Harish-Chandra's Tempered Representations and Geometry I

Is rep theory useful for global analysis on a manifold? —Multiplicity: Approach from PDEs

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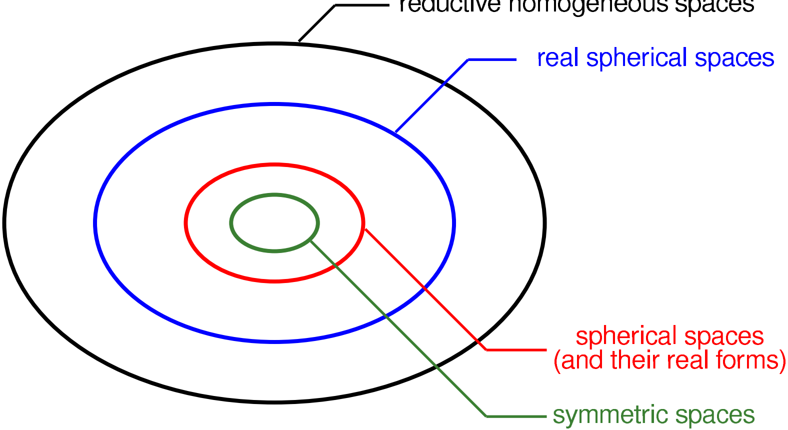
Reductive homogeneous space G/H

G: real reductive groups
H: reductive subgroup



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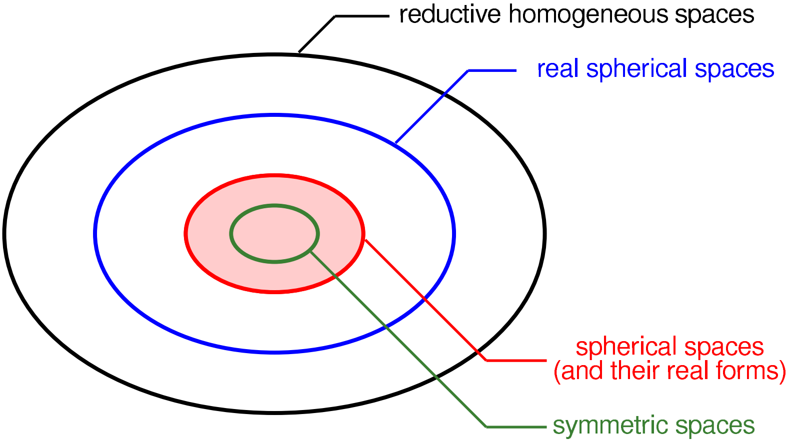

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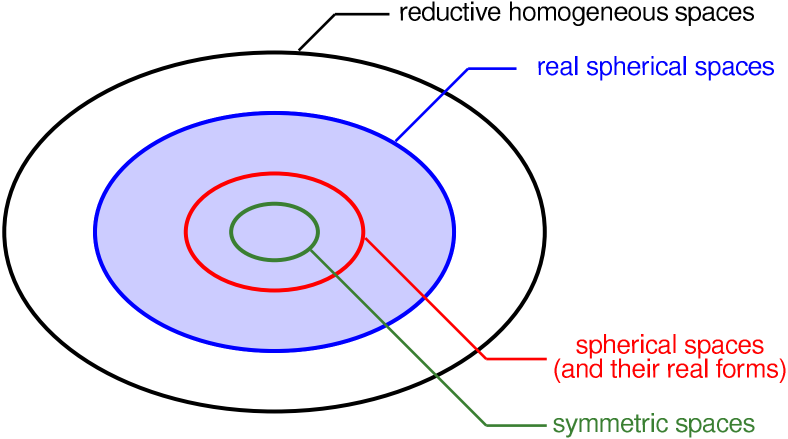

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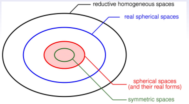
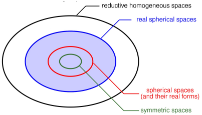
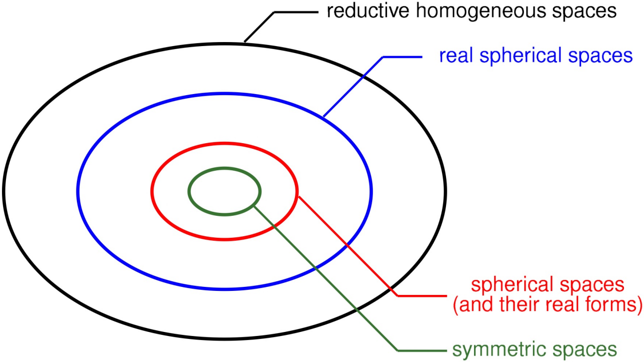
Plan of Lectures

- Talk 1: Is rep theory useful for global analysis?
  —Multiplicity: Approach from PDEs







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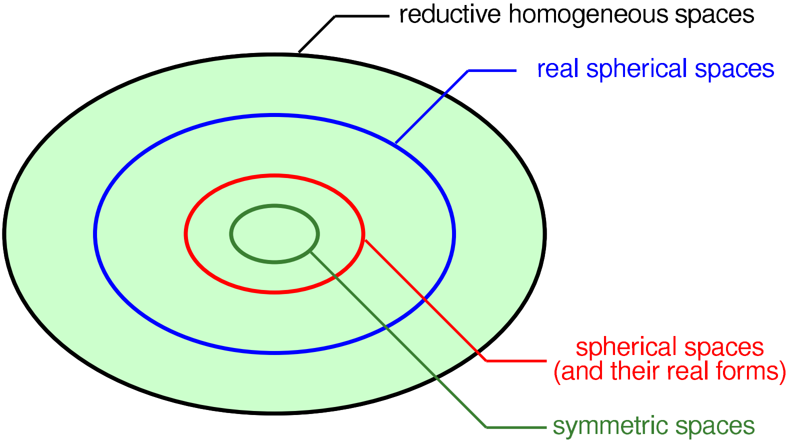

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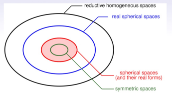
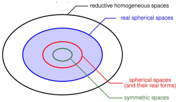
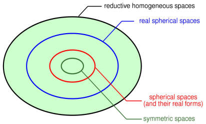




- Talk 2: Tempered homogeneous spaces
  —Dynamical approach

- Talk 3: Classification theory of tempered G/H
  —Combinatorics of convex polyhedra



- Talk 4: Tempered homogeneous spaces
  —Interaction with topology and geometry

Is rep theory useful for global analysis on manifolds?

$$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X)$$

Geometry — Functions

Basic Problem 1

Does the group G " control well " the function space $C^\infty(X)$?

