Singular support & Characteristic cycle of an ℓ-adic sheaf

0. Introduction

X smooth / k perfect field char $p \geq 0$

$\ell \neq p$ $\mathcal{F}$ constructible

- $SS\mathcal{F} \subset T^*X$ closed conical subset
- $CC\mathcal{F} = \sum m_a C_a$ $SS\mathcal{F} = \bigcup C_a$, $m_a \in \mathbb{Z}$

Definition & properties.

Refinement ~~coefficients~~ $m_a \in \bar{\mathbb{Q}}_\ell^\times \otimes \mathbb{Q}$ Takeuchi,

& product formula for $\det(-\mathrm{Fr}, H^*(X_{\bar{k}}, \mathcal{F}))$

~~Ex~~ample. 1. $\mathcal{F}$ locally const, nonzero rank.

$$\Rightarrow \quad SS\mathcal{F} = T^*_X X \quad \text{0-section.}$$

$$CC\mathcal{F} = \underbrace{(-1)^n}_{\text{perverse}} \mathrm{rk}\,\mathcal{F} \cdot T^*_X X \qquad n = \dim X.$$

2. $\dim X = 1$. $U \subset X$ largest open $\mathcal{F}|_U$ loc. const

$$\Rightarrow \cdot\ SS\mathcal{F} = T^*_X X \cup \bigcup_{x \in X-U} T^*_x X \qquad \text{fiber}$$

$$\cdot\ CC\mathcal{F} = -\left(\mathrm{rk}\,\mathcal{F} \cdot T^*_X X + \sum a_x \mathcal{F} \cdot T^*_x X\right)$$

$a_x\mathcal{F}$ Artin conductor

$= \mathrm{rk}\,\mathcal{F} - \mathrm{rk}\,\mathcal{F}_{\bar{x}} + \mathrm{Sw}_x\mathcal{F}$

wild ramification

Lagrangian. not in general.

Defin.

SS $\quad C \subset T^*X$ closed conical subset

$f: X \to Y$ $\quad$ good rel to C — transversality
$\quad\Rightarrow$ good rel to $\mathcal{F}$ — local acyclicity

smallest such C.

CC $\quad$ Functorial characterization

compatibility with $\left\{\begin{array}{l}\text{push forward}\\ \text{pull-back}\end{array}\right.$

1. trans & l.a.
2. Singular support
3. char cycle

1. SS

1.1. transversality

k field X smooth$/k$. T^*X cotangent b'dle

Def 1.1.1. Let $C \subset T^*X$ closed conical subset
W smooth$/k$ $\boxed{h: W \to X}$ morphism $/k$

1. $h^*C = W \times_X C \subset W \times_X T^*X$ closed con. subset

2. We say that h is C-trans if

$$h^*C \cap \ker(W \times_X T^*X \to T^*W) \subset W \times_X T^*_X X$$
0-sect

Ex 1.1.2. 1. h smooth $\Leftrightarrow$ h C-tran for $C = T^*_X X$
on a nbd of $Z \subset X$

2. $C = T^*_X X \Rightarrow \forall h.$ C-trans

3. For $C \subset C'$ C'-tran $\Rightarrow$ C-tran

Lemma 1.1.3 If h C-tran $\Rightarrow$ $W \times_X T^*X \to T^*W$ is finite on h^*C
Im $h^*C = h^\circ C \subset T^*W$ is a closed conical subset

Def. 1.1.4 $C \subset T^*X$,
W, Y smooth $X \xleftarrow{h} W \xrightarrow{f} Y$ morph / k

1. We say (h, f) is C-trans. if
$(h, f): W \to X \times Y$ is $C \times T^*Y \subset T^*(X \times Y)$-tran

2. If $h = 1_X$, we say f is C-tran if $C(1_X, f)$ is C-tra

Lemma 1.1.5.

1 $\boxed{f: X \to Y}$ C-tran $\iff$ $C \times_{T^*X} (X \times_Y T^*Y) \subset X \times_Y T_Y^*Y$

2. (h, f) C-trans $\Leftarrow$ h. C-trans &
f $h^\circ C$-trans.

Ex. 1 For $C = T_X^*X$, C-trans $\iff$ f. smooth.

2. C-trans $\Rightarrow$ f smooth on a nbd of $B = B(C)$
$= C \cap T_X^*X \subset$

1.2 local acyclicity = l.a.

