Wild ramification and the characteristic cycle of an ℓ-adic sheaf

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Abstract

The graded quotients of the logarithmic higher ramification groups of a local field of characteristic $p > 0$ with arbitrary residue field are abelian groups killed by p. Their character groups are canonically embedded in some spaces of twisted differential forms.

Using the embeddings, we define the characteristic cycle of an ℓ-adic sheaf, satisfying certain conditions, as a cycle on the logarithmic cotangent bundle and prove that the intersection with the 0-section gives the characteristic class.

Contents

1 Ramification along a divisor

Let k be a perfect field of characteristic $p > 0$, X be a smooth scheme of dimension d over k and $U = X \setminus D$ be the complement of a divisor D with simple normal crossings. We consider a smooth ℓ-adic sheaf $\mathcal{F}$ on U.

Let $D_1, \ldots, D_m$ be the irreducible components of D. We define the log blow up

$$(X \times X)' \to X \times X$$

to be the blow-up at $D_1 \times D_1, D_2 \times D_2, \ldots, D_m \times D_m$. Namely the blow-up by the ideal sheaf $(\mathrm{pr}_1^*\mathcal{I}_{D_1} + \mathrm{pr}_2^*\mathcal{I}_{D_1}) \cdot (\mathrm{pr}_1^*\mathcal{I}_{D_2} + \mathrm{pr}_2^*\mathcal{I}_{D_2}) \cdots (\mathrm{pr}_1^*\mathcal{I}_{D_m} + \mathrm{pr}_2^*\mathcal{I}_{D_m})$. We define the log product $(X \times X)^\sim \subset (X \times X)'$ to be the complement of the proper transforms of $D \times X$ and $X \times D$. The scheme $(X \times X)^\sim$ is affine over $X \times X$ and is defined by the quasi-coherent $\mathcal{O}_{X \times X}$-algebra

$$\mathcal{O}_{X \times X}[\mathrm{pr}_1^*\mathcal{I}_{D_i}^{-1} \cdot \mathrm{pr}_2^*\mathcal{I}_{D_i},\ \mathrm{pr}_1^*\mathcal{I}_{D_i} \cdot \mathrm{pr}_2^*\mathcal{I}_{D_i}^{-1}; i = 1, \ldots, m] \subset j_*\mathcal{O}_{U \times U}$$

where $j : U \times U \to X \times X$ denotes the open immersion. The diagonal map $\delta : X \to X \times X$ is uniquely lifted to the log diagonal map $\delta' : X \to (X \times X)'$. The projections $(X \times X)^\sim \to X$ are smooth. The conormal sheaf $\mathcal{N}_{X/(X\times X)^\sim}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)$ of rank d.

Let $R = r_1D_1 + \cdots + r_mD_m$ be an effective divisor with rational coefficients $r_1, \ldots, r_m \geq 0$. For simplicity, in the lecture, we assume that the coefficients $r_1, \ldots, r_m$ are integers. We define

$$(X \times X)^{[R]} \to (X \times X)'$$

to be the blow-up at the divisor $R \subset X$ in the log diagonal $X \subset (X \times X)'$. We define an open subscheme $(X \times X)^{(R)} \subset (X \times X)^{[R]} \times_{(X\times X)^\sim} (X \times X)' \subset (X \times X)^{[R]}$ to be the complement of the support of $(p^*\mathcal{I}_R O_{(X\times X)^{[R]}} + \mathcal{J}_X O_{(X\times X)^{[R]}})/p^*\mathcal{I}_R O_{(X\times X)^{[R]}}$ where $\mathcal{I}_R \subset \mathcal{O}_X$ and $\mathcal{J}_X \subset \mathcal{O}_{(X\times X)'}$ are the ideal defining $R \subset X$ and the log diagonal $X \subset (X \times X)'$ and $p : (X \times X)^{[R]} \to X$ denotes the projection. The scheme $(X \times X)^{(R)}$ is an affine scheme over $(X \times X)^\sim$ defined by the quasi-coherent $\mathcal{O}_{(X\times X)^\sim}$-algebra