Warming up: Analysis and Synthesis

Philosophy — Analysis and Synthesis : Try to understand

- how things are built up from the "smallest" objects;
- what are the "smallest" ones.

- Chemistry: understand a substance from the "smallest particle" (molecule, atom, $\cdots$).

- Lie groups: "built up" from simple Lie groups ($SL(n, \mathbb{R})$, $SO(p, q)$, $\cdots$) and one-dimensional ones ($\mathbb{R}$ or $\mathbb{T}$).

- Representations: "decompose" into irreducible representations.

- Functions: "expand" functions into "basic" functions.

First viewpoint $\cdots$ Spectral Analysis on Riemannian manifolds

Without "group theory"

X : complete Riemannian manifold

$\rightsquigarrow \quad \Delta_X = -\operatorname{div} \circ \operatorname{grad}$ (Laplacian)

The Laplacian Δ_X is essentially self-adjoint on $L^2(X)$.

$\rightsquigarrow \quad L^2(X) \simeq \int_0^\infty \mathcal{H}_\lambda d\tau(\lambda)$ (spectral decomposition of Δ_X).

$\cdots$ any L^2-function on X can be expanded into eigenfns of Δ_X.

Second viewpoint $\cdots$ Group Representation 1

Without specific geometric structure such as Riemannian structure.

$$G \curvearrowright X \text{ (manifold)} \rightsquigarrow G \curvearrowright C^\infty(X), L^2(X), \cdots$$

Geometry | Functions

$G \curvearrowright C^\infty(X)$

One defines a rep of G on $C^\infty(X)$ by $\pi_X(g) \colon f(x) \mapsto f(g^{-1}x)$.

$G \curvearrowright L^2(X)$

- If X has a G-invariant Radon measure μ_X, then G acts unitarily on $L^2(X) := L^2(X, \mu_X)$.
- More generally, let $\mathcal{L}$ be the half density bundle of X.

  $\rightsquigarrow$ G acts unitarily on $L^2(X) := L^2(X, \mathcal{L})$.

Second viewpoint $\cdots$ Group Representation 1

$G \curvearrowright X$ (manifold) $\rightsquigarrow$ $G \curvearrowright L^2(X)$

Geometry Functions

$G \curvearrowright L^2(X)$

- Let $\mathcal{L}$ be the half density bundle of X.

$\rightsquigarrow$ G acts unitarily on $L^2(X) := L^2(X, \mathcal{L})$.

Alternative definition

$G \curvearrowright L^2(X)$ (multiplier representation)

We set $L^2(X) := L^2(X, \mu_X)$ by choosing a volume form μ_X on X.
One defines a unitary operator $\pi_X(g)\colon L^2(X) \to L^2(X)$ by

$$(\pi_X(g)f)(x) := c(g,x)^{\frac{1}{2}} f(g^{-1}x) \in L^2(X),$$

where $c(g,x)$ is defined by $g_*\mu_X = c(g,x)\mu_X$ (Radon–Nykodim derivative).

$\rightsquigarrow$ π_X gives a unitary representation of G on $L^2(X)$.

Second viewpoint — Group Representation 2

Fact (Mautner) Any unitary rep Π of G can be disintegrated into irreducibles:

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi) \qquad \text{(direct integral)}$$

$\widehat{G}$:= {irreducible unitary representations}/ $\sim$ (unitary dual),

$m \colon \widehat{G} \to \mathbb{N} \cup \{\infty\}$, $\pi \mapsto m_\pi$ (multiplicity).

$$m_\pi \pi = \underbrace{\pi \oplus \cdots \oplus \pi}_{m_\pi}$$

In our setting

$G \curvearrowright X$ (manifold) $\rightsquigarrow$ $G \overset{\pi_X}{\curvearrowright} L^2(X)$ (Hilbert space)

$L^2(X) \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$ (Plancherel-type theorem)

Connection of the two viewpoints

(Without group theory)

X : pseudo-Riemannian manifold

Spectral analysis of Δ_X: $L^2(X) \simeq \int \mathcal{H}_\lambda d\tau(\lambda)$.

"generalize" ⇣ ⇡ if $m_\pi = 1$

Representation Theory: $L^2(X) \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$.

Symmetry: $G \curvearrowright X$

(No geometric structure specified)

<u>Example</u> Special cases for which both settings occur:

$G = \mathrm{Isom}(X) \cdots$ the groups of isometries of a pseudo-Riemannian manifold X.

Example: Spherical harmonics expansion on S^n

Two viewpoints give the same expansion for $X = S^n$ or H^n:

(1) (Spectral analysis: eigenfunctions of the Laplacian Δ_{S^n})
Any $f \in C^\infty(S^n)$ has an eigenfunction expansion:

$$f = \sum_{j=0}^{\infty} \varphi_j$$

where $\Delta_{S^n} \varphi_j = j(n+j-1) \varphi_j$ $({}^\forall j)$.