$f : X \to Y$. x. y geometric pt

specialization $f(x) \leftarrow y$. $= y \to \mathcal{O}^{sh}_{f(x)}$ strict localiz.

$X_{(x)} \times_{Y_{f(x)}} y$ Milnor fiber

Def 1.2.1

1. f l.a rel to $\mathcal{F}$ constructible

$\forall\, f(x) \leftarrow y$

$$\mathcal{F}_x \to R\Gamma(X_{(x)} \times_{Y_{f(x)}} y, \mathcal{F})$$

is a quasi-isom.

2. U.l.a.

If Y curve vanishing cycles $= 0$.

1.2.2 (). étale local.

Example. 1. $f = 1_X : X \to X$.

1_X l.a rel to $\mathcal{F} \Leftrightarrow \mathcal{F}$. locally const

2. $f = 0 : X \to 0 \subset \mathbb{A}^1 = Y$.

f l.a rel to $\mathcal{F} \Leftrightarrow \mathcal{F} = 0$.

Theorem 1.2.3.

1. generic local acyclicity SGA $4\frac{1}{2}$. Th. finitude

$f: X \to S$. $\mathcal{F}$. const. on X. $\exists\, U \subset X$ dense open

s.t $f: X \times_S U \to U$ is U l.a. rel to $\mathcal{F}|_{X_U}$.

2. local acyclicity of smooth morphism SGA 4-3.

$f: X \to Y$. smooth $\mathcal{F}$ loc. const

$\Rightarrow$ f l.a rel to $\mathcal{F}$.

Prop 1.2.4 $f: X \to Y$ l.a. rel to $\mathcal{F}$ $g: Y \to Z$ sm

$\Rightarrow$ $g \circ f: X \to Z$ l.a rel to $\mathcal{F}$

1.3 microsupport (rel'n between trans & l.a.)

Def 1.3.1. $C \subset T^*X$. $\mathcal{F}$ on X

We say $\mathcal{F}$ is micro supported on C if

$\forall (h,f)$. C-trans $\Rightarrow$ $f : W \to Y$ is u.l.a rel to $h^*\mathcal{F}$.

$C \subset C'$ $\mathcal{F}$ m.s on C $\Rightarrow$ $\mathcal{F}$ m.s on C'

Prop 1.3.3.

1 $\forall \mathcal{F}$ micro supported on $T^*X \times_X \operatorname{supp}\mathcal{F}$

2. $\mathcal{F}$ loc. const'_t $\Leftrightarrow$ $\mathcal{F}$ m.s on $T^*_X X$.

3. $\mathcal{F}$ m.s on C $\Rightarrow$ $B(C) \supset \operatorname{supp}\mathcal{F}$.

1. (h,f) $T^*X \times_X \operatorname{supp}\mathcal{F}$-tran $\overset{\text{defn 1.1.4}}{\Leftrightarrow}$ $(h,f) : W \to X \times Y$ $(T^*X \times_X \operatorname{supp}\mathcal{F} \times T^*Y)$-tran

$\Leftrightarrow$ $(h,f) : W \to X \times Y$ smooth on a nbd of $\operatorname{supp}\mathcal{F} \times Y$

Ex 1.1.2.1

$$\begin{tikzcd}
\mathcal{F} & & h^*\mathcal{F} \\[-20pt]
X \arrow[d] & X \times Y \arrow[l] \arrow[d] & W \arrow[l] \arrow[dl, "\text{u.l.a}"'] \arrow[d, "\text{l.a}"] \\
\cdot & Y \arrow[l] & \mathbb{A}^1_Y \arrow[l, "\text{sm}"']
\end{tikzcd}$$

Prop. 1.2.4

$(T^*X \times_X Z) \times T^*Y$

$Z = \operatorname{supp}\mathcal{F}$

2. $\Rightarrow$

(h, f) $T_x^{\prime} X$ - trans $\Leftrightarrow$ ~~$\forall h$~~ no condition f smooth

$f: W \to Y$ smooth Th. 1.2.3.2.

h $\mathcal{F}$-l.c

$\Leftarrow$ $(1_X, 1_X)$ 1_X l.a rel to $\mathcal{F}$ $\Rightarrow$ $\mathcal{F}$ l.c

Example 1.2.2.1

3 $U = X - B$

$X \xleftarrow{j} U \xrightarrow{0} \mathbb{A}^1$ C-trans

$0: U \to \mathbb{A}^1$ l.a rel to $\mathcal{F}|_U$ $\Rightarrow$ $\mathcal{F}|_U = 0$.

