


We shall also discuss when G and H are not nesssarily reductive.

Plan of Lectures

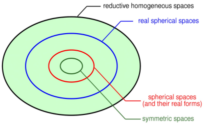
- Talk 1: Is rep theory useful for global analysis?
  —Multiplicity: Approach from PDEs





- Talk 2: Tempered homogeneous spaces
  —Dynamical approach

- Talk 3: Classification theory of tempered G/H
  —Combinatorics of convex polyhedra



- Talk 4: Tempered homogeneous spaces
  —Interaction with topology and geometry

Plan for Today

Beyond spherical cases and "coarse information".

Basic Problem (Today) Find a geometric criterion for $G \curvearrowright X$ that assures $L^2(X)$ to be almost L^p.

– Change of approach

PDE $\rightsquigarrow$ Dynamical approach

Plan of Today (Lecture 2)

- Methods and elementary examples
  - Optimal constant $q(G; X)$ for L^q-estimate $\mathrm{vol}(gS \cap S)$.
  - Almost L^p-representation.
- Tempered homogeneous spaces
- Tempered subgroups

Learn from Dynamical System

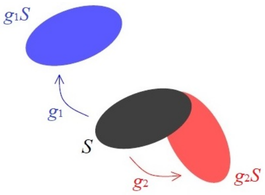
G : locally compact group

X : locally compact space

Definition A continuous action $G \curvearrowright X$ is called proper if the subset

$$G_S := \{g \in G : S \cap gS \neq \emptyset\}$$

is compact for any compact subset $S \subset X$.



$\Longrightarrow$ $g_1 \notin G_S$, $g_2 \in G_S$

Definition The action is free $\Longleftrightarrow$ $G_{\{x\}} = \{e\}\ \forall x \in X$.

Criterion for proper actions — topology

Basic problem (topology) Given a geometry X.
Find a criterion for a group L $(\subset \mathrm{Aut}(X))$ to act properly on X.

Group theoretic approach:

- Properness criterion was established for a homogeneous space X of a reductive group G (1989*–1996).
  $\cdots$ Applications include a solution (1989*) to the Calabi-Markus phenomenon (Ann. Math., 1962).
- Properness criterion for nilpotent Lie groups G up to 3-step (1995–**).
- Open problems in general.***

* T. Kobayashi (Math. Ann., ’89 and JLT ’96), Benoist (Ann. Math., ’96);
** R. Lipsman (JLT ’95), S. Nasrin (’01), T. Yoshino (IJM, ’07), Baklouti–Khlif (IMM, ’05) et al;
*** T. Kobayashi, Conjectures on reductive homogeneous spaces, Lect. Notes in Math., (2023).

Non-proper action — delicate example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

<u>Example</u> Let $\mathbb{R} \ni t$ act on $\mathbb{R}^2$ by

$$(x, y) \mapsto (e^t x, e^{-t} y).$$

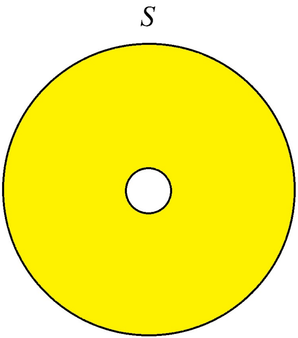
1) This action is neither free nor proper because the origin $(0,0)$ is a fixed point. The removal of the origin makes the situation slightly better.
2) The action on $X := \mathbb{R}^2 \setminus \{(0,0)\}$ is free, but is not proper.

Non-proper action — delicate example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

<u>Example</u> Let $\mathbb{R} \ni t$ act on $\mathbb{R}^2$ by

$$(x, y) \mapsto (e^t x, e^{-t} y).$$

1) This action is neither free nor proper
   because the origin $(0,0)$ is a fixed point.
   The removal of the origin makes the situation slightly better.
2) The action on $X := \mathbb{R}^2 \setminus \{(0,0)\}$ is free, but is not proper.



