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The second S-W class of ℓ-adic coh.

ℓ-adic coh of middle degree of a variety of even dim
→ orthogonal rep'n of Gal gp
→ St. Wh class

Compare them with the inv. of de Rham coh.

1st SW class determinant v.s disc.

2nd : e.g $k/\mathbb{Q}_p$ finite $p \neq \ell$
sign of the local ε-const Deligne.
dim = 0 Serre's formula ~~for~~ involving the trace form.

1. Second SW class.

k field. X proper smooth $/k$
$\ell \neq$ char k q integer
$H^q(X_{\bar k}, \mathbb{Q}_\ell)$ ℓ-adic rep'n of $G_k = \mathrm{Gal}(\bar k/k)$

$$\det H^q = e_q \cdot \chi_\ell^{\frac{q \cdot b_q}{2}} \qquad \chi_\ell : G_k \to \mathbb{Z}_\ell^\times \text{ } \ell\text{-adic cyclo. ch}$$

$b_q = \dim H^q$ even if q odd (Suh) ~~e_q~~
$e_q^2 = 1$ conseq. of Weil conj

$q = n$ $H^q \times H^q \to \mathbb{Q}_\ell(-n)$ non deg bilinear
n. odd ⇒ ~~alte~~ symplectic ⇒ $e_n = 1$.
n. even $V = H^n(X_{\bar k}, \mathbb{Q}_\ell(\frac{n}{2}))$. orthogonal rep'n.

e.g $n=0$ $X = \mathrm{Spec}\, L$. L/k fin sep. ext'n. $V = \mathrm{Ind}_{G_L}^{G_k} 1$.

$\det V : G_k \to \{\pm 1\}$ elt. of $H^1(G_k, \mathbb{Z}/2\mathbb{Z}) \underset{\uparrow}{\simeq} k^\times/k^{\times 2}$
char $k \neq 2$

$sw_2(V) \in H^2(G_k, \mathbb{Z}/2\mathbb{Z})$ ($\simeq Br_2(k)$ ch $k \neq 2$) 2nd sw. class

the class of the pull-back of the central ext'n

$$\begin{array}{ccccccccc} & & & & & & G_k & & \\ & & & & & & \downarrow & & \\ 1 & \to & \{\pm 1\} & \to & \tilde O(V) & \to & O(V) & \to & 1 \end{array}$$

central ext'n of alg gp $/\mathbb{Q}_\ell$

defined by using the Clifford alg

$\dim Cl(V) = \dim \Lambda^\cdot(V)$

$Cl(V) = T^\cdot(V)/(x\cdot x - q(x);\ x \in V)$

$C(V) \subset Cl(V)^\times$ alg ~~gp~~ subgp gen by $V \cap Cl(V)^\times = \{x \in V \mid q(x) \neq 0\}$

$C(V) \to \mathbb{G}_m$ can. gp hom sending $x \in V \cap Cl(V)^\times$ to $q(x)$

$\tilde{O}(V) = \mathrm{Ker}(C(V) \to \mathbb{G}_m)$

$\tilde{O}(V) \to O(V)$ " " $x \in V \cap \tilde{O}(V)$ to the reflexion $v \mapsto v - 2b(v, x)x$

$b(x, y)$ sym b.f. $q(x) = b(x, x)$

2. Conjecture.

$D = H^n_{dR}(X/k)$ fin. dim k-v.sp.

$\cup$ defines nondeg symm bil. form.

E.g. $n = 0$ $X = \mathrm{Spec}\, L$ L/k finite sep. ext'n.

$D = L$. $\mathrm{Tr}_{L/k}(xy)$.

char $k \neq 2$

$d = \mathrm{disc}\, D \in k^\times/k^{\times 2} = H^1(G_k, \mathbb{Z}/2\mathbb{Z})$

$= \sum_{i=1}^{b_n} \{a_i\}$ $x_1, \dots, x_n$ orthogonal basis

$a_i = q(x_i)$ $a \mapsto \{a\}$

$\hat{k}^\times \longmapsto \hat{k}^\times/(k^\times)^2$.

$hw_2 D \in H^2(G_k, \mathbb{Z}/2\mathbb{Z})$

$= \sum_{i<j} \{a_i, a_j\}$ $\{a, b\} = \{a\} \cup \{b\} \in H^2(G_k, \mathbb{Z}/2\mathbb{Z})$.

