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Poisson vertex algebras in the theory of Hamiltonian equations

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Abstract. We lay down the foundations of the theory of Poisson vertex algebras aimed at its applications to integrability of Hamiltonian partial differential equations. Such an equation is called integrable if it can be included in an infinite hierarchy of compatible Hamiltonian equations, which admit an infinite sequence of linearly independent integrals of motion in involution. The construction of a hierarchy and its integrals of motion is achieved by making use of the so called Lenard scheme. We find simple conditions which guarantee that the scheme produces an infinite sequence of closed 1-forms ω_j, $j \in \mathbb{Z}_+$, of the variational complex Ω. If these forms are exact, i.e., ω_j are variational derivatives of some local functionals $\int h_j$, then the latter are integrals of motion in involution of the hierarchy formed by the corresponding Hamiltonian vector fields. We show that the complex Ω is exact, provided that the algebra of functions $\mathcal{V}$ is "normal"; in particular, for arbitrary $\mathcal{V}$, any closed form in Ω becomes exact if we add to $\mathcal{V}$ a finite number of antiderivatives. We demonstrate on the examples of the KdV, HD and CNW hierarchies how the Lenard scheme works. We also discover a new integrable hierarchy, which we call the CNW hierarchy of HD type. Developing the ideas of Dorfman, we extend the Lenard scheme to arbitrary Dirac structures, and demonstrate its applicability on the examples of the NLS, pKdV and KN hierarchies.

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