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Rank and duality in representation theory*

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Abstract. There is both theoretical and numerical evidence that the set of irreducible representations of a reductive group over local or finite fields is naturally partitioned into families according to analytic properties of representations. Examples of such properties are the rate of decay at infinity of "matrix coefficients" in the local field setting, and the order of magnitude of "character ratios" in the finite field situation.

In these notes we describe known results, new results, and conjectures in the theory of "size" of representations of classical groups over finite fields (when correctly stated, most of them hold also in the local field setting), whose ultimate goal is to classify the above mentioned families of representations and accordingly to estimate the relevant analytic properties of each family.

Specifically, we treat two main issues: the first is the introduction of a rigorous definition of a notion of size for representations of classical groups, and the second issue is a method to construct and obtain information on each family of representation of a given size.

In particular, we propose several compatible notions of size that we call *U-rank, tensor rank and asymptotic rank*, and we develop a method called *eta correspondence* to construct the families of representation of each given rank.

Rank suggests a new way to organize the representations of classical groups over finite and local fields—a way in which the building blocks are the "smallest" representations. This is in contrast to Harish-Chandra's philosophy of cusp forms that is the main organizational principle since the 60s, and in it the building blocks are the cuspidal representations which are, in some

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sense, the “largest”. The philosophy of cusp forms is well adapted to establishing the Plancherel formula for reductive groups over local fields, and led to Lusztig’s classification of the irreducible representations of such groups over finite fields. However, the understanding of certain analytic properties, such as those mentioned above, seems to require a different approach.

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