(2) (Representation theory: Irreducible decomposition)
$O(n+1)$ acts unitarily on $L^2(S^n)$, which decomposes

$$O(n+1) \curvearrowright L^2(S^n) \simeq \bigoplus_{j=0}^{\infty} \mathcal{H}_j$$ (multiplicity-free irreducible decomposition).

Likewise for $G = O(n,1) \curvearrowright H^n$ (hyperbolic space).

Laplacian $\Delta_X \rightsquigarrow$ Invariant differential operators

Generalization of Δ_X by using "symmetry"

Setting $G \curvearrowright X$ (manifold) $\rightsquigarrow G \curvearrowright C^\infty(X), L^2(X), \cdots$

X: no geometric structure specified.

Definition A differential operator D on X is G-invariant if

$$D \circ \pi_X(g) = \pi_X(g) \circ D \qquad \text{on } C^\infty(X), {}^\forall g \in G.$$

Note: $D, \pi_X(g) \in \mathrm{End}(C^\infty(X))$.

$\mathbb{D}_G(X)$:= ring of G-invariant differential operators on X

Example For a (pseudo-)Riemannian manifold X,
take $G := \mathrm{Isom}(X) \cdots$ the group of isometries of X.

(1) $\Delta_X \in \mathbb{D}_G(X)$.

(2) For $X = S^n$, $G \simeq O(n+1)$ and $\mathbb{D}_G(X) \simeq \mathbb{C}[\Delta_X]$.

Multiplicities in regular representations

$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X)$

Geometry — Functions

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Formulation Consider the dimension of

$$\mathrm{Hom}_G(\pi, C^\infty(X)) \quad \text{for } \pi \in \mathrm{Irr}(G).$$

infinite, finite, bounded, 0 or 1

→

control better

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Smooth admissible representations

G : real reductive linear Lie gp $\supset K$: max. compact, $\mathfrak{g}$: Lie alg

Example $G = GL(n, \mathbb{R}) \supset K = O(n)$, $\mathfrak{g} = M(n, \mathbb{R})$

- Analytic rep theory (Fréchet space, Hilbert space, $\cdots$)

$\mathrm{Irr}(G) :=$ { irred admissible reps of G of moderate growth}/ $\sim$

$\widehat{G}$:= { irred unitary representations of G}/ $\sim$ (unitary dual)

$$\widehat{G} \hookrightarrow \mathrm{Irr}(G), \quad \pi \mapsto \pi^{\infty} \quad \text{(smooth rep)}$$

$\mathrm{Irr}(G)_f :=$ { irred finite-dim'l reps of G}.

Spherical vs real spherical

$G_{\mathbb{C}}$ complex reductive $\curvearrowright X_{\mathbb{C}}$ complex manifold (connected)

Definition $X_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

Def. Borel subgroup B of $G_{\mathbb{C}}$
$\underset{\text{def}}{=}$ maximal connected solvable subgp of $G_{\mathbb{C}}$.

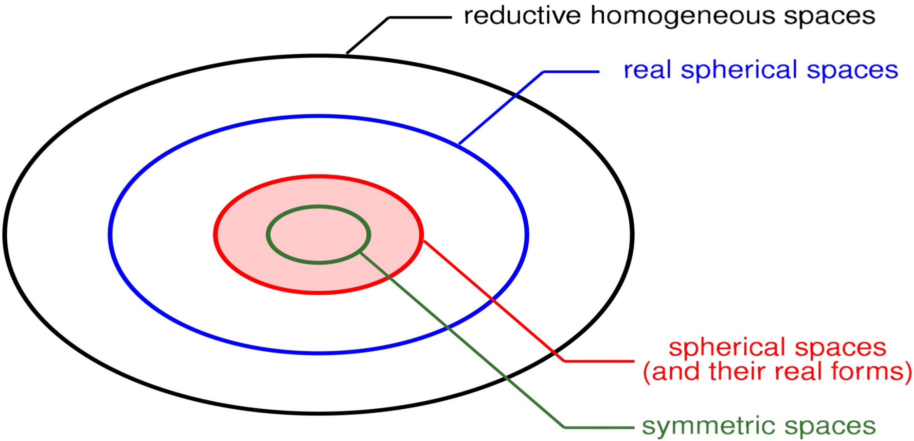
e.g. $$B = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\right\} \subset GL(3, \mathbb{C}) = G_{\mathbb{C}}.$$

Example Grassmannian varieties, flag varieties, symmetric spaces, complexification of weakly symmetric spaces (à la Selberg), $\cdots$ are spherical.

Spherical vs real spherical

$G_{\mathbb{C}}$ complex reductive $\supset H_{\mathbb{C}}$ complex subgroup

<u>Definition</u> $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$.



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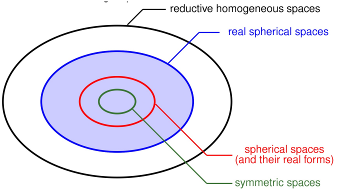
G real reductive $\supset H$ subgroup

Definition** We say G/H is real spherical if a minimal parabolic P of G has an open orbit in G/H.



** T. Kobayashi, Introduction to harmonic analysis on spherical homogeneous spaces, 22–41, 1995.

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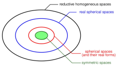

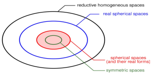

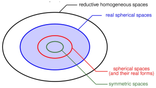

G/H symmetric space $\overset{\Rightarrow}{\not\Leftarrow}$ $G_{\mathbb{C}}/H_{\mathbb{C}}$ spherical $\overset{\Rightarrow}{\not\Leftarrow}$ G/H real spherical

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Definition $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$.

