Wild ramification and Cotangent bundle

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Ramification of an ℓ-adic sheaf

on a variety

over a(n algebraically closed) field k

char $k = p > 0$, $\ell \neq p$

Smooth ℓ-adic sheaf $\mathcal{F}$

on a smooth connected scheme U over k

$$\Leftrightarrow$$

ℓ-adic representation

of the algebraic fundamental group $\pi_1(U, \bar{x})$

Ramification along the boundary;

$X \supset U$ smooth over k,

$j : U = X - D \to X$: open immersion

of the complement of

a divisor D with simple normal crossings

$\mathcal{F}$: smooth on U ramifies along D

divisor $D \subset X$ with simple normal crossings

Locally, $X = \mathbf{A}^n_k = \text{Spec } k[T_1, \ldots, T_n]$

$$\supset D = (T_1 \ldots T_r) \ (0 \leqq r \leqq n)$$

Want: Characteristic cycle

Char $j_!\mathcal{F}$

Linear combination of conic subvarieties

of dimension $d = \dim X$

of the cotangent bundle $T^*X = \mathbf{V}(\Omega^1_{X/k})$

of dimension $2d$

conic subvarieties: irreducible components

of support of Char $j_!\mathcal{F}$

classified by dimension of the fibers

0: 0-section coefficient $=$ rank $\mathcal{F}$

1: subject of this talk (non-degenerate)

$\geqq 2$: ???

Analogy with $\mathcal{D}$-modules in char. 0

microlocal analysis

Char $\mathcal{M}$ for holonomic $\mathcal{D}$-module $\mathcal{M}$

wild ramification in char $p > 0$

vs

irregular singularity in char 0

Related works

Deligne (unpublished) : using jet bundle

Laumon : Euler number for surfaces

Kato: rank 1

Abbes-S.: Ramification group of local field
with imperfect residue field

Expected properties of Char $j_!\mathcal{F}$:

(1) Determined by wild ramification

(2) Compute the characteristic class (SGA5)

and the Euler number

(3) Controls nearby cycles

(4) Compatible with the pull-back

by non-characteristic morphism

. . . . . .

Example 1: X curve (classical)

(1) Char $j_!\mathcal{F}$

$$= -(\text{rank } \mathcal{F} \cdot [T^*_X X] + \sum_{x \in D} \dim \text{tot}_x(\mathcal{F}) \cdot [T^*_x X])$$

$T^*_Y X$ conormal bundle of subscheme $Y \subset X$

$T^*_X X$: 0-section, $T^*_x X$: fiber at x

$\dim \text{tot}_x(\mathcal{F})$: total dimension

$$= \text{rank} + \text{Swan conductor } \text{Sw}_x(\mathcal{F})$$

$$\|$$

measure of wild ramification

(2) $C(j_!\mathcal{F})$ (characteristic class) $=$ [Char $j_!\mathcal{F}$]

in $H^2(T^*X, \mathbf{Q}_\ell(1)) = H^2(X, \mathbf{Q}_\ell(1))$

Consequently, if X is proper,

$$\chi_c(U, \mathcal{F}) = (\text{Char } j_!\mathcal{F}, [T^*_X X])_{T^*X}$$

(Grothendieck-Ogg-Shafarevich)

$$(\chi_c(U, \mathcal{F}) = \sum_{q=0}^{2} (-1)^q \dim H^q_c(U, \mathcal{F}))$$

(3) Induction formula: $\pi\colon X \to Y$

finite generically étale morphism of curves

$$\dim \operatorname{tot}_y \pi_* \mathcal{F} = \dim \operatorname{tot}_x \mathcal{F} + \operatorname{rank} \mathcal{F} \cdot \operatorname{length} \Omega^1_{X/Y,x}$$

Example 2: $\mathcal{F}$ tamely ramified along D (easy)

(1) Char $j_!\mathcal{F} = (-1)^d$rank $\mathcal{F} \cdot \sum_I [T^*_{X_I} X]$

$d = \dim X$, $D = \bigcup_i D_i$, $X_I = \bigcap_{i \in I} D_i$

$T^*_{X_I} X \subset T^* X \times_X X_I$: conormal bundle

(2) $C(j_!\mathcal{F}) = [\text{Char } j_!\mathcal{F}]$,

$\chi_c(U, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U, \mathbf{Q}_\ell)$