Non-proper action — delicate example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

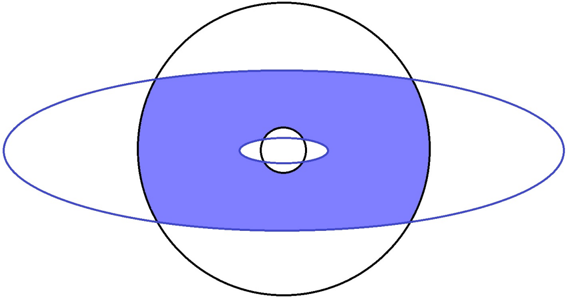
Example Let $\mathbb{R} \ni t$ act on $\mathbb{R}^2$ by

$$(x,y) \mapsto (e^t x, e^{-t} y).$$

1) This action is neither free nor proper
   because the origin $(0,0)$ is a fixed point.
   The removal of the origin makes the situation slightly better.
2) The action on $X := \mathbb{R}^2 \setminus \{(0,0)\}$ is free, but is not proper.

e.g.
$(g = 2 \in \mathbb{R})$

$S \cap gS$



Idea: Quantify proper actions

Locally compact group $G \curvearrowright X$ locally compact space

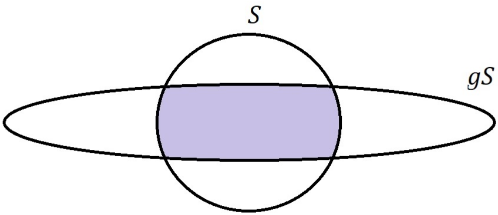
$G \curvearrowright X$ proper $\overset{\text{def}}{\Leftrightarrow} \{g \in G : S \cap gS \neq \emptyset\}$ is compact $^{\forall}S \subset X$ compact,

$\Leftrightarrow \mathrm{vol}(S \cap gS) \in C_c(G)$ $^{\forall}S \subset X$ compact,

where we fix an appropriate Radon measure on X.

Idea: Quantitative estimate for non-proper actions.

Look at asymptotic behavior of $\mathrm{vol}(S \cap gS)$ as g goes to infinity.



Volume estimate $\mathrm{vol}(t \cdot S \cap S)$: example $\quad \mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$

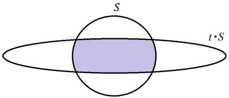
Example Let $\mathbb{R} \ni t$ act on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ by

$$(x, y) \mapsto (e^t x, e^{-t} y)$$

- This action is free, but is not proper.
- Asymptotic behavior of $\mathrm{vol}(S \cap t \cdot S)$.

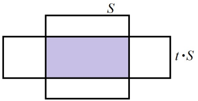
For any compact neighbourhood S of the origin in $\mathbb{R}^2$, one has

$$C_1 e^{-|t|} \leq \mathrm{vol}(t \cdot S \cap S) \leq C_2 e^{-|t|}.$$



For instance, if $S = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$,

$$\mathrm{vol}(t \cdot S \cap S) = 4e^{-|t|}.$$



Almost L^p function

Z: locally compact space equipped with a Radon measure.
Eg. a locally compact group G with (left) Haar measure.

Definition A measurable function f on Z is almost L^p if

$$f \in \bigcap_{\varepsilon>0} L^{p+\varepsilon}(Z).$$

Remark For $p \leq p'$, one has
f is almost $L^p \Longrightarrow f$ is almost $L^{p'}$.

We are interested in the best possible p for which f is almost L^p, in particular, when Z is a semisimple Lie group G.
(*e.g.*, $G = SL(n,\mathbb{R}), SU(p,q), SO(p,q), Sp(n,\mathbb{R}). \cdots$).

Example 1. L^p-estimate of K-finite eigenfunctions

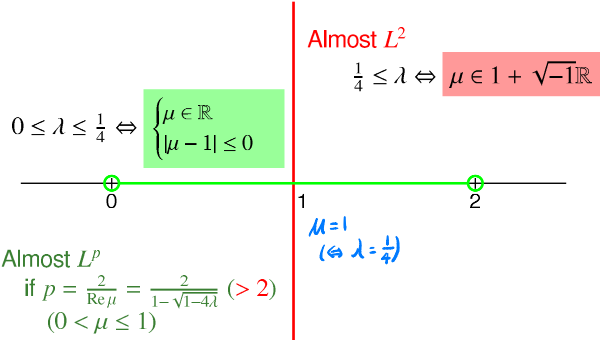
$$D = \{z \in \mathbb{C} : |z| < 1\} \quad ds^2 = \frac{4(dx^2+dy^2)}{(1-|z|^2)^2} \quad \text{(Poincaré disc)}$$

