The chearacteristic class and the Swan class of an ℓ-adic sheaf (with Abbes and with Kato)

December 20, 2004

1982: Galois theory. undergraduate, seminar.
1985-6: Intersection theory. First time to study it.
Report on application of intersection theory to etale cohomology.
Plan:
0. Outline.
1. Swan class and Grothendieck-Ogg-Shafarevich formula. (with Kato)
2. Characteristic class and its relation with the Swan class. (with Abbes)
Notation: F field of characteritic $p > 0$.
$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$ etc. $(\ell \neq p)$
X variety over F.
$\mathcal{F}$ Λ-sheaf on X, or more generally, an object in a suitable derived category.
0.1. X variety, $U \subset X$ dense open smooth over F.
$\mathcal{F}$ smooth on U.
The Swan class $\mathrm{Sw}(\mathcal{F})$ is defined in $CH_0(X \setminus U)_{\mathbb{Q}}$. If X is proper,

$$\chi_c(U_{\bar F}, \mathcal{F}) \left(= \sum_{q=0}^{2d} (-1)^q \dim H^q_c(U_{\bar F}, \mathcal{F}) \right) = \mathrm{rank}\mathcal{F} \cdot \chi_c(U_{\bar F}) - \mathrm{deg}\mathrm{Sw}(\mathcal{F}).$$

0.2. The characteristic class $C(\mathcal{F}) \in H^0(X, K_X)$ is defined by Abbes. Implicitly in SGA5. In complex geometry, it is defined by Kashiwara-Schapira.

$K_X = Ra^!\Lambda, a : X \to \mathrm{Spec}\ F$. If X is smooth of dimension d, $C(\mathcal{F})$ is defined in $H^{2d}(X, \Lambda(d))$ If X is proper,

$$\mathrm{Tr}\ C(\mathcal{F}) = \chi(X_{\bar F}, \mathcal{F}) \left(= \sum_{q=0}^{2d} (-1)^q \dim H^q(X_{\bar F}, \mathcal{F}) \right).$$

Let $j : U \to X$ be the open immersion. Then, the relation

$$C(j_!\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot C(j_!\Lambda) - \mathrm{cl}\ \mathrm{Sw}(\mathcal{F})$$

in $H^0(X, K_X)$ is verified in many cases. $\mathrm{cl} : CH_0(X) \to H^0(X, K_X)$ cycle class map.

1. $U \subset X$: smooth over F, $\mathcal{F}$ on U smooth.

For simplicity, assume $\mathcal{F}$ is trivialized by a finite Galois covering $V \to U$ of Galois group G. M : representation of G corresponding to $\mathcal{F}$.

Further assume there is a commutative diagram

$$\begin{array}{ccc} Y & \overset{\supset}{\longleftarrow} & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \overset{\supset}{\longleftarrow} & U \end{array}$$

where $f : Y \to X$ is proper, Y is smooth and V is the complement of a divisor with simple normal crossings. (In general, we consider $\mathcal{F}$ mod ℓ and use the Brauer trace and also consider alteration.)

$\sigma \in G = \mathrm{Gal}(U/V), \sigma \neq 1$.

Figure 1.

Γ_σ : graph of σ.

$(Y \times Y)' \to Y \times Y$: Blow up at $D_1 \times D_1, \ldots, D_m \times D_m$ where $D_1, \ldots, D_m$ are the irreducible components of D.

$\Delta_Y : Y \to (Y \times Y)'$: the log diagonal map.

Figure 2.

$\overline{\Gamma_\sigma}$: closure of $\Gamma_\sigma \subset V \times_U V$ in $(Y \times Y)'$.

tame ramification : no intersection.

wild ramification : non-empty intersection.