Example 3: Artin-Schreier sheaf (typical)

$X = \mathbf{A}^2 = \text{Spec } k[x,y] \supset U = \text{Spec } k[x^{\pm 1}, y]$

(1) (i) $\mathcal{F}$ defined by $t^p - t = \dfrac{1}{x^n}$, $p \nmid n$

$$\text{Char } j_!\mathcal{F} = [T^*_X X] + (n+1) \cdot [T^*_D X]$$

(rank $\mathcal{F} = 1$, $(-1)^2 = 1$, dim $\text{tot}_0 = n+1$)

Example 3: Artin-Schreier sheaf (typical)

$X = \mathbf{A}^2 = \text{Spec } k[x,y] \supset U = \text{Spec } k[x^{\pm 1}, y]$

(1) (ii) $\mathcal{F} : t^p - t = \dfrac{y}{x^n}, \ p \mid n,$

$((p,n) \neq (2,2)$: non-exceptional case)

$\text{Char } j_!\mathcal{F} = [T^*_X X] + n \cdot [\langle dy \rangle \text{ over } D \ (\simeq T^*D)]$

Points in Definition of Char $j_!\mathcal{F}$:

1. Invariant of wild ramification

(New method: Blow-up at the ram. locus

$R \subset X$ in the diagonal $X \to X \times X$)

(More detail at the end, if time permits)

- Ramification of $\mathcal{F}$ is bounded by $R+$

for $R = \sum_i r_i D_i$, $r_i \geqq 1$ rational

(assume integer for simplicity)

Pts in Def. of Char $j_!\mathcal{F}$ (cnt'd.):

1. Invariant of wild ramification

- characteristic forms $\quad (r = \text{rank } \mathcal{F})$

$$\omega_1^{(i)}, \cdots, \omega_r^{(i)} \in \Gamma(D_i, \Omega^1_{X/k}(R) \otimes \mathcal{O}_{D_i})$$

(precisely speaking, defined over purely inseparable coverings of étale schemes D_{ij} over D_i)

Pts in Def. of Char $j_!\mathcal{F}$ (cnt'd.):

Examples of characteristic forms

(i) $t^p - t = \dfrac{1}{x^n}$, $p \nmid n$

$$\omega = d\frac{1}{x^n} = \frac{-n\,dx}{x^{n+1}} \in \Gamma(D, \Omega^1_{X/k}((n+1)D) \otimes \mathcal{O}_D)$$

(ii) $t^p - t = \dfrac{y}{x^n}$, $p \mid n$, $(p, n) \neq (2, 2)$

$$\omega = d\frac{y}{x^n} = \frac{dy}{x^n} \in \Gamma(D, \Omega^1_{X/k}(nD) \otimes \mathcal{O}_D)$$

The exceptional case in (ii) : $t^2 - t = \dfrac{y}{x^2}$

$$\omega = \frac{\sqrt{y}dx + dy}{x^2} \in \Gamma(D', \Omega^1_{X/k}(2D) \otimes \mathcal{O}_{D'})$$

$D' \to D$: inseparable covering of degree 2

Pts in Def. of Char $j_!\mathcal{F}$ (cnt'd.):

2. Non-degenerate (Assumption)

(⇒ component of support of Char $j_!\mathcal{F}$ ($\subset T^*X$) has fiber dim $\leq$1 over X)

D_{ij} are finite étale over D_i and $\omega_1^{(i)}, \cdots, \omega_r^{(i)}$ are nowhere vanishing

(copy : characteristic forms

$$\omega_1^{(i)}, \cdots \omega_r^{(i)} \in \Gamma(D_{ij}, \Omega^1_{X/k}(R) \otimes \mathcal{O}_{D_{ij}})$$

defined over purely inseparable coverings of étale schemes D_{ij} over D_i, $r = \mathrm{rank}\mathcal{F}$)

Pts in Def. of Char $j_!\mathcal{F}$ (cnt'd.):

characteristic forms

$\omega_j^{(i)} \in \Gamma(D_{ij}, \Omega^1_{X/k}(R) \otimes \mathcal{O}_{D_{ij}})$ define

$$\omega_j^{(i)} \colon L(R) \times_X D_{ij} \to T^*X \times_X D_{ij} \to T^*X$$

$L(R)$: line bundle defined by the divisor R

E.g. $t^p - t = \dfrac{y}{x^n}$ $(p \mid n)$:

$$L(nD) \times_X D \to T^*X \times_X D : x^n \mapsto dy$$

Definition of Char $j_!\mathcal{F}$:

Char $j_!\mathcal{F}$

$$= (-1)^d \left((\text{rank } \mathcal{F}) \cdot [T^*_X X] + \sum_i r_i \cdot \sum_{j=1}^{\text{rk } \mathcal{F}} \frac{[\text{Im } \omega_j^{(i)}]}{[D_{ij} : D_i]} \right)$$

$$\omega_j^{(i)} : L(R) \times_X D_{ij} \to T^* X$$

defined by characteristic forms

Example of Char $j_!\mathcal{F}$:

$t^p - t = \dfrac{y}{x^n}$ $(p \mid n,\ (p,n) \neq (2,2))$:

$$L(nD) \times_X D \to T^*X \times_X D : x^n \mapsto dy$$

Char $j_!\mathcal{F}$

$= (-1)^2\Big(1 \cdot [T^*_X X] + n \cdot [\langle dy \rangle \text{ over } D(\simeq T^*D)]\Big)$

Results on Char $j_!\mathcal{F}$:

Characteristic class and the Euler number

$$C(j_!\mathcal{F}) = [\text{Char } j_!\mathcal{F}]$$

in $H^{2d}(X, \mathbf{Q}_\ell(d))$ ($d = \dim X$).

Results on Char $j_!\mathcal{F}$ (cnt'd.):

If X is proper,

$$\chi_c(U, \mathcal{F}) = (\text{Char } j_!\mathcal{F}, [T^*_X X])_{T^*X}$$

(general'n of Grothendieck-Ogg-Shafarevich)

$$(\chi_c(U, \mathcal{F}) = \sum_{q=0}^{2d} (-1)^q \dim H^q_c(U, \mathcal{F}))$$

Results on Char $j_!\mathcal{F}$ (cnt'd.):

Pull-back by non characteristic morphism

$f\colon X' \to X$ of smooth schemes

- $j'\colon U' = f^{-1}(U) = X' - D' \to X'$
- $f^*(\text{Char } j_!\mathcal{F}) \cap \text{Ker}(df\colon T^*X \times_X X' \to T^*X')$

is in the 0-section

$$T^*X \xleftarrow{f} T^*X \times_X X' \xrightarrow{df} T^*X'$$

Results on Char $j_!\mathcal{F}$ (cnt'd.):

Pull-back by non characteristic morphism

$f\colon X' \to X$ of smooth schemes

$$\text{Char } j'_! f^*\mathcal{F} = df(f^*(\text{Char } j_!\mathcal{F}))$$

$$T^*X \xleftarrow{f} T^*X \times_X X' \xrightarrow{df} T^*X'$$

Consequence: Char $j_!\mathcal{F}$ is

char'zed by restriction to curves $C \subset X$

Results on Char $j_!\mathcal{F}$ (cnt'd.):

Local acyclicity for non characteristic

smooth morphism $f\colon X \to Y$ of smooth schemes

• Char $j_!\mathcal{F} \cap df(T^*Y \times_Y X) \subset T^*X$

is in the 0-section $T^*_X X$

$$T^*Y \times_Y X \xrightarrow{df} T^*X \supset \text{Char } j_!\mathcal{F}$$

Results on Char $j_! \mathcal{F}$ (cnt'd.):

Local acyclicity for non characteristic

smooth morphism $f\colon X \to Y$ of smooth schemes

$$j_! \mathcal{F} \text{ is locally acyclic rel. to } f$$

local acyclicity:

vanishing cycles $\phi^q = 0$ ($\dim Y = 1$) or

H^q(Milnor fibers)=0 (in general) for $q > 0$

Conjecture (Deligne):

morphism $f\colon X \to \mathbf{A}^1 = \text{Spec } k[t]$

non characteristic except at a closed pt x

Then,

$$-\dim\ \text{tot}_x\phi(j_!\mathcal{F}) = (\text{Char } j_!\mathcal{F}, dt(X))_{T^*X}$$