$$\sum_{l \geq 0} \mathcal{I}_{lR}^{-1} \cdot \mathcal{J}_X^l \subset \tilde{j}_* \mathcal{O}_{U\times U} \qquad (1.1)$$

where $\tilde{j} : U \times U \to (X \times X)^\sim$ denotes the open immersion. The log diagonal map $\delta' : X \to (X \times X)'$ is uniquely lifted to a closed immersion $\delta^{(R)} : X \to (X \times X)^{(R)}$. The projections $(X \times X)^{(R)} \to X$ are smooth. The conormal sheaf $\mathcal{N}_{X/(X\times X)^{(R)}}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)(R)$.

We consider the commutative diagram

$$\begin{array}{ccc} U \times U & \xrightarrow{j = j^{(R)}} & (X \times X)^{(R)} \\ \uparrow{\scriptstyle \delta_U} & & \uparrow{\scriptstyle \delta^{(R)}} \\ U & \xrightarrow{j_0} & X \end{array}$$

of open immersions and the diagonal immersions.

Definition 1 *Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$. We define a smooth sheaf $\mathcal{H}$ on $U \times U$ by $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})$. Let $R = \sum_i r_iD_i \geq 0$ be an effective divisor with rational coefficients.*

We say the log ramification of $\mathcal{F}$ along D is bounded by $R+$ if the identity $1 \in \mathrm{End}_U(\mathcal{F}) = \Gamma(U, \mathcal{E}nd_U(\mathcal{F})) = \Gamma(X, j_{0*}\mathcal{E}nd_U(\mathcal{F}))$ is in the image of the base change map

$$\Gamma(X, \delta^{(R)*} j_* \mathcal{H}) \longrightarrow \Gamma(X, j_{0*}\mathcal{E}nd_U(\mathcal{F})) = \mathrm{End}_U(\mathcal{F}). \qquad (1.2)$$

Definition 1 is related to the filtration by ramification groups in the following way. Let D_i be an irreducible component and K_i be the fraction field of the completion $\widehat{\mathcal{O}}_{X,\xi_i}$ of the local ring at the generic point ξ_i of D_i. We will often drop the index i in the sequel. The sheaf $\mathcal{F}$ defines an ℓ-adic representation $\mathcal{F}_{\bar\eta_i}$ of the absolute Galois group $G_{K_i} = \mathrm{Gal}(\overline{K}_i/K_i)$. The filtration $G^r_{K,\log} \subset G_K, r \in \mathbb{Q}, r > 0$ by the logarithmic ramification groups is defined. We put $G^{r+}_{K,\log} = \overline{\bigcup_{q>r} G^q_{K,\log}}$.

Lemma 2 *The following conditions are equivalent.*

(1) *There exists an open neighborhood of ξ_i such that the log ramification of $\mathcal{F}$ along D is bounded by $R+$.*

(2) *The action of $G^{r_i+}_{K_i,\log}$ on $\mathcal{F}_{\bar\eta_i}$ is trivial.*

Proposition 3 *Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$ such that the log ramification of $\mathcal{F}$ is bounded by $R+$. Let C and C' be smooth curves in X and x be a closed point in $C \cap C' \cap D$. Assume the following conditions are satisfied:*

(1) *For every irreducible component D_i of D, we have $(C, D_i)_x = (C', D_i)_x$.*

(2) $\mathrm{length}_x \mathcal{O}_{C\cap C',x} \geq (C, R + D)_x$.

Then, étale locally at x, there exist an étale morphism $f : C \to C'$ and an isomorphism $f^*\mathcal{F}|_{C'} \to \mathcal{F}|_C$.