Conjecture. X proper smooth$/k$. $n = \dim X$ even

char $k \neq 2, \ell$

$sw_2(H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\tfrac{n}{2})) + \{e, -1\} + \beta \cdot c_\ell$

$$= hw_2(D) + \begin{cases} r \cdot \{d, -1\} + \binom{r}{2}\{-1, -1\} & n \equiv 0\ (4) \\ (r + b_{dR,n} - 1)\{d, -1\} + \binom{r + b_{dR,n}}{2}\{-1, -1\} & \equiv 2\ (4) \end{cases}$$

$+ \{2, d\} + \eta(c_\ell - c_2)$.

Prop. X proper smooth$/k$ $n = \dim X$ even ch $k \neq 2, \ell$

$\det V$

$$= \mathrm{disc}\, D + \begin{cases} r \cdot \{-1\} & n \equiv 0\ (4) \\ (r + b_{dR,n})\{-1\} & \equiv 2\ (4) \end{cases}$$

$r = \sum_{q<n} (-1)^q b_{dR,q}$.

$e = \sum_{q<n} e_q \qquad \det H^q = e_q \cdot \chi_\ell^{\frac{q \cdot b_q}{2}}$.

$\beta = \frac{1}{2} \sum_{q<n} (-1)^q (n-q) b_q$.

$c_\ell \qquad H^2(\mathbb{Z}_\ell^\times, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\chi_\ell^*} H^2(G_k, \mathbb{Z}/2\mathbb{Z})$

$\quad\in$ uniq non-trivial $\longmapsto$ $c_\ell \in$

$\mathbb{Z}_\ell^\times \simeq \mathbb{Z}_\ell \times$ cyclic of even order. $\qquad 0$ if char $k \neq 0$

$d = \operatorname{disc} D$.

$\eta = \sum_{q<\frac{n}{2}} (-1)^q \left(\frac{n}{2} - q\right) \chi(X, \Omega^q_{X/k})$

A complex version $\quad sw_2(H^\cdot) = hw_2(D^\cdot) + \{2, d\} + \eta(c_\ell - c_2)$

3. Evidence.

Theorem Conjecture is true if one of the following is satisfied

1. $k/\mathbb{Q}_p$ finite $p \neq 2, \ell$. $\exists X/\mathcal{O}_k$ proj. reg flat, s.t. X_F has at most odp as sing.
2. $k/\mathbb{Q}_p$ finite unr. $p = \ell > n+1$. good reduction.
3. $k = \mathbb{R}$. X proj.
4. $k \supset \overline{\mathbb{Q}}$.
5. smooth hypersurface in $\mathbb{P}^{n+1}_k$. $\ell > n+1$

Rmk 1. Theorem 5, $n=0$ $\Rightarrow$ Th of Serre $\quad sw_2(\mathrm{Ind}_{G_L}^{G_k} 1) = hw_2(\mathrm{Tr}_{L/k} x^2) + \{d, 2\}$

2. $k = \mathbb{Q}_p$ $p > 3$. A abelian surface $\quad sw_2(H^2(A_{\bar{k}}, \mathbb{Q}_\ell(1))) = 0 \neq sw_2(H^2(A_{\bar{k}}, \mathbb{Q}_p(1))) \in Br_2(\mathbb{Q}_p)$

Sketch of Pf.

1. Picard-Lefschetz formula + formula for det
2. p-adic Hodge theory with integral coeff. (= Fontaine-Laffaille) + a similar argument to prove Th of Serre
3. Hodge theory + polarization
4. Lefschetz principle $k \supset \mathbb{C}$. transcendental argument.
5. moduli space + 1~4.

$U/\mathbb{Z}$ moduli space.

$$\begin{array}{lll}
 & & \xrightarrow{1} H^1(D[\tfrac{1}{2\ell}], \mathbb{Z}/2\mathbb{Z}) \\
P\text{-}D \quad \mathrm{Ker}(1 \oplus 2 \oplus 4) \subset H^2(U[\tfrac{1}{2\ell}], \mathbb{Z}/2\mathbb{Z}) & & \xrightarrow{2} H^1(U \otimes \mathbb{F}_\ell, \mathbb{Z}/2\mathbb{Z}) \quad (\ell \neq 2) \\
\qquad \| & & \xrightarrow{3} H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \\
H^2(\mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\sim} & \xrightarrow{4} H^2(U_{\overline{\mathbb{Q}}}, \mathbb{Z}/2\mathbb{Z})
\end{array}$$