$\Longleftrightarrow \#(B\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$ (Brion, Vinberg) (~1986)

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G/H symmetric space $\underset{\not\Leftarrow}{\Rightarrow}$ $G_{\mathbb{C}}/H_{\mathbb{C}}$ spherical $\underset{\not\Leftarrow}{\Rightarrow}$ G/H real spherical

Spherical vs real spherical

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G real reductive $\supset H$ reductive subgroup

<u>Definition</u> We say G/H is real spherical if a minimal parabolic P of G has an open orbit in G/H.

For reductive H, Tanaka recently settled* a conjecture since '95:

$$G/H \text{ real spherical} \iff G = K^{\exists}AH$$

* Y. Tanaka, A Cartan decomposition for a reductive real spherical homogeneous space, Kyoto J. Math., 95–102, (2022).

** T. Kobayashi, Introduction to harmonic analysis on spherical homogeneous spaces, 22–41, 1995.

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infinite,　finite,　bounded,　0 or 1

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Answer to Basic Problem 1 (multiplicity)

$G \supset H$ real reductive linear groups, $X := G/H$ (algebraic)

Theorem A* (i) and (ii) are equivalent on $(G, X) = (G, G/H)$.

(i) (Analysis and Rep Theory: finite multiplicities)

$$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) < \infty \qquad ({}^\forall \pi \in \mathrm{Irr}(G)).$$

(ii) (Geometry) X is real spherical.

Recall X real spherical $\Leftrightarrow P \curvearrowright X = G/H$ has an open orbit

$\Leftrightarrow H \curvearrowright G/P$ has an open orbit

* T. Kobayashi and T. Oshima, Adv. Math., **248** (2013), 921–944.

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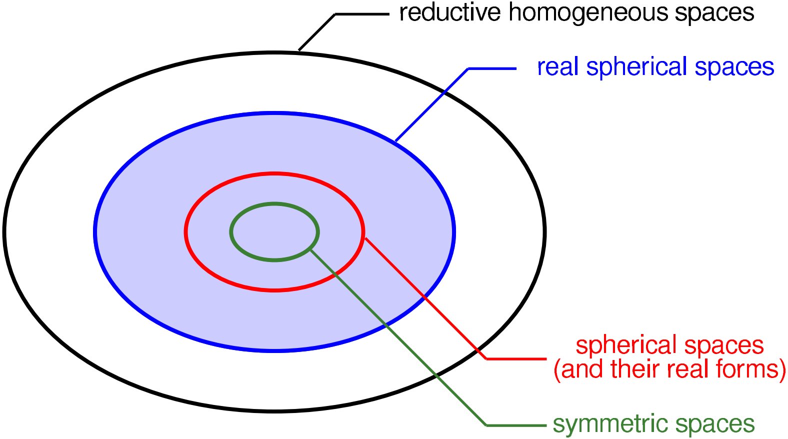

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Methods of proof

(ii) $\Rightarrow$ (i) Reduction to geometry of boundaries

Equivariant compactification + hyperfunction-valued boundary maps for a system of partial differential equations.

(i) $\Rightarrow$ (ii)

Construction of integral intertwining operators from boundaries

(Cf. Knapp–Stein, Poisson–Fourier, Jacquet integral, . . .)

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Remark 1) Theorem A holds in a more general setting:

- non-reductive H,
- sections for any G-equivariant vector bundle $\mathcal{V} \to X$.

2) ("qualitative results" (Thm A) $\rightsquigarrow$ "quantitative estimate")

Upper/lower estimates of the multiplicities are obtained.

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Ex. Kostant–Lynch theory (’79) for Whittaker model, when H := maximal unipotent subgroup

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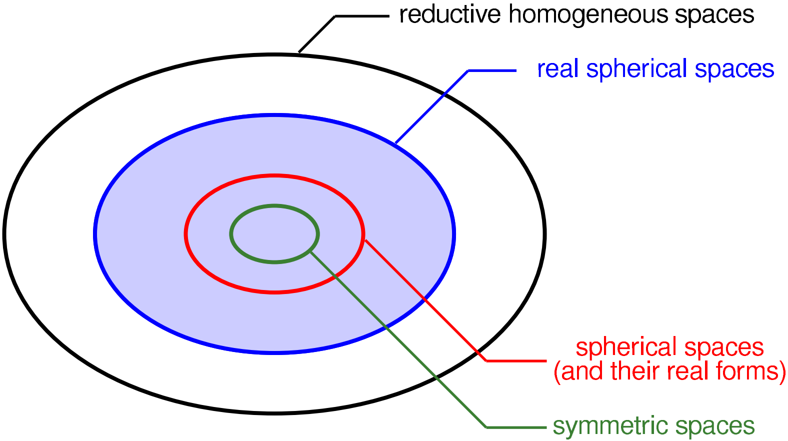

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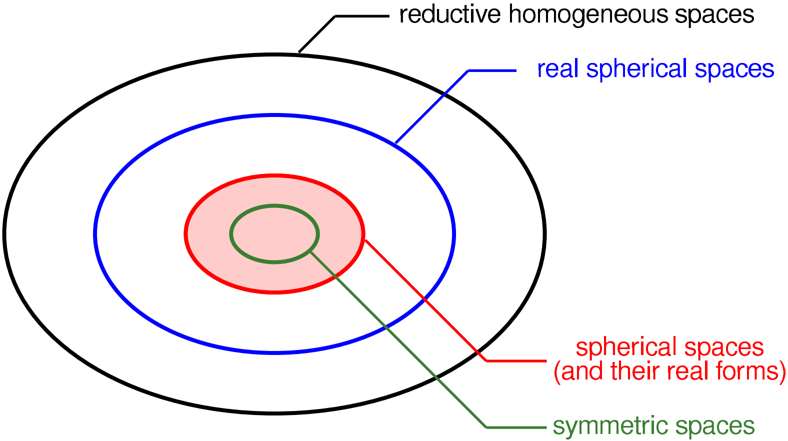

We shall also discuss when G and H are not nesssarily reductive.