Let $j = j^{(R)} : U \times U \to (X \times X)^{(R)}$ be the open immersion. We put $D^+ = \bigcup_{i:r_i>0} D_i$. The open subscheme $U \times U \subset (X \times X)^{(R)}$ is the complement of the inverse image $E^+ = (X \times X)^{(R)} \times_X D^+$. The inverse image E^+ is canonically identified with the vector bundle $\mathbf{V}(\Omega^1_X(\log D)(R)) \times_X D^+$. For a sheaf $\mathcal{H}$ on E, we consider the Fourier transform $F_\psi\mathcal{H} = R\mathrm{pr}_{2!}(\mathrm{pr}_1^*\mathcal{H} \otimes \langle\ ,\ \rangle^*\mathcal{L}_\psi)(d)[2d]$ on the dual E^*.

Proposition 4 *Assume that the log ramification is bounded by $R+$. Let $S_\mathcal{F} \subset E^{+*}$ be the support of the Fourier transform $F_\psi(j_*\mathcal{H})|_{E^+}$ of the restriction of $j_*\mathcal{H}$ on $E^+ = \mathbf{V}(\Omega^1_X(\log D)(R)) \times_X D^+$. Then, the projection $S_\mathcal{F} \to D^+$ has finite fibers.*

Idea of proof. There exists a unique smooth map $\mu : (X \times X)^{(R)} \times_X (X \times X)^{(R)} \to (X \times X)^{(R)}$ that makes the diagram

$$\begin{array}{ccc} (X \times X)^{(R)} \times_X (X \times X)^{(R)} & \xrightarrow{\mu} & (X \times X)^{(R)} \\ \downarrow & & \downarrow \\ (X \times X) \times_X (X \times X) & \xrightarrow{\mathrm{pr}_{13}} & (X \times X) \end{array}$$

commutative. The composition

$$\mathcal{H} \boxtimes \mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F}) \otimes \mathcal{H}om(\mathrm{pr}_3^*\mathcal{F}, \mathrm{pr}_2^*\mathcal{F}) \to \mathrm{pr}_{13}^*\mathcal{H} = \mathcal{H}om(\mathrm{pr}_3^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})$$

induces a map $j_*\mathcal{H} \boxtimes j_*\mathcal{H} \to \mu^* j_*\mathcal{H}$. It further induces $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ on $\mathcal{G} = F_\psi\mathcal{H}$. Using this map, we prove Proposition 4.

Proposition 4 has the following consequence. Let D_i be an irreducible component of D. The graded piece $\mathrm{Gr}^{r_i}_{\log} G_{K_i} = G^{r_i}_{\log,K}/G^{r_i+}_{\log,K}$ is abelian. The restriction of $\mathcal{F}_{\bar\eta_i}$ to $G^{r_i}_{\log,K}$ is decomposed into direct sum of characters $\bigoplus_\chi \chi^{n_\chi}$. The fiber $\Theta^{(r_i)}_{\log} = E^+ \times_{D^+} \xi_i$ at the generic point ξ_i is a vector space over the function field F_i of D_i. The restriction of $j_*\mathcal{H}$ on the geometric fiber $\Theta^{(r_i)}_{\log,\overline{F}_i}$ is decomposed as $\bigoplus_\chi \mathrm{End}_{I_i}(\mathcal{F}_{\bar\eta_i}) \otimes \mathcal{L}_\chi$ where $\mathcal{L}_\chi$ is a smooth rank one sheaf defined by the Artin-Schreier equation $T^p - T = f_\chi$ where $f_\chi = \mathrm{rsw}\ \chi$ is a linear form on $\Theta^{(r_i)}_{\log,\overline{F}_i}$ called the refined Swan character of χ.