Any K-finite function f satisfying $\Delta f = \lambda f$ is almost $L^{p(\lambda)}$ ($\lambda > 0$), where $p(\lambda) := \frac{2}{1-\sqrt{1-4\lambda}}$ $(0 \leq \lambda \leq \frac{1}{4})$; $= 2$ $(\frac{1}{4} \leq \lambda)$. In fact, one has

$$f(\tanh t(\cos\varphi, \sin\varphi)) \sim Ae^{-\mu_+ t} + Be^{-\mu_- t}$$

where $\mu_\pm := 1 \pm \sqrt{1-4\lambda}$ and λ is generic (Lecture 1).

Figure in the μ-plane with $\lambda = -\frac{1}{4}(\mu^2 - 2\mu)$.



Example 2. L^p-estimate of $\mathrm{vol}(gS \cap S)$ for $G \curvearrowright G/N$

The example $\mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$, $(x,y) \mapsto (e^t x, e^{-t} y)$ is interpreted as

$$A \hookrightarrow G \curvearrowright G/N \Longleftrightarrow \mathbb{R} \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}.$$

($A \simeq \mathbb{R}$, $G/N \simeq \mathbb{R}^2 \setminus \{(0,0)\}$)

$$A = \{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R}\} \subset G = SL(2,\mathbb{R}) \supset N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}.$$

> $\mathrm{vol}(gS \cap S)$ is almost $L^2(G)$ for any compact subset $S \subset G/N$.

- For any compact $S \subset G/N$ and $g = k_1 a_t k_2$ with $k_1, k_2 \in SO(2)$,

$$\mathrm{vol}(gS \cap S) \sim e^{-|t|} \quad \text{(previous example).}$$

- Haar measure on $g = k_1 a_t k_2 \in G = SL(2,\mathbb{R})$: One has

$$dg = \sinh(2t) dk_1 dt dk_2 \sim e^{2|t|} dk_1 dt dk_2.$$

Hence

$$\mathrm{vol}(gS \cap S) \in L^{p+\varepsilon}(G) \Longleftrightarrow 2 - p - \varepsilon < 0.$$

Optimal constant $q(G;X)$ of volume estimate

$$G \curvearrowright X$$

Suppose X admits a G-invariant Radon measure.

Definition We write $q(G;X)$ for the optimal constant $q > 0$ such that

$$\mathrm{vol}(S \cap gS) \text{ is an almost } L^q\text{-function on } G$$

for every compact subset $S \subset X$.

Example $q(G;X) = 2$ if $(G, X) = (SL(2,\mathbb{R}), \mathbb{R}^2)$.

General Problem Find an explicit formula of $q(G;X)$.

Finding the optimal L^p-estimate of $\mathrm{vol}(gS \cap S)$

Let G be a semisimple Lie group acting on X.
$q(G;X)$: the optimal constant for L^q-estimate of $\mathrm{vol}(gS \cap S)$.

We shall give an explicit formula of $q(G;X)$
when $X = V$ (linear action) or $X = G/H$ (H: reductive).

			Method
(Theorem E)	Case 1	$G \curvearrowright V$ (G: semisimple, V: linear)	Dynamical approach
(Theorem F)	Case 2	$G \supset H$ (G: semisimple, H: reductive)	Global geometry + Case 1

L^p-estimate of $\mathrm{vol}(gS \cap S) \cdots$ Case 1. $H \curvearrowright V$ linear

Notation : $G \curvearrowright X \rightsquigarrow H \curvearrowright V$ (linear)

Let H be a semisimple Lie group, and $\tau\colon H \to SL_{\mathbb{R}}(V)$ a representation. Assume τ has a compact kernel.

The optimal constant $q(H;V)$ for $\mathrm{vol}(gS \cap S)$ to be almost L^q is given as follows.

