Analogy between ℓ-adic sheaf in char $p>0$ and $\mathcal{D}$ modules on $\mathbb{C}$x mfd

wild ramification — irregular sing.

Char cycle $\xrightarrow[\text{or}]{\text{pull-back by 0-section}}$ Char class

cch class

Cycles of dim d on the cotangent b'dle — $H^{2d}(X, \mathbb{Q}_\ell(d))$

Deligne

How to define it.? A possible approach -- vanishing cycles

X/k smooth dim d — $\mathcal{F}$ ℓ-adic sheaf

k alg closed char $p>0$ — constructible $\mathbb{F}_\ell$-sheaf, $\ell \neq p$

$f: X \to C$ flat morphism to smooth curve

$\mathcal{F}$. $u \in X$ — $\phi_u(\mathcal{F}, f)$ ~~space~~ complex of vanishing cycles

$\psi_u(\mathcal{F}, f)$. nearby .

$\phi^q_u(\mathcal{F}, f)$. fin. dim. action of the absolut Galois gp of K_v $v = f(u)$

$\to$ $\dim \mathrm{tot}_v\, \phi^q_u(\mathcal{F}, f)$ $=0$ unless $0 \leq q \leq \dim X$

$\dim + \mathrm{Sw}$

local acyclicity of smooth morphism.

$\phi_u(\mathcal{F}, f) = 0$ if f is smooth & $\mathcal{F}$ locally constant

Use nonvanishing / total dim of $\phi_u(\mathcal{F}, f)$ to measure sing of f and ramification of $\mathcal{F}$

qualitatively $\to$ Singular Support — Conic closed subset of T^*X

$\uparrow$ supp

quantitatively $\to$ Characteristic Cycle — cycle on T^*X.

Existence of Sing Supp $\Rightarrow$ Def of Characteristic cycle

$\Downarrow$ Milnor formula. — Euler-Poincaré formula.

$S \underset{\text{closed}}{\subset} T^*X = V(\Omega^1_X)$ conic! stable under scalar multipl.

Suppose there exists a closed conic subset S of dim d. satisfying

$X \to C$ non char. loc. acyc

$X \times_C T^*C \to T^*X \Rightarrow$ $\underset{S}{\cup}$

(SS1)

$$\begin{array}{ccc} W & \xrightarrow{f} & C \\ \text{étale} \downarrow & & \downarrow \text{smooth curve} \\ X \times B & \longrightarrow & B \end{array}$$

B smooth

$W \times_X T^*X \xrightarrow{\sim} T^*W/B \longleftarrow W \times_C T^*C/B$

family

f. flat, non characteristic w.r.t S

the inv. image of the pull-back is contained in the 0-section

$\Rightarrow$ f is (univ.) locally acyclic rel to the pull-back of $\mathcal{F}$

generalization of the vanishing of vanishing cycles

S $SS\mathcal{F}$ singular support.

Example ① $X \supset D = \cup D_i$ SNC $U = X - D$. $\mathcal{F}$ l.c.c. + tamely ram along D

$D_I = \bigcap_{i \in I} D_i$

$SS(j_!\mathcal{F})$ $\bigcup_I T^*_{D_I}X$ conormal b'dle

② X curve $\mathcal{F}$ l.c.c on $U \subset X$, $U = X - Z$

$$SS(\mathcal{F}) = \underset{\text{0-sect}}{T^*_X X} \cup \bigcup_{x \in Z} \underset{\text{fibers}}{T^*_x X}$$

①. ② Lagrangean. $\dim X \geqq 2$, wildly ram.

some example non Lagrangean.

Ramification theory $\Rightarrow \exists Z \subset X$ codim 2. on $X - Z$ $SS\mathcal{F}$ is defined

$\Rightarrow \dim X \leqq 2 \Rightarrow SS$ is defined.

Thm Suppose SSK exists. Then there exists a unique linear combination $CC K = \sum a_i [T^*_{S_i}]$ of irred comp of $S = \bigcup S_i$ with $a_i \in \mathbb{Z}[\frac{1}{p}]$ such that

$$-\dim \mathrm{tot}\, \phi_u(K, f) = (CC K, df)_{T^*X, u}$$

for every morphism $f: V \to C$ on an étale nbd of a closed pt such that u is an isolated char point of f w.r.t the pull-back of SSK

$$\begin{array}{ccc} V \times_C T^*C & \longrightarrow & T^*V \supset SSK \\ \uparrow f^*dt & & \nabla \\ & & \end{array}$$

isolated ch $\Rightarrow$ $SSK \cap \mathrm{Im}\, f^*dt \subset T^*_u V$.

r.h.s. int. multiplicity

Definition Coefficients a_i

Assume X quasi-projective. $\mathcal{L}$ very ample.

$$X \times_{\mathbb{P}} H = \mathbb{P}(X \times_{\mathbb{P}} T^*\mathbb{P}) \supset \mathbb{P}(\tilde{S}_i)$$

$$\begin{array}{ccccc} & & \downarrow & & \downarrow \\ \mathbb{P} & \longrightarrow & \mathbb{P}^\vee & \supset & S_i^\vee \end{array}$$

univ. family of hyperplane sections

$\uparrow$ univ. line

$G = G(1, \mathbb{P}^\vee)$ $\quad a_i = (-1)^{d-1}$ ~~Artin conductor~~ total dim of the vanishing cycle on ~~the~~ generic line

Milnor formula for generic pencil.

Continuity of the Swan conductor Deligne-Laumon

Milnor formula for every morphism t. indep of $\mathcal{L}$.

More assumption $\Rightarrow$ Compatibility with generic restriction

$\Rightarrow$ Euler-Poincaré by induction on dim