Lie Group and Representation Theory Seminar
Mini-course

Title: Co-isotropic actions

Date: March 7 (Wed) 10:00-11:30 (Room 005)
March 9 (Fri) 10:00-11:30 (Room 402)
Mach 13 (Tue) 10:00-11:30 (Room 402)
March 14 (Wed) 10:00-11:30 (Room 005)
March 16 (Fri) 10:00-11:30 (Room 402)

Speaker: Tilmann Wurzbacher (Metz)

Abstract: The first aim of this course is to explain the context and the basic properties of co-isotropic actions of Lie groups on symplectic manifolds (i.e. actions having generically co-isotropic orbits), as well as of spherical varietes (i.e. complex-algebraic varieties with an action of a complex reductive Lie group such that all Borel subgroups thereof have an open orbit). After interludes on geometric quantization resp. on lagrangian actions, we prove the equivalence of the two above conditions in the complex-algebraic set-up. Finally, we give applications of this theorem to, e.g., geometric quantization of Kähler manifolds and remark on connections to related subjects.

Resumé of the five lectures

I. Symplectic reminders

I.a Normal forms

We review very quickly the basic properties of finite dimensional symplectic vector spaces and the normal forms for distinguished types of linear subspaces (lagrangian, isotropic, co-isotropic). We then proceed to the normal form theorems for certain types of submanifolds of symplectic manifolds (notably the theorems of Darboux, Moser and Weinstein).

I.b The moment map

We motivate "moment maps"via their use in hamiltonian dynamical systems and recall several classical examples. In the context of Lie group actions on symplectic manifolds, we define moment maps, give classes of examples and show their elementary properties. We comment on the co-adjoint action, homogeneous symplectic manifolds and symplectic reduction relative to a moment value. Furthermore we mention Kähler-isometry homogeneous manifolds and the Vinberg conjecture on their structure, proven dy Dorfmeister and Nakajima.

II. Geometric quantization in 90 minutes
After motivation quantization via the so-called "canonical quantization" often used as a recipe for the passage from classical to quantum mechanics in physics, we explain the main points of geometric quantization à la Souriau and Kostant: integral symplectic forms and complex line bundles-with-connection, pre-quantization, quantization, metaplectic structures, and quantization of symmetries (i.e. equivariant geometric quantization). We also remark on lifting hamiltonian symmetries to line bundles.

III. Lagrangian actions
We first recall the Arnold-Liouville normal form of integrable classical hamiltonian systems and discuss then the periodic case, i.e. symplectic manifolds with a lagrangian (torus) action. We sketch the basics of complex-algebraic toric varieties and the proof of Delzant's theorem on the equivalence of the two conditions (lagrangian torus action resp. toric variety). We also comment on the quantization of these varieties, notably on holomorphic quantization and on Bohr-Sommerfeld quantization.

IV. Co-isotropic actions
We explain in some detail the notions of co-isotropic resp. multiplicity-free symplectic group action. We also recall the basic properties of spherical complex-algebraic varieties. We then show the equivalence of these two concepts in the context of complex-algebraic varieties acted upon transitively by complex reductive Lie groups and deduce several useful versions of this basic equivalence, e.g. for the compact kählerian case.

V. Applications, remarks and outlook
We apply the equivalence theorem (see fourth lecture) to refine and proof a conjecture of Guillemin and Sternberg in equivariant geometric quantization. We discuss general "constrained quantization" (and the creed "quantization commutes with reduction"), and comment on topics as non-Kählerian multiplicity-free symplectic spaces. Furthermore, we connect co-isotropicity to other notions, such as (co-)polarity and visibility for a group action on a manifold.

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