Theorem E For a linear action $H \curvearrowright V$, one has

$$q(H;V) = p_V .$$

analysis combinatorics

$$p_V := \max_{Y \in \mathfrak{h}\setminus\{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$$

$\rho_{\mathfrak{h}}$, ρ_V $\cdots$ next page.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Piecewise linear function ρ_V associated to $\tau\colon \mathfrak{h} \to \mathrm{End}(V)$

For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduce:

<u>Definition</u> (non-negative function ρ_V on the Lie algebra $\mathfrak{h}$)

$$\rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\mathrm{Re}\, \lambda(Y)|.$$

gen. eigenvalues of $\tau(Y) \in \mathrm{End}(V_{\mathbb{C}})$

Let $\mathfrak{a}$ be a maximal split abelian subspace of the Lie algebra $\mathfrak{h}$.

$\rightsquigarrow$
- ρ_V is determined by its restriction to $\mathfrak{a}$,
- $\rho_V|_{\mathfrak{a}}$ is a piecewise linear function.

<u>Remark</u> For a reductive $\mathfrak{h}$ and for $(\tau, V) = (\mathrm{ad}, \mathfrak{h})$,

$\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ = twice the <u>usual ρ</u> on the dominant Weyl chamber,

however, our $\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ is not linear whereas the <u>usual ρ</u> is linear.

A constant p_V associated to $\tau\colon \mathfrak{h} \to \mathrm{End}(V)$

Let $\mathfrak{a}$ be a maximally split abelian subspace of a Lie algebra $\mathfrak{h}$. For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduce:

Definition $p_V := \max_{Y\in\mathfrak{h}\setminus\{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} = \max_{Y\in\mathfrak{a}\setminus\{0\}} \frac{\sum|\text{eigenvalue of } \mathrm{ad}(Y)\in\mathrm{End}(\mathfrak{h})|}{\sum|\text{eigenvalue of } \tau(Y)\in\mathrm{End}(V)|}$.

Short Summary $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$

$\rightsquigarrow$ ρ_V $\cdots$ piecewise linear function

p_V $\cdots$ positive number

Example $H_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\mathfrak{a} = \mathbb{R}H_0 \subset \mathfrak{h} = \mathfrak{sl}(2,\mathbb{R}) \curvearrowright V = \mathbb{R}^2$

$\rho_{\mathfrak{h}}(tH_0) = \frac{1}{2}(|2t| + 0 + |-2t|) = 2|t|$.

$\rho_V(tH_0) = \frac{1}{2}(|t| + |-t|) = |t|$.

$p_V = 2$.

Sketch of Proof for Theorem E: $H \curvearrowright V$ (linear)

Let H be a semisimple Lie group. Suppose $\tau: H \to GL_{\mathbb{R}}(V)$ has a compact kernel. As in the case $(H, V) = (SL(2,\mathbb{R}), \mathbb{R}^2)$, one has

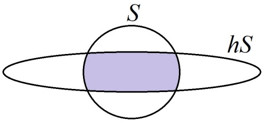
<u>Theorem E</u> For a linear action $H \curvearrowright V$, one has

$$q(H;V) = p_V .$$

analysis   combinatorics

Proof. • For $H \ni h = k_1 e^Y k_2$, one has

$$\mathrm{vol}(hS \cap S) \sim e^{-\rho_V(Y)}.$$



• For the Haar measure dh on H, one has

$$dh \sim e^{\rho_{\mathfrak{h}}(Y)} dk_1 dY dk_2 \quad \text{(away from wall).}$$

Therefore the $L^{q+\varepsilon}$-estimate of $\mathrm{vol}(hS \cap S)$ amounts to

$$\mathrm{vol}(hS \cap S)^{q+\varepsilon} dh \sim e^{\rho_{\mathfrak{h}}(Y) - (q+\varepsilon)\rho_V(Y)} dk_1 dY dk_2.$$

□

Strategy: finding the optimal L^p-estimate of $\mathrm{vol}(gS \cap S)$

Let $G \curvearrowright X$.

$q(G;X)$: the optimal constant for L^q-estimate of $\mathrm{vol}(gS \cap S)$.