When does the group "control" better the function space?

$G \supset H$ real reductive linear groups, $X := G/H$ (algebraic).

Theorem B * The following conditions are all equivalent:

(i) (Analysis & rep theory) There exists $C > 0$ s.t.

$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq C$ for all $\pi \in \mathrm{Irr}(G)$.

(ii) (Complex geometry) $X_\mathbb{C}$ is $G_\mathbb{C}$-spherical.

(ii)′ (Algebra) The ring $\mathbb{D}_G(X)$ is commutative.

(ii)″ (Algebra) The ring $\mathbb{D}_G(X)$ is a polynomial ring.

The equivalence (ii) ~ (ii)″ is classical (Vinberg, Knop, $\cdots$).
The main point that we emphasize on here is an interaction of

(i) Analysis $\Longleftrightarrow$ (ii)~(ii)″ Algebra & Geometry.

- Surprisingly, uniform boundedness of the multiplicity in $C^\infty(X)$ is detected only by the complexification $X_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C}$.

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$X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ detects "bounded multiplicity property"

The uniform boundedness of the multiplicity in $C^\infty(X)$ is detected, surprisingly, only by the complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$.

Example. Let $n \geq 2m$, and consider

$G/H = SL(n,\mathbb{R})/Sp(m,\mathbb{R}),\ SU(n)/Sp(m),$

$SU(p,q)/Sp(p',q')\ (p+q=n, p'+q'=m, p \geq 2p', q \geq 2q'),$

$SU^*(\frac{n}{2})/Sp(p',q')$ (n even, $p'+q'=m$), $\ldots$

These homogeneous spaces have the isomorphic complexification

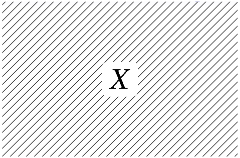
$$G_{\mathbb{C}}/H_{\mathbb{C}} = SL(n,\mathbb{C})/Sp(m,\mathbb{C}).$$

The bounded multiplicity property for $C^\infty(G/H)$ holds

$\iff$ $n = 2m$ or $2m+1$ (depending only on $G_{\mathbb{C}}/H_{\mathbb{C}}$).

Remark Finite multiplicity property depends on real forms.

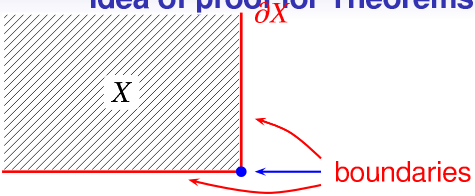
Idea of proof for Theorems A and B $\cdots$ PDEs



$$P_1 f = P_2 f = \cdots = P_\ell f = 0$$

System of partial differential eqns

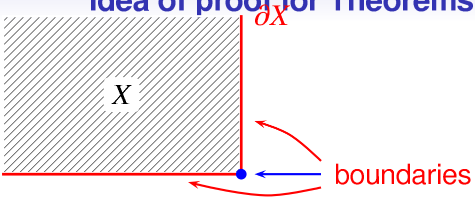
Idea of proof for Theorems A and B $\cdots$ PDEs



$$P_1 f = P_2 f = \cdots = P_\ell f = 0 \quad \Longrightarrow \quad f|_{\partial X}$$

System of partial differential eqns → "boundary value"

Idea of proof for Theorems A and B $\cdots$ PDEs



$$P_1 f = P_2 f = \cdots = P_\ell f = 0 \quad \Longrightarrow \quad f|_{\partial X}$$

System of partial differential eqns — "boundary value"

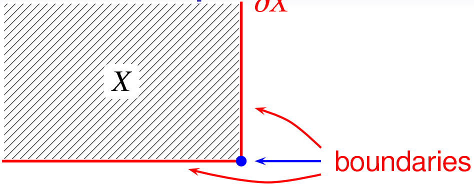
<u>Example</u> X = unit disk, $\partial X = S^1$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x, y) = 0 \quad \Longrightarrow \quad f|_{\partial X}$$

holomorphic function — "hyperfunction"

(Idea of Sato (1928–2023)).

Idea of proof for Theorems A and B $\cdots$ PDEs



$P_1 f = P_2 f = \cdots = P_\ell f = 0 \quad \Longrightarrow \quad f|_{\partial X}$

System of partial differential eqns — "boundary value"

<u>Example</u> X = unit disk, $\partial X = S^1$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x, y) = 0 \quad \Longleftrightarrow \quad f|_{\partial X}$$

harmonic function — "hyperfunction"

(Idea of Sato (1928–2023)).

$\cup$		$\cup$
$\cdots$	$\longleftrightarrow$	distributions
Hardy space	$\longleftrightarrow$	L^2-functions
$\cdots$	$\longleftrightarrow$	continuous functions

Classical example: $\Delta f = \lambda f$ in the Poincaré disc

Poincaré disc $D = \{z \in \mathbb{C} : |z| < 1\}$ $ds^2 = \frac{4(dx^2+dy^2)}{(1-|z|^2)^2}$

Laplacian $$\Delta = -\frac{1}{4}(1 - x^2 - y^2)^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

$$\sim -\tfrac{1}{4}(\theta^2 - 2\theta)$$ near the boundary $(s = 0)$

where $\theta = s\frac{\partial}{\partial s}$, $s := \sqrt{1 - x^2 - y^2}$

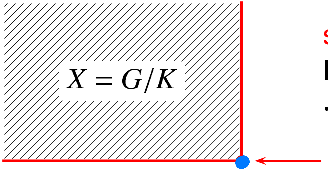
Suppose $\triangle f = \lambda f$ in D and f is K-finite.
Look at near the boundary ∂D.
The theory of ODE with regular singularity tells us

$$f(\sqrt{1 - s^2}(\cos\varphi, \sin\varphi)) = {}^{\exists}A(s, \varphi)s^{1+\sqrt{1-4\lambda}} + {}^{\exists}B(s, \varphi)s^{1-\sqrt{1-4\lambda}}$$

for generic λ correspondingly to $(-\frac{1}{4}(\theta^2 - 2\theta) - \lambda)s^{1\pm\sqrt{1-4\lambda}} = 0$.