Conjecture:

$$-\text{dim tot}_x \phi(j_! \mathcal{F}) = (\text{Char } j_! \mathcal{F}, dt(X))_{T^*X}$$

Deligne-Milnor formula: $X = U, \mathcal{F} = \mathbf{Q}_\ell$,

x; isolated singularity of $X \to Y$

Consequence (Hasse-Arf):

Char $j_! \mathcal{F}$ has integral coefficients

New Method to study ramification:

- Partial blow-up $P^{(R)} \to X \times X$ at the ram. locus $R \subset X$ in the diagonal $X \to X \times X$ $(N_{X/P^{(R)}} = \Omega^1_{X/k}(R),\ N_{X/X\times X} = \Omega^1_{X/k})$
- Groupoid structure $P^{(R)} \times_X P^{(R)} \to P^{(R)}$ lifting

$$(X \times X) \times_X (X \times X) = X \times X \times X \xrightarrow{\mathrm{pr}_{13}} X \times X$$

Blow-up to study ramification

(R integer coefficients > 1 for simplicity):

$$
\begin{array}{ccccc}
T(R) \times_X D & \overset{\subseteq}{\longrightarrow} & P^{(R)} & \overset{\supseteq}{\longleftarrow} & U \times U \\
\downarrow & & \downarrow & & \downarrow \\
D & \overset{\subseteq}{\longrightarrow} & X & \overset{\supseteq}{\longleftarrow} & U = X - D
\end{array}
$$

Groupoid structure on $P^{(R)}$ induces

addition on a twisted tangent bundle

$T(R) \times_X D = V(\Omega^1_{X/k}(R) \otimes \mathcal{O}_D)$

Blow-up to study ramification:

V: G-torsor over U, G finite

$$\begin{array}{ccccc} & & W^{(R)} & \overset{\supseteq}{\longleftarrow} & V \times V/\Delta G \\ & & \downarrow & & \downarrow \\ T(R) \times_X D & \overset{\subseteq}{\longrightarrow} & P^{(R)} & \overset{\supseteq}{\longleftarrow} & U \times U \end{array}$$

$W^{(R)}$ largest open in the normalization étale over $P^{(R)}$

Blow-up to study ramification:

Ram'n of G-torsor V over U bounded by $R+$:

$$\begin{array}{ccccc} W^{(R)} & \overset{\supset}{\longleftarrow} & V \times V/\Delta G & \longleftarrow & V/G \\ \downarrow & & \downarrow & & \| \\ P^{(R)} & \overset{\supset}{\longleftarrow} & U \times U & \xleftarrow{\Delta_U} & U \end{array}$$

is extended to $X \to W^{(R)}$

(largest open étale over $P^{(R)}$)

Blow-up to study ramification:

Ram'n of G-torsor V over U bounded by $R+$:

$U = V/G \to V \times V/\Delta G$ is ext'd to $X \to W^{(R)}$

$$\begin{array}{ccccc} & & W^{(R)} & \stackrel{\supset}{\longleftarrow} & V \times V/\Delta G \\ & & \downarrow & & \downarrow \\ T(R) \times_X D & \stackrel{\subset}{\longrightarrow} & P^{(R)} & \stackrel{\supset}{\longleftarrow} & U \times U \end{array}$$

$W^{(R)}$ largest open étale over $P^{(R)}$

$\Rightarrow W^{(R)} \to P^{(R)}$ morphism of groupoids

Blow-up to study ramification:

$$\begin{array}{ccccc}
E^{(R)} & \overset{\subseteq}{\longrightarrow} & W^{(R)} & \overset{\supseteq}{\longleftarrow} & V \times V/\Delta G \\
\downarrow & & \downarrow & & \downarrow \\
T(R) \times_X D & \overset{\subseteq}{\longrightarrow} & P^{(R)} & \overset{\supseteq}{\longleftarrow} & U \times U
\end{array}$$

Non-degenerate:

étale morphism $W^{(R)} \to P^{(R)}$ of groupoids

induces a finite étale morphism

$$E^{(R)} \longrightarrow T(R) \times_X D$$

of smooth group schemes over D

Blow-up to study ramification:

Non-degenerate:

Finite étale morphism $E^{(R)} \to T(R) \times_X D$

of smooth group schemes over D

Classification of étale isogeny

to vector bundle by linear forms

(defined over inseparable covering)

defines characteristic forms