We discussed when $X = V$ (linear). Now consider $X = G/H$.

			Method
(Theorem E)	<u>Case 1</u>	G (semisimple) $\curvearrowright$ V (linear)	Dynamical approach
(Theorem F)	<u>Case 2</u>	G (semisimple) $\supset$ H (reductive)	Global geometry + Case 1

Case 2 $G \supset H$

G semisimple, H reductive

Recall $q(G;X)$ is the optimal constant q for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset X$.

Theorem F* Let G be a semisimple Lie group, and H a reductive subgroup. Then one has

$$q(G;G/H) = p_{\mathfrak{g}/\mathfrak{h}} + 1.$$

analysis — combinatorics

Recall $p_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_{\mathfrak{g}/\mathfrak{h}}(Y)}$ is defined for a linear action $H \curvearrowright V$.

Point It turns out that one can control $\mathrm{vol}(gS \cap S)$ for $g \in G$ only by "ρ-function" for the subgroup H acting on $\mathfrak{g}/\mathfrak{h}$.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

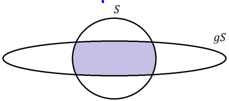
Case 2 $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$ Global geometry + Case 1

Asymptotic estimate of volume

For any compact $S \subset G/H$, we want to find $m(g)$ and $M(g)$:

$$m(g) \le \mathrm{vol}(gS \cap S) \le M(g) \quad \text{for all } g \in G.$$

for $g \in H$



$$H \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h} \underset{\text{locally}}{\fallingdotseq} G/H.$$

Some difficulties to overcome:

- Need a lower bound $m(g)$ for $g \in G$, not only for $g \in H$.
-

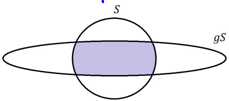
Case 2 $\underset{\text{semisimple}}{G} \supset \underset{\text{reductive}}{H}$ Global geometry + Case 1

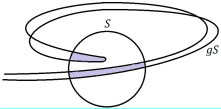
Asymptotic estimate of volume

For any compact $S \subset G/H$, we want to find $m(g)$ and $M(g)$:

$$m(g) \le \mathrm{vol}(gS \cap S) \le M(g) \quad \text{for all } g \in G.$$

for $g \in H$





$$H \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h} \underset{\text{locally}}{\fallingdotseq} G/H.$$

Some difficulties to overcome:

- Need a lower bound $m(g)$ for $g \in G$, not only for $g \in H$.
- An upper bound $M(g)$ is more involved.

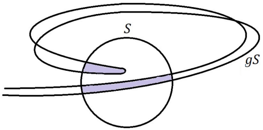
Case 2 $G \supset H$ Global geometry + Case 1

semisimple reductive

Theorem F* Let G be a semisimple Lie group, and H a reductive subgroup. Then one has

$$q(G; G/H) = p_{\mathfrak{g}/\mathfrak{h}} + 1.$$

analysis combinatorics



Key idea: Quantify the proof of the properness criterion ** for subgroups L of G acting on G/H.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

** T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann., **285** (1989), 249–263.

Plan

- Methods and elementary examples
  - Optimal constant $q(G;X)$ for L^q-estimate $\mathrm{vol}(gS \cap S)$.
  - Almost L^p-representation, tempered representations.
- Tempered homogeneous spaces.
- Tempered subgroups.

Almost L^p representations

Almost L^p functions

$\rightsquigarrow$

Almost L^p representations

Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$.

Definition For $p \geq 1$, $(\pi, \mathcal{H})$ is called almost L^p if there is a dense subspace $D \subset \mathcal{H}$ such that matrix coefficients for $x, y \in D$ are almost L^p, namely,

$$(\pi(g)x, y)_{\mathcal{H}} \in \bigcap_{\varepsilon>0} L^{p+\varepsilon}(G) \qquad {}^\forall x, {}^\forall y \in D$$

Harish-Chandra's tempered representation — Definition

Let G be a locally compact group.

Def A unitary rep π of G is called tempered if $\pi \ll L^2(G)$.

$\ll$ $\cdots$ weakly contained

i.e., every matrix coefficient of π is a uniform limit on every compacta of G by a sequence of sum of coefficients of $L^2(G)$.