$\rightsquigarrow A(0, \varphi)$, $B(0, \varphi) \cdots$ "boundary values" of f.

Helgason's conjecture $\cdots$ symmetric case



simple prototype

Helgason's conjecture (theorem)

$\cdots$ Kashiwara–Okamoto–Oshima et al.

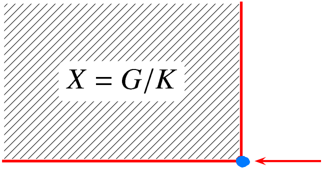
G/P

Geometry

$X = G/K$ $\overset{\text{compactification}}{\rightsquigarrow}$ ∂X

Riemannian symmetric space — normal crossing

Helgason's conjecture $\cdots$ symmetric case



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Geometry

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Riemannian symmetric space — normal crossing

Analysis

$f \in \mathcal{A}(G/K)$ s.t.

$Df = \lambda_{(D)} f$ $({}^\forall D \in \mathbb{D}(G/K))$

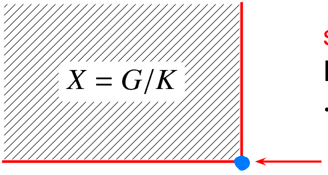
Poisson transform $\circlearrowleft$ "boundary map"

"$f|_{G/P}$" $\in \mathcal{B}(G/P, \mathcal{L}_\lambda)$

$\Uparrow$

micro-local analysis, PDE with regular singularities

Helgason's conjecture $\cdots$ symmetric case



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micro-local analysis, PDE with regular singularities

with K-finiteness assumption $\cdots$ goes back to Harish-Chandra

Sketch of proof of Theorem A

General case : $G/H \quad HP \underset{\text{open}}{\subset} G, \quad H$ possibly non-reductive

- Consider $\tau\colon H \to GL(V)$

$$\mathcal{V} := G \times_H V \to G/H$$

$$\rightsquigarrow G \curvearrowright C^\infty(G/H, \mathcal{V}) \subset \mathcal{B}(G/H, \mathcal{V})$$

- $U(\mathfrak{g}) \supset Z(\mathfrak{g})$: center of enveloping algebra

Sketch of proof of Theorem A

General case : G/H $\quad HP \underset{\text{open}}{\subset} G$, H possibly non-reductive

Fix $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ infinitesimal character

$$\mathcal{B}_\chi(G/H;\mathcal{V}) := \{f \in \mathcal{B}(G/H;\tau) : Df = \chi(D)f \ ({}^\forall D \in Z(\mathfrak{g}))\}$$

Sketch of proof of Theorem A

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By using an appropriate compactification, ${}^\exists(\mathfrak{g}, K)$-filtration

$$\mathcal{B}_\chi(G/H;\mathcal{V})_K \equiv \mathcal{B}^0 \supset \mathcal{B}^1 \supset \cdots \supset \mathcal{B}^N = \{0\}$$

such that

$$\mathcal{B}^j/\mathcal{B}^{j+1} \underset{\text{“boundary map”}}{\hookrightarrow} \mathcal{B}(HP/P; {}^\exists\sigma_j)$$

Sketch of proof of Theorems A and B

General case : $G/H \quad HP \underset{\text{open}}{\subset} G$, H possibly non-reductive

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Corollary $\mathcal{B}_\chi(G/H;\mathcal{V})_K$ is of finite length as a $(\mathfrak{g}, K)$-module for ${}^\forall\chi$ and ${}^\forall\tau$, namely

#(irred. subquotients) $< \infty$

Induction $H \uparrow G$ *vs* Restriction $G \downarrow H$

Let $H \subset G$.

So far, we have discussed "Basic Problem" for the "induction"

$$C^\infty(G/H) = \mathrm{Ind}_H^G(\mathbf{1})^\infty.$$

Basic Problem 1
Does the group G " control well " the function space $C^\infty(X)$?

$G \curvearrowright X$ ⇝ $G \curvearrowright C^\infty(X)$
Geometry   Functions

We may also think of the "restriction" $G \downarrow H$ ("branching problem") which is much more involved.

Basic Problem 2   Single out nice pairs (G, H) for which detailed study of the restriction $G \downarrow H$ ("branching problem") is "fruitful".

Comparison: $GL(n,\mathbb{R}) \downarrow O(n)$ vs $GL(n,\mathbb{R}) \downarrow O(p, n-p)$

Harish-Chandra's admissibility theorem concerns the restriction with respect to a Riemannian symmetric pair

$$G \supset K, \qquad \text{e.g., } GL(n,\mathbb{R}) \supset O(n)$$

and asserts

$$[\Pi|_K : \pi] < \infty \qquad {}^\forall\Pi \in \mathrm{Irr}(G), {}^\forall\pi \in \mathrm{Irr}(K).$$

In contrast,

For a reductive symmetric pair

$$G \supset G', \qquad \text{e.g., } GL(n,\mathbb{R}) \supset O(p, n-p)$$

it may happen that

$[\Pi|_{G'} : \pi] = \infty$ for some $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$.