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1.4 Singular support

Def 1.4.1. $C \subset T^*X$ $\mathcal{F}$ on X

C is the S.S of $\mathcal{F}$ means

$C \subset C' \iff \mathcal{F}$ is m.s on C'.

C smallest.

Example 1.4.2

1. $SS\mathcal{F} = \phi \iff \mathcal{F} = 0$

   (0 is m.s on ϕ.)

2. $SS\mathcal{F} = T^*_X X \iff \mathcal{F} = $ l. const & supp $\mathcal{F} = X$

$\Rightarrow$ $\mathcal{F}$ m.s on $T^*_X X \overset{1.3.2.2}{\Rightarrow} \mathcal{F}$ l.c.

$\mathcal{F}$ m.s on $T^*X \times_X \mathrm{supp}\,\mathcal{F} \overset{1.3.2.1}{\Rightarrow} T^*X \times_X \mathrm{supp}\,\mathcal{F} \supset T^*_X X$

$\Downarrow$

$\mathrm{supp}\,\mathcal{F} = X$

$\Leftarrow$ $\mathcal{F}$ m.s on $T^*_X X \overset{1.3.2.2}{\Leftarrow} \mathcal{F}$ l.c

$\mathcal{F}$ m.s on $C' \Rightarrow B(C') \supset \mathrm{supp}\,\mathcal{F} = X$

$\Rightarrow C' \supset T^*_X X$

Theorem 1.4.3 (Beilinson)

1. $SS\mathcal{F}$ exists.

2. $SS\mathcal{F} = \bigcup C_a$ irred. cpt $\forall a$ $\dim C_a = \dim X$.

~~Outline of~~ Main ingredient of Pf.

1. Reduction to $\mathbb{P}^n$.

Radon. $$\mathbb{P} \xleftarrow{p} Q \quad = \{(x,H) \in \mathbb{P}\times\mathbb{P}^\vee \mid x \in H\} = \mathbb{P}(T^*\mathbb{P})$$
$$\downarrow p^\vee$$
$$\mathbb{P}^\vee = \{H \subset \mathbb{P}\}$$

$$R\mathcal{F} = Rp^\vee_* p^* \mathcal{F} \qquad \text{(naive)}$$

$\mathcal{F}$ is m.c on C

$\Rightarrow$ $p^\vee$ is u.l.a rel to $p^*\mathcal{F}$ outside $E = \mathbb{P}(C) \subset \mathbb{P}(T^*\mathbb{P}) = Q$

$\Rightarrow$ $R(\mathcal{F}) = Rp^\vee_* p^*\mathcal{F}$ is m.c on $L C \cup T^*_{\mathbb{P}^\vee}\mathbb{P}^\vee$.

$R^\vee R(\mathcal{F})$ is almost $\mathcal{F}$

2. Lefschetz pencil + Zariski-Nagata purity.

2. CC.

2.1. push forward and pull-back.

$f : X \to Y$ proper $\quad n = \dim X$, $m = \dim Y$

$$\begin{array}{ccccc} T^*X & \xleftarrow{p} & X \times_Y T^*Y & \xrightarrow{q} & T^*Y \\ \cup & & \cup & & \cup \\ C & \longleftarrow & p^{-1}(C) & \longrightarrow & f_\circ(C) = q(p^{-1}(C)) \end{array}$$

closed conical subset

$A = \sum m_a C_a$ $\quad$ $f_! A = p_! \, p^! A$ intersection theory
$\in Z_n(C)$ $\quad$ $\in CH_m(f_\circ C)$
$\quad$ $\| \; Z_m(f_\circ C)$ if $\dim f_\circ C \leqq m$.
(Satisfied if $\dim = 0$, C Lagrangian.)

$h : W \to X$ C-transversal $\quad \dim X = n \quad \dim W = m$

$$\begin{array}{ccccc} T^*X & \xleftarrow{q} & W \times_X T^*X & \xrightarrow{p} & T^*W. \\ \cup & & \cup & & \cup \\ C & \longleftarrow & h^* C & \xrightarrow{\text{finite}} & h^\circ C. \end{array}$$