Almost L^2 representation *vs* tempered representations

Definition A unitary representation π of G is called tempered if $\pi \ll L^2(G)$.

- For a semisimple Lie group G, one has

Fact G (Cowling–Haagerup–Howe)* One has the equivalence:

$$\pi \text{ is tempered} \iff \pi \text{ is almost } L^2.$$

* M. Cowling–M. Haagerup–R. Howe, Almost L^2 matrix coefficients, J. Reine Angew. Math. **387**, (1988), 97–110.

Almost L^2 representation *vs* tempered representations

Definition A unitary representation π of G is called tempered if $\pi \ll L^2(G)$.

- For a solvable Lie group G,

all unitary reps π are tempered (Hulanicki–Reiter), but are not always almost L^2.
E.g. the trivial one-dimensional rep is not almost L^p $(1 \leq p < \infty)$ if G is non-compact.

- For a semisimple Lie group G, one has

Fact G (Cowling–Haagerup–Howe)* One has the equivalence:
π is tempered $\iff$ π is almost L^2.

* M. Cowling–M. Haagerup–R. Howe, Almost L^2 matrix coefficients, J. Reine Angew. Math. **387**, (1988), 97–110.

Temperedness under disintegration

Mautner: Any unitary rep Π can be decomposed into irreducibles:

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi \, \pi \, d\mu(\pi) \qquad \text{(direct integral).}$$

<u>Fact</u> Π is tempered $\Leftrightarrow$ <u>irreducible</u> reps π are tempered for μ-a.e.

$$\begin{aligned} \widehat{G} \quad &= \{\text{irreducible unitary reps}\} \\ \cup \quad & \\ \widehat{G}_{\text{temp}} &:= \{\text{irreducible tempered reps}\}. \end{aligned}$$

That is,

$$\Pi \text{ is tempered} \iff \int_{\widehat{G}_{\text{temp}}}^{\oplus} m_\pi \pi d\mu(\pi).$$

Classification theory of the unitary dual $\widehat{G}$

<u>Fact</u> (Kirillov, Duflo) Classification of the unitary dual $\widehat{G}$ for real algebraic groups G is reduced to that for real reductive Lie groups.

Suppose G is a real reductive Lie group (*e.g.*, $GL(n,\mathbb{R})$, $O(p,q)$).

irred. admissible		irred. unitary		irred. tempered
$\mathrm{Irr}(G)$	$\supset$	$\widehat{G}$	$\supset$	$\widehat{G}_{\mathrm{temp}}$
classified		(still mysterious)		classified
Langlands		over 70 years		Knapp–Zuckerman

reduce

Tempered representations (warming up)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$= \{\mathbf{1}\} \amalg \{$ principal series $\} \amalg \{$ complementary series $\}$

$\amalg \{$ discrete series $\} \amalg \{$ limit of discrete series $\}$

Tempered representations (warming up)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$= \{ \mathbf{1} \} \amalg \{$ principal series $\} \amalg \{$ complementary series $\}$

$\amalg \{$ discrete series $\} \amalg \{$ limit of discrete series $\}$

$-\frac{1}{2}$ Casimir operator acts on them as scalars

$\{0\}$, $[\frac{1}{4}, \infty)$, $(0, \frac{1}{4})$, $\{\frac{1}{4}(n^2 - 1) : n \in \mathbb{N}_+\}$, $\{0\}$

Γ: congruence subgroup of $G = SL(2,\mathbb{R})$

Selberg's $\frac{1}{4}$ eigenvalue conjecture *:

All eigenvalues of Δ on Maas wave forms for $\Gamma \geq \frac{1}{4}$.

$\Longleftrightarrow$ The unitary rep of $G \curvearrowright L^2_{\mathrm{cusp}}(\Gamma \backslash G)$ is tempered.

Just one irred non-tempered rep would deny the conjecture.

* A. Selberg, On the estimate of Fourier coefficients of modular forms, Proc. Symp. Pure Math. 1965.

Irreducible tempered reps — semisimple Lie groups

Def A unitary representation π of G is called tempered if $\pi \ll L^2(G)$.