Branching problems

$$
\begin{array}{ccc}
G & \xrightarrow[\text{irreducible}]{\pi} & GL(V) \\
\cup & \nearrow_{\pi|_{G'}} & \\
G' & &
\end{array}
$$

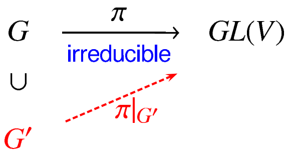
Branching problems

$$\begin{array}{ccc} G & \xrightarrow[\text{irreducible}]{\pi} & GL(V) \\ \cup & \nearrow_{\pi|_{G'}} & \\ G' & & \end{array}$$

<u>Example</u> (tensor product of two representations)

$$\begin{array}{ccc} G_1 \times G_1 & \xrightarrow[\text{outer tensor product}]{\pi' \boxtimes \pi''} & GL(V) \\ \cup & \nearrow_{\pi' \otimes \pi''} & \\ \operatorname{diag} G_1 & & \end{array}$$

Branching problems



Branching problem (in a wider sense than the usual)

$\cdots$ wish to understand

how the restriction $\pi|_{G'}$ behaves as a G'-module.

Application of Thms A & B (Induction) to branching problems

$G \quad \supset \quad G'$ reductive groups

Application of Thms A & B (Induction) to branching problems

$$\begin{array}{ccccl} G & \supset & G' & & \text{reductive groups} \\ \text{irred} \circlearrowleft & & \circlearrowright \text{irred} & & \\ \pi & & \tau & & \end{array}$$

Application of Thms A & B (Induction) to branching problems

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reductive groups

symmetry breaking operator
(continuous G'-homomorphism)

$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) :=$ { symmetry breaking operators }

In general,

the dimension of $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ might be infinite

even when G' is a maximal reductive subgroup in G.

Application of Thms A & B (Induction) to branching problems

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<u>Theorem C</u>* (criterion for finite multiplicity)

$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ is finite-dimensional $({}^{\forall}\pi \in \mathrm{Irr}(G), {}^{\forall}\tau \in \mathrm{Irr}(G'))$

$\iff (G \times G')/\operatorname{diag}(G')$ is $(G \times G')$-real spherical.

* T. Kobayashi, Shintani functions, real spherical manifolds, . . . , Perspective Math. (2014).

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Theorem D* (criterion for bounded multiplicity)

${}^\exists C > 0, \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \le C$ $({}^\forall \pi \in \mathrm{Irr}(G), {}^\forall \tau \in \mathrm{Irr}(G'))$

$\iff (G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag}(G'_{\mathbb{C}})$ is $(G_{\mathbb{C}} \times G'_{\mathbb{C}})$-spherical.

* T. Kobayashi, Shintani functions, real spherical manifolds, . . . , Perspective Math. (2014).

a priori estimate $\rightsquigarrow$ construction

$$\begin{array}{ccc} G & \supset & G' \\ \text{irred} \circlearrowleft & & \circlearrowright \text{irred} \\ \pi & \dashrightarrow & \tau \end{array}$$

reductive groups

Theorems C and D (criterion for finite / bounded mult.)

1) $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ is finite-dimensional $({}^{\forall}\pi \in \mathrm{Irr}(G),\ {}^{\forall}\tau \in \mathrm{Irr}(G'))$
$\iff (G \times G')/\operatorname{diag}(G')$ is real spherical.

2) ${}^{\exists}C > 0$, $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \le C$ $({}^{\forall}\pi \in \mathrm{Irr}(G),\ {}^{\forall}\tau \in \mathrm{Irr}(G'))$
$\iff (G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag}(G'_{\mathbb{C}})$ is spherical.

$\leftarrow$ classification of (G, G') satisfying (1)

techniques: linearization, prehomogeneous sp., quivers

a priori estimate $\rightsquigarrow$ construction

$G \supset G'$ reductive groups

irred $\circlearrowleft$ $\circlearrowright$ irred

$\pi \dashrightarrow \tau$

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$\rightsquigarrow$ $\leftarrow$ classification of (G, G') satisfying (1)

<u>Example 1</u> * $(G, G') = (GL(n+1, \mathbb{F}),\ GL(n, \mathbb{F}) \times GL(1, \mathbb{F}))$

$\mathbb{F} = \mathbb{R}, \mathbb{C} \cdots$ both (1) and (2) hold.

$\mathbb{F} = \mathbb{H} \quad \cdots$ (1) holds, but (2) fails.

* T. Kobayashi–T. Matsuki, Transformation Groups, 2014 (Dynkin volume).

a priori estimate $\rightsquigarrow$ construction

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Classification Theory (classical) compact (G, G') satisfying (2)

$\cdots$ Cooper (Kostant), Krämer (1970s)

$\cdots$ $(G_{\mathbb{C}}, G'_{\mathbb{C}}) \approx (GL_n(\mathbb{C}), GL_{n-1}(\mathbb{C}))$ or $(O_n(\mathbb{C}), O_{n-1}(\mathbb{C}))$.

cf. Gan–Gross–Prasad conjecture for (GL_n, GL_{n-1}) or (O_n, O_{n-1}).

a priori estimate $\rightsquigarrow$ construction

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$\rightsquigarrow$ $\leftarrow$ classification of (G, G')

<u>Further Problem</u>

(1) <u>Construct explicitly</u> symmetry breaking operators

(2) <u>Gan–Gross–Prasad conjecture</u> for (GL_n, GL_{n-1}) or (O_n, O_{n-1}).

Induction $H \uparrow G$ *vs* Restriction $G \downarrow H$

$$X := G/H$$

$G \curvearrowright X$ $\rightsquigarrow$ $G \curvearrowright C^\infty(X)$

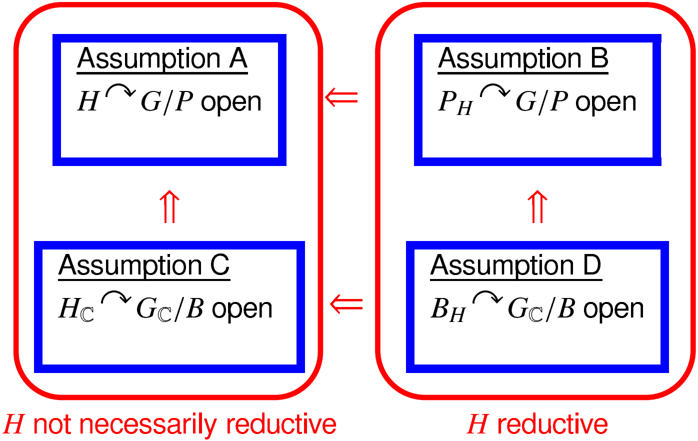
Geometry — Functions

Basic Problem 1

Does the group G " control well " the function space $C^\infty(X)$?