Define

$$s_{V/U}(\sigma) = -(\overline{\Gamma_\sigma}, \Delta_Y)_{(Y\times Y)'} \in CH_0(Y - V),$$

$s_{V/U}(1) = -\sum_{\sigma \neq 1} s_{V/U}(\sigma)$ and

$$\mathrm{Sw}(\mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G} f_* s_{V/U}(\sigma) \mathrm{Tr}(\sigma : M) \in CH_0(X - U) \otimes \mathbb{Q}. \tag{1}$$

In fact, $\mathrm{Sw}(\mathcal{F})$ is defined as an element of $CH_0(E)_{\mathbb{Q}}$ where $E \subset X - U$ is the wild ramification locus.

Problem: Compute the Swan class in terms of Abbes-Saito filtration. (Partial answer in the rank 1 case.)

We have a generalization of the Grothendieck-Ogg-Shafarevich formula.

Theorem 1 *If X is proper,*

$$\chi_c(U, \mathcal{F}) = \chi_c(U) \cdot \mathrm{rank}\ \mathcal{F} - \deg\ \mathrm{Sw}(\mathcal{F}).$$

Main ingredient of proof. Lefschetz trace formula for an open variety, proved using a method of Pink-Faltings.

Variant: We may also define $\mathrm{Sw}(\mathcal{F})$ in a mixed characteristic situation. We have a relative version of Theorem 1 that gives a conductor formula with a coefficient sheaf.

2. More generally, the characteristic class is defined for a cohomological correspondence.

X variety over F. $c : C \to X \times X$ closed immersion, $p_i : C \to X$ $(i = 1, 2)$ compositions with the projections.

$\mathcal{F}$ on X, $u : p_2^*\mathcal{F} \to p_1^!\mathcal{F}$ a cohomological correspondence (direction is the inverse of that in SGA 5).

We put $\mathcal{H} = R\mathcal{H}om(pr_2^*\mathcal{F}, pr_1^!\mathcal{F})$. Then, u defines a map $\Lambda_C \to c^!\mathcal{H}$ and hence $u \in H^0_C(X \times X, \mathcal{H})$.

On the other hand, the canonical isomorphism $\mathcal{F} \boxtimes D\mathcal{F} \to \mathcal{H}$ and the evaluation map $\mathcal{F} \otimes D\mathcal{F} \to K_X$ induce a map $e : \delta^*\mathcal{H} \to K_X$.

We define a class $C(\mathcal{F}, C, u) \in H^0_{C\cap X}(X, K_X)$ as $e \circ \delta^* u$.

Proposition 2 *If X is proper over F,*

$$\mathrm{Tr}(u^* : H^*(X_{\bar{F}}, \mathcal{F})) = \mathrm{Tr}\ C(\mathcal{F}, C, u).$$

$C(\mathcal{F}, C, u)$ is the pairing $\langle \mathrm{id}, u\rangle$ in the notation of SGA5. A reformulation of the Lefschetz trace formula in SGA5. A special case of the compatibility of the construction of the characteristic class with proper push-forward.

$$\begin{array}{ccccc}
C(j_!\mathcal{F}) \in & H^0(X, K_X) & \xleftarrow{\text{cycle map}} & CH_0(X-U)_{\mathbb{Q}} \ni \mathrm{Sw}(\mathcal{F}) \\
\downarrow & \Big\downarrow{\scriptstyle \mathrm{Tr}} & & \Big\downarrow{\scriptstyle \deg} \\
\chi_c(U_{\bar{F}}, \mathcal{F}) \in & \mathbb{Q}_\ell & \supset & \mathbb{Q}
\end{array}$$

Conjecture 3 *$U \subset X$: smooth over F, $\mathcal{F}$ smooth $\mathbb{Q}_\ell$-sheaf on U. Then, we have*

$$C(j_!\mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot C(j_!\Lambda) - \mathrm{Sw}\mathcal{F} \tag{2}$$

in $H^0(X, K_X)$.