Theorem 5 *The graded quotient* $\mathrm{Gr}^r_{\log} G_K$ *is annihilated by* p *and the map*

$$\mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\overline{F}_i}(\Theta^{(r)}_{\log}, \overline{F}_i) \tag{1.3}$$

sending a character χ *to the refined Swan character* $f_\chi = \mathrm{rsw}\ \chi$ *is an injection.*

2 Characteristic cycle

We recall the definition of the characteristic class. Let X be a scheme over k and $\mathcal{F}$ be a constructible Λ-sheaf. We put $K_X = Ra^!\Lambda$ and $D_X\mathcal{F} = R\mathcal{H}om(\mathcal{F}, K_X)$. We define the characteristic class $C(\mathcal{F}) \in H^0(X, K_X)$ to be the image of $1 \in \mathrm{End}(\mathcal{F})$ by the composition

$$\mathrm{End}(\mathcal{F}) \to H^0_X(X \times X, R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F})) \to H^0(X, \mathcal{F} \otimes^L D_X\mathcal{F}) \to H^0(X, K_X).$$

The first map is a natural identification of $H^0_X(X \times X, R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F}))$ with $\mathrm{End}(\mathcal{F})$. The second is induced by the inverse of the canonical isomorphism $\mathcal{F} \boxtimes D_X\mathcal{F} \to R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F})$ and the pull-back map. The last map is induced by the evaluation map $\mathcal{F} \otimes^L D_X\mathcal{F} \to K_X$.

If X is proper, by the Lefschetz trace formula in SGA 5, the characteristic class computes the Euler number: $\mathrm{Tr}\ C(\mathcal{F}) = \chi(X_{\overline{F}}, \mathcal{F})$. If X is smooth of dimension d and if $\mathcal{F}$ is smooth, the canonical class $C(\mathcal{F})$ is equal to rank $\mathcal{F}$ times the self intersection class $(X, X) = (-1)^d c_d(\Omega^1_X) \in H^{2d}(X, \Lambda(d))$.

Let $\chi : \mathrm{Gr}^r_{\log} G_K \to \mathbb{F}_p$ be a non-trivial character. The refined Swan character $\mathrm{rsw}\ \chi : \Theta^{(r)}_{\log} \to \overline{F}_i$ defines an $\overline{F}_i$-rational point $[\mathrm{rsw}\ \chi]$ of $\mathbf{P}(\Omega^1_X(\log D)^*)(\overline{F}_i)$. We define a reduced closed subscheme $T_\chi \subset \mathbf{P}(\Omega^1_X(\log D)^*)$ to be the Zariski closure $\overline{\{[\mathrm{rsw}\ \chi](\mathrm{Spec}\ \overline{F}_i)\}}$ and let $L_\chi = \mathbf{V}(\mathcal{O}_{T_\chi}(1))$ be the pull-back to T_χ of the tautological sub line bundle $L \subset T^*X(\log D) \times_X \mathbf{P}(\Omega^1_X(\log D)^*)$. It defines a commutative diagram

$$\begin{array}{ccccc} L_\chi & \longrightarrow & T^*X(\log D) \times_X D_i & \longrightarrow & T^*X(\log D) = \mathbf{V}(\Omega^1_X(\log D)^*) \\ \downarrow & & \downarrow & & \downarrow \\ T_\chi & \xrightarrow{\pi_\chi} & D_i & \longrightarrow & X. \end{array} \tag{2.1}$$

We put

$$SS_\chi = \frac{1}{[T_\chi : D_i]} \pi_{\chi *}[L_\chi] \tag{2.2}$$

in $Z_d(T^*X(\log D) \times_X D_i)_{\mathbb{Q}}$.

Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on $U = X \setminus D$ and $R = \sum_i r_i D_i$ be an effective divisor with rational coefficients $r_i \geq 0$. In the rest of talk, we assume that $\mathcal{F}$ satisfies the following conditions:

(R) The log ramification of $\mathcal{F}$ along D is bounded by $R+$.

(C) For each irreducible component D_i of D, the closure $\overline{S_{\mathcal{F}} \times F_i}$ is finite over D_i and the intersection $\overline{S_{\mathcal{F}} \times F_i} \cap D_i$ with the 0-section is empty.

The conditions imply $\mathcal{F}_{\bar{\eta}_i} = \mathcal{F}_{\bar{\eta}_i}^{(r_i)}$ for every irreducible component D_i of D.