$A = \sum m_a C_a$ $\quad$ $h^! A = (-1)^{n-m} p_! \, q^! A$
$\in Z_n(C)$ $\quad$ $\in CH_m(h^\circ C)$ always $\geqq$.
$\quad$ $\| \; Z_m(h^\circ C)$ if $\dim h^\circ C = m$
properly transversal

h smooth $\Rightarrow$ properly transversal for $\forall C$

2.2. Characteristic cycle.

Theorem 2.2.1. k perfect. There exists a unique way to associate $CC\mathcal{F} = \sum m_a C_a$. ($SS\mathcal{F} = \bigcup C_a$, $m_a \in \mathbb{Z}$) to constructible $\mathcal{F}$ on X satisfying the following conditions

1. (normalization) $X = \operatorname{Spec} k$. $\mathcal{F} = \Lambda \Rightarrow CC\mathcal{F} = 1 \cdot [T^*_X X]$.
2. (additivity). For ~~dist. triangle~~ $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$,
$$CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''.$$
3. (closed imm) For $i : X \to P$ closed imm,
$$CC\, i_*\mathcal{F} = i_! CC\mathcal{F}$$
4. (pull-back) For properly $SS\mathcal{F}$-trans. $h : W \to X$,
$$CC\, h^*\mathcal{F} = h^! CC\mathcal{F}.$$
5. (Radon) For $\mathcal{F}$ on $\mathbb{P} = \mathbb{P}^n$
$$CC\, R\mathcal{F} = L\, CC\mathcal{F} \; (= p^\vee_! p^! CC\mathcal{F})$$

Cor 2.2.2 If X is proj. smooth
$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}.$$

$\dim X = 1$ Grothendieck-Ogg-Shafarevich
$$= \operatorname{rk}\mathcal{F}\, \chi(X_{\bar{k}}) - \sum_{x \in X - U} a_x \mathcal{F}$$

Thm 1 ⇒ Cor 2. Reduce to $X = \mathbb{P}^n$, $n \neq 1$.

$$CC(R^\vee R\mathcal{F}) - CC\mathcal{F} = \chi(X_{\bar{\varepsilon}}, \mathcal{F})(n-1) \cdot T^*_{\mathbb{P}}\mathbb{P}$$

$$L^\vee L C - C = (C, T^*_X X)_{T^*X}(n-1) \cdot T^*_{\mathbb{P}}\mathbb{P}$$

$C = CC\mathcal{F} \Rightarrow$ 2×5 $L^\vee L C = CC R^\vee R\mathcal{F}$ //

Conjecture 2.2.3 If f is proper,

$$CC\, Rf_*\mathcal{F} = f_! CC\mathcal{F} \quad \text{in } CH_n(f_\circ SS\mathcal{F})$$

Better axiom. replace 3&5 by

For $f: X \to Y$ proper & $\dim f_\circ SS\mathcal{F} \leq m$,

$$CC\, Rf_*\mathcal{F} = f_! CC\mathcal{F}.$$

Theorem 2.2.4 Conj 3 is true if f projective
$\dim Y = 1$ & $\dim f_\circ SS\mathcal{F} \leq 1$

Cor 2.2.5 Conj 3 is true if X, Y proj & $\dim f_\circ SS\mathcal{F} \leq m$

Cor 2.2.6 $f: X \to Y$ proj flat, X, Y smooth,
$\dim Y = 1$ $V \subset Y$ dense open set $X_V \to V$ smooth, $y \in Y - V$
$\dim X = n$

$$\Rightarrow \; -a_y(Rf_*\mathbb{Q}_\ell) = (-1)^n \deg c_n{}^{X}_{X_y}(\Omega^1_{X/Y})$$

loc. Chern class

Bloch's conductor formula

2.3 Construction of C.C

Thm 2.3.1 Another set of axioms $m_a \in \mathbb{Q}$

1. (curve) If $\dim X = 1$. $CC\mathcal{F} = -(\mathrm{rk}\,T_X^* X + \sum a_x \mathcal{F})\cdot[T_x^* X]$

2. (Milnor formula) Assume that $f : X \to \mathbb{P}^1$ is proj. & has at most isolated $SS\mathcal{F}$-characteristic pt., Then

$$CC\, Rf_*\mathcal{F} = f_* CC\mathcal{F} \cdot \quad \underline{\text{except}}\ 0\text{-section.}$$

3 (closed imm.) Same as 2.2.1.3

4 (pull-back) h. étale Same as 2.2.1.4.

Under 1, 2 means

$$-\dim \mathrm{tot}\, R\Phi_x \mathcal{F} = (CC\mathcal{F}, df)_{T^*X, x}.$$

uniqueness Milnor formula.

existence. m_a well-defined.