Theorem 4 *Conjecture 3 is true if there exists a finite etale Galois covering $V \to U$ satisfying one of the following conditions.*

(Res) *There exist a proper smooth scheme Y over F, a divisor $D \subset Y$ with simple normal crossings, an isomorphism $V \to Y \setminus D$ and an action of G on Y extending that on V. The pull-back $\mathcal{F}_V$ of $\mathcal{F}$ on V is tamely ramified along D.*

(Triv) *The pull-back $\mathcal{F}_V$ is constant.*

Proof is similar to that of Theorem 1.

Assume X is smooth and $D = X - U$ has simple normal crossings. If $\mathcal{F}$ is tamely ramified, we have

$$C(j_!\mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot (-1)^d c_d(\Omega^1_{X/F}(\log D)) \tag{3}$$

in $H^{2d}(X, \Lambda(d))$. In particular,

$$C(j_!\Lambda) = (-1)^d c_d(\Omega^1_{X/F}(\log D)).$$

If $\dim U = 1$ and $\mathrm{rank}\mathcal{F} = 1$, we can prove Theorem 4 integrally.

Theorem 5 *Let X be a smooth curve and $U \subset X$ be a dense open. Let $\mathcal{F}$ be a smooth Λ-sheaf of rank 1. Then, we have*

$$C(j_!\mathcal{F}) = C(j_!\Lambda) - \mathrm{Sw}\mathcal{F} \tag{4}$$

in $H^2(X, \Lambda(1))$.

Sketch of Proof. Assume for simplicity $U = X - \{x\}$. Put $n = \mathrm{Sw}_x\mathcal{F} \geq 0$.

$(X \times X)^{(0)} \to X \times X$ the blow-up at the image of x by the diagonal map $X \to X \times X$.

The diagonal map $X \to X \times X$ is extended to the log diagonal map $X \to (X \times X)^{(0)}$.

We define blow-up $(X \times X)^{(i)} \to (X \times X)^{(i-1)}$ for $i = 1, 2, \ldots, n$ inductively.

$\delta^{(n)} : X \to (X \times X)^{(n)}$: immersion induced by the diagonal

E_i: exceptional divisor.

$(U \times U)^{(n)}$: complement in $(X \times X)^{(n)}$ of the union of the proper transforms of $X \times x$, $x \times X$, and the exceptional divisors E_i for $i = 0, 1 \ldots, n-1$.

In the commutative diagram

$$\begin{array}{ccc} (X \times X)^{(n)} & \xleftarrow{j^{(n)}} & (U \times U)^{(n)} \\ {\scriptstyle f^{(n)}}\big\downarrow & & \big\uparrow{\scriptstyle k^{(n)}} \\ X \times X & \xleftarrow{j} & U \times U, \end{array}$$

the left vertical arrow is the composition of blow-ups and the others are open immersions.

Proposition 6 *We put $\mathcal{H} = \mathcal{H}om(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})$. Then, we have the following.*

1. *The Λ-sheaf $\mathcal{H}^{(n)} = k_*^{(n)}\mathcal{H}$ is a smooth Λ-sheaf of rank 1 on $(U \times U)^{(n)}$.*
2. *The restriction $\mathcal{H}^{(n)}|_{E_n}$ is an Artin-Schreier sheaf.*
3. *If $\mathcal{F}$ is ramified at x, the canonical map $j_!^{(n)}\mathcal{H}^{(n)} \to Rj_*^{(n)}\mathcal{H}^{(n)}$ is an isomorphism.*

Proof of Proposition. Identify $H^1(K_x, \mathbb{Z}/p^m\mathbb{Z}) = W_m(K_x)/F - 1$ and consider the filtration of Brylinski inducing the filtration by ramification.

Proof of Theorem. The characteristic class $C(j_!\mathcal{F})$ is defined by the composition $\delta_!^{(n)}\Lambda_X \to \mathcal{H} \otimes pr_1^*K_X \to \delta_*^{(n)}K_X$ and hence equal to the intersection product $(X, X)_{(X \times X)^{(n)}}$.

Application of Theorem. Proof of the GOS formula without using the Weil formula. (Brauer induction).