- For a semisimple Lie group G and irreducible $\pi \in \widehat{G}$, tempered representations π have been studied extensively.

Known results on tempered reps and beyond ...

- Many equivalent definitions, *e.g.*, $L^{2+\varepsilon}(G)$,
- Harish-Chandra's theory towards Plancherel formula,
- Knapp–Zuckerman's classification *,
- A cornerstone of Langlands' classification,
- Selberg $\frac{1}{4}$ eigenvalue conjecture (1965-),
- Gan–Gross–Prasad conjecture, $\cdots$

* A. W. Knapp–G. Zuckerman, Classification of irreducible tempered representations of semisimple Lie groups, Ann. Math., (1980), 389–455; 457–501.

Tempered homogeneous spaces and tempered subgroups

$G \supset H$ Lie groups

- Induction

Definition We say G/H is a tempered homogeneous space if $L^2(G/H)$ is a tempered rep of G.

- Restriction

Definition We say H is a G-tempered subgroup if $\pi|_H$ is tempered for any $\pi \in \widehat{G} \setminus \{\mathbf{1}\}$.

cf. Margulis used the terminology "G-tempered subgroup" in a stronger sense by using an L^1-estimate rather than an $L^{2+\varepsilon}$-estimate.

Basic questions on Harish-Chandra's tempered representations

$G \supset H$ Lie groups

<u>Problem 2</u> (induction) Find a criterion for (G, H) such that $L^2(G/H)$ is a tempered rep of G.

<u>Problem 3</u> (restriction) Find a criterion for (G, H) such that the restriction $\pi|_H$ is a tempered rep of H $\quad {}^\forall \pi \in \widehat{G} \setminus \{\mathbf{1}\}$.

We shall see that Problem 3 is related to the existence problem of cocompact discontinuous groups Γ for G/H.

Tempered homogeneous space $X = G/H$, *i.e.*, $L^2(X) \ll L^2(G)$

Problem 2 When is the unitary rep on $L^2(X)$ tempered?

$\Downarrow\!\!\!\!/$

cf. $L^2(X)$ can be disintegrated by irred X-tempered reps (this is almost 'tautology'). (Harish-Chandra, Oshima, Bernstein $\sim$ 80s).

Towards a temperedness criterion

Problem 2 For which pair $G \supset H$, is the unitary rep of G on $L^2(G/H)$ tempered?

For semisimple Lie groups G, we have already discussed a refinement of Problem 2 as below:

Problem 1 Find the optional constant $q(G;G/H)$ for which $\mathrm{vol}(gS \cap S)$ is almost L^q for all compact subset $S \subset G/H$.

$$q(G;G/H) \leq 2 \Longleftrightarrow L^2(G/H) \text{ is tempered.}$$

Temperedness criterion in the reductive case

G semisimple Lie group,

H any reductive subgroup.

Since we know from Theorem F that

$$q(G; G/H) = p_{\mathfrak{g}/\mathfrak{h}} + 1$$

analysis — combinatorics

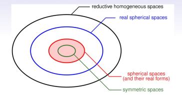
where $p_V = \max_{\mathfrak{h} \ni Y \neq 0} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$ is defined for a linear action $H \curvearrowright V$, one obtains from the volume estimate:

> Theorem F′* For a pair of real reductive Lie groups, one has
> $L^2(G/H)$ is G-tempered $\iff p_{\mathfrak{g}/\mathfrak{h}} \leq 1$.

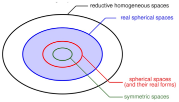
Remark. $p_{\mathfrak{g}/\mathfrak{h}} \leq 1 \iff 2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{h}$.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Plan of Lectures

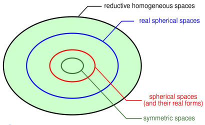


- Talk 1: Is rep theory useful for global analysis?
  —Multiplicity: Approach from PDEs



- Talk 2: Tempered homogeneous spaces
  —Dynamical approach

- Talk 3: Classification theory of tempered G/H
  —Combinatorics of convex polyhedra



- Talk 4: Tempered homogeneous spaces
  —Interaction with topology and geometry

Thank you for your attention!