Basic Problem 2 Single out nice pairs (G, H) for which detailed study of the restriction $G \downarrow H$ ("branching problem") is "fruitful".

Four assumptions Theorems A–D



P_H and B_H are minimal parabolic and Borel for H and $H_{\mathbb{C}}$, respectively.

Four assumptions Theorems A–D

Theorem A

$H \curvearrowright G/P$ open

$\Leftarrow$

Theorem B

$P_H \curvearrowright G/P$ open

$\Uparrow$ $\Uparrow$

Theorem C

$H_{\mathbb{C}} \curvearrowright G_{\mathbb{C}}/B$ open

$\Leftarrow$

Theorem D

$B_H \curvearrowright G_{\mathbb{C}}/B$ open

Induction

$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \tau)$

$\dim \tau < \infty$

Restriction

$\mathrm{Hom}_H(\pi|_H, \tau)$

Allow $\dim \tau = \infty$

Four assumptions Theorems A–D

Finite multiplicity theorems

Theorem A

$$\dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \tau) < \infty$$
$$({}^\forall \pi \in \mathrm{Irr}(G),\ {}^\forall \tau \in \mathrm{Irr}(H)_f)$$

$\Leftarrow$

Theorem B

$$\dim \mathrm{Hom}_G(\pi|_H, \tau) < \infty$$
$$({}^\forall \pi \in \mathrm{Irr}(G),\ {}^\forall \tau \in \mathrm{Irr}(H))$$

$\Uparrow$ $\Uparrow$

Theorem C

$$\sup_{\pi \in \mathrm{Irr}(G)} \sup_{\tau \in \mathrm{Irr}(H)_f} \dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \tau) < \infty$$

$\Leftarrow$

Theorem D

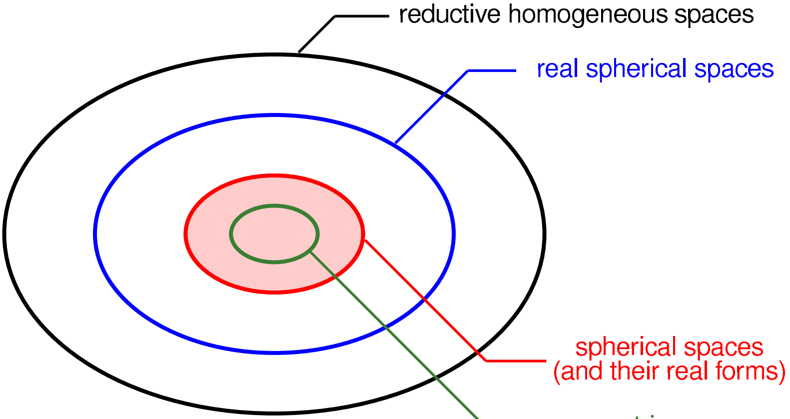
$$\sup_{\pi \in \mathrm{Irr}(G)} \sup_{\tau \in \mathrm{Irr}(H)} \dim \mathrm{Hom}_H(\pi|_H, \tau) < \infty$$

Bounded multiplicity theorems

Reductive homogeneous space G/H

G: real reductive groups

H: reductive subgroup



We shall also discuss when G and H are not nesssarily reductive.

Reductive homogeneous space G/H

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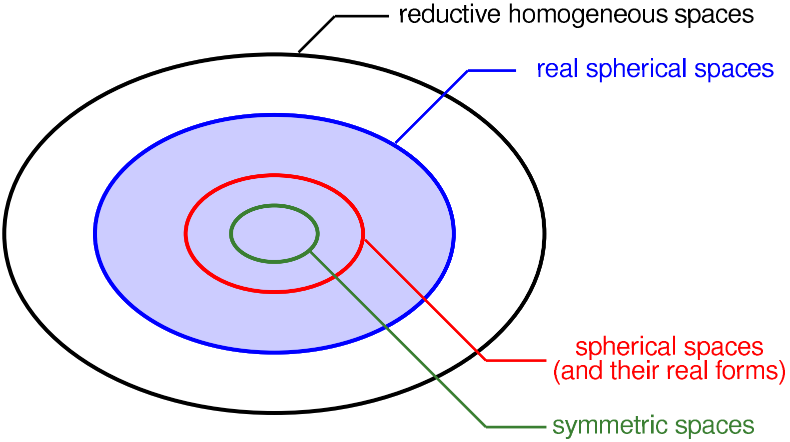

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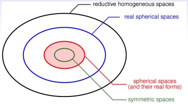
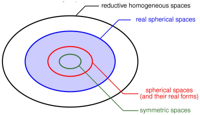
Plan of Lectures

- Talk 1: Is rep theory useful for global analysis?
  —Multiplicity: Approach from PDEs





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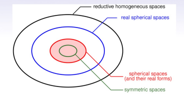
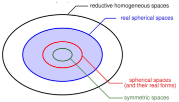
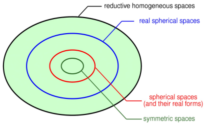




- Talk 2: Tempered homogeneous spaces
  —Dynamical approach

- Talk 3: Classification theory of tempered G/H
  —Combinatorics of convex polyhedra



- Talk 4: Tempered homogeneous spaces
  —Interaction with topology and geometry

Thank you for your attention!