Definition 6 *Let $\mathcal{F}$ be a smooth Λ-sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C).

For an irreducible component D_i of D with $r_i > 0$, let $\mathcal{F}_{\bar{\eta}_i} = \sum_\chi n_\chi \chi$ be the direct sum decomposition of the representation induced on $\mathrm{Gr}^{r_i}_{\log} G_{K_i}$. We define the characteristic cycle by

$$CC(\mathcal{F}) = (-1)^d \left(\mathrm{rank}\, \mathcal{F} \cdot [X] + \sum_{i, r_i > 0} r_i \cdot \sum_\chi n_\chi \cdot [SS_\chi] \right) \tag{2.3}$$

in $Z^d(T^*X(\log D))_{\mathbb{Q}}$.

Theorem 7 *Let X be a smooth scheme over k and D be a divisor with simple normal crossings. Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C).

Then we have

$$C(j_{0!}\mathcal{F}) = [CC(\mathcal{F})]$$

in $H^{2d}(X, \Lambda(d)) = H^{2d}(T^*X(\log D), \Lambda(d))$. In other words, we have

$$C(j_{0!}\mathcal{F}) = (CC(\mathcal{F}), X)_{T^*X(\log D)}.$$

Idea of Proof. Under the assumption (C), we see

$$SS_\chi = \big(c(\Omega^1_X(\log D)) \cap (1+R)^{-1} \cap [T^*X(\log D) \times_X D_i]\big)_{\dim d}.$$

Hence we have

$$CC(\mathcal{F}) = (-1)^d \cdot \mathrm{rank}\, \mathcal{F} \cdot \big([X] + c(\Omega^1_X(\log D)) \cap (1+R)^{-1} \cap [T^*X(\log D) \times_X R]))_{\dim d}\big)$$

and

$$\begin{aligned} & (CC(\mathcal{F}), X)_{T^*X(\log D)} \\ = \; & (-1)^d \cdot \mathrm{rank}\, \mathcal{F} \cdot \big((X, X)_{T^*X(\log D)} + c(\Omega^1_X(\log D)) \cap (1+R)^{-1} \cap [R]))_{\dim d}\big) \end{aligned}$$

On the other hand, the image of the identity $\mathrm{id}_{j_{0!}\mathcal{F}} \in \mathrm{End}(j_{0!}\mathcal{F})$ by the pull-back map

$$\begin{aligned} & H^0_X(X \times X, R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, R\mathrm{pr}_1^!\mathcal{F})) \\ & \longrightarrow H^0_{f^{-1}(X)}((X \times X)^{(R)}, j_*^{(R)}\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})(d)[2d]) \end{aligned}$$

is equal to the image of $H^0_X((X \times X)^{(R)}, j_*^{(R)}\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})(d)[2d])$ defined by the cup-product $e \cup [X]$ of the section $e \in H^0(X, \delta^{(R)*}j_*^{(R)}\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F}))$ lifting the identity of $\mathcal{F}$ with the cycle class $[X] \in H^0_X((X \times X)^{(R)}, \Lambda(d)[2d])$. Thus the left hand side is equal to rank $\mathcal{F} \cdot (X, X)_{(X \times X)^{(R)}}$. Therefore, it follows from the equality

$$(X, X)_{(X \times X)^{(R)}} = (-1)^d \cdot c_d(\Omega^1_X(\log D)) - (c(\Omega^1_X(\log D))^* \cap (1 - R)^{-1} \cap [R])_{\dim d}.$$

Closing the lecture by raising questions:

1. What can one say if there is more than one R? What happens if the ramification is not controled by the points of codimension 1?

2. Analogy with $\mathcal{D}$-modules with irregular singularities. The same construction seems to work. Relation with the existing theory of characteristic cycles, irregularities etc.?

3. Mixed characteristic case. The same results on the graded quotient seems within reach. How one can define characteristic cycle?

4. Epsilon factors. One can compute the Euler number using the characteristic cycle. What can one do for the determinant of cohomology.