- continuity of Swan conductor. (Deligne-Laumon)
- nearby cycles over general base scheme

X affine (4), X $\mathbb{A}^n$ (3), X $\mathbb{P}^n$ (4)

Lefschetz pencil (4)

~~2.3~~ 2.2.1.

uniqueness. ~~...~~ $\mathcal{F}$ - loc. const.

$\dim X = 1$. $a_x\mathcal{F}$. G.O.S + ramified covering.

X affine (4), $X = \mathbb{A}^n$ (3) $X = \mathbb{P}^n$ (4)

Lefschetz + Radon (5) + (4) $\Rightarrow$ $X = \mathbb{P}^1$.

existence $2.3.1 \Rightarrow 2.2.1$.

- 1, 2, 3,
- 4 smooth Thom-Sebastiani formula Illusie

$$CC(\mathcal{F} \boxtimes \mathcal{G}) = CC\mathcal{F} \boxtimes CC\mathcal{G}.$$

- $m_a \in \mathbb{Z}$ find Lefschetz pencil. if $p \neq 2$, a

  $p = 2$ + excep $\Rightarrow$ use $p=2$ & non excep ~~...~~ pull-back to $X \times \mathbb{A}^1$

- 4 imm, 5 except 0-section. Radon-trans h
- Remaing 5 for 0-section.

  Prove by induction together with index formula:

  using c.c.

2.4 Characteristic class

$\mathcal{F} \longmapsto CC\mathcal{F} = \sum m_a C_a \quad C_a \subset T^*X$

projective completion $\qquad\qquad\qquad \cap \qquad \cap$

$\overline{CC\mathcal{F}} = \sum m_a \overline{C_a} \quad \overline{C_a} \subset \mathbb{P}(T^*X \oplus \mathbb{A}^1_X)$

Def 2.4.1. Characteristic class

$$C(\mathcal{F}) = [\overline{CC\mathcal{F}}] \in CH_n(\mathbb{P}(T^*X \oplus \mathbb{A}^1_X))$$

$$\cap \qquad \qquad \parallel$$

$$\bigoplus_{q=0} CH_q(X) = CH_\bullet(X)$$

$$cc_X : K(X, \Lambda) \longrightarrow CH_\bullet(X)$$

If char $= 0$, MacPherson's Chern class

For $f : X \to Y$

$$\begin{array}{ccc} K(X,\Lambda) & \xrightarrow{cc_X} & CH_\cdot(X) \\ {\scriptstyle f_*}\downarrow & \circlearrowright & \downarrow {\scriptstyle f_*} \\ K(Y,\Lambda) & \xrightarrow[cc_Y]{} & CH_\cdot(Y) \end{array}$$

However char $p > 0$ counter example $f =$ Frobenius

Conj 2.2.3 implies

Conj 2.4.2

$$\begin{array}{ccc} K(X,\Lambda) & \xrightarrow{cc_X} & CH_0(X) \\ \downarrow & & \downarrow \\ K(Y,\Lambda) & \longrightarrow & CH_0(Y) \end{array}$$

Conj 2.4.2 is proved if k finite. X & Y proj

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2.5
~~Proo~~ End of Pf

Remaining

(A) $\begin{array}{ccc} K(\mathbb{P}, A) & \xrightarrow{cc} & CH_\bullet(\mathbb{P}) \\ R\downarrow_{v} & & \downarrow L \\ K(\mathbb{P}^\vee, A) & \xrightarrow{cc} & CH_\bullet(\mathbb{P}^\vee) \end{array}$ (B) $\begin{array}{ccc} K(\mathbb{P}, A) & \xrightarrow{\chi} & \mathbb{Z} \\ \downarrow & \nearrow_{(\ ,\ T_X^* X)_{\mathbb{P}(T^*)}} & \\ CH_\bullet(\mathbb{P}) & & \end{array}$

Induction

$$B(n-1)$$
$$A(n) \Rightarrow B(n) \quad \text{except } n=1.$$

$\Leftarrow$ generic $\mathrm{rk} = \chi(H, \mathcal{F}|_H)$

$\Rightarrow$ $[\check{R}R \mathcal{F}] - [\mathcal{F}] = (n-1)\,\chi(\mathbb{P}_{\hat{e}}, \mathcal{F})$

$A(0)$ $0=0$ $B(0)$ normalization

$A(1)$ $1=1$. $B(1)$ GOS for $X = \mathbb{P}^1$