THE CHARACTERISTIC CYCLE AND THE SINGULAR SUPPORT OF AN ETALE SHEAF

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Abstract. These are the notes for a series of lectures given by T. Saito at Freie Universität Berlin in July 2015. The typist takes full responsibility for all mistakes and inaccuracies.

Singular support due to A. Beilinson, *Characteristic cycle* due to T. Saito.

1. Lecture 1

1.1. Introduction.

- k a field of characteristic $p > 0$. Mostly perfect or even algebraically closed.
- X a smooth k-scheme, $n = \dim X$. Let Λ be a finite extension of $\mathbb{F}_\ell$, $\ell \neq p$.
- $\mathcal{F}$ a constructible complex of Λ-modules.
- We can take cohomology sheaves $\mathcal{H}^q(\mathcal{F})$; they are constructible and $= 0$ except for finitely many q.
- T^*X the cotangent bundle of X associated to Ω^1_X, which is a vector bundle of rank n. Thus T^*X has dimension $2n$.
- $C \subseteq T^*X$ a closed conical subset, where conical means: stable under the action of $\mathbb{G}_m$, which naturally acts by multiplication on the vector bundle T^*X.
- $T^*X = \operatorname{Spec} S^\bullet(\Omega^1_X)^\vee$ and C is defined by some ideal of $S^\bullet(\Omega^1_X)^\vee$. From this perspective *conical* means that C is defined by a *graded* ideal.
- The *Singular support of* $\mathcal{F}$ is denoted $SS(\mathcal{F}) = C \subseteq T^*X$. It is a closed conical subset of T^*X. Moreover, we can write it as union of irreducible components

$$C = \bigcup C_a$$

where C_a is an irreducible component of $\dim C_a = \dim X$[1].
- Today we explain $SS(\mathcal{F})$. Later the characteristic cycle $\operatorname{Char}(\mathcal{F})$.
- $\operatorname{Char}(\mathcal{F}) = \sum_a m_a[C_a]$ with $m_a \in \mathbb{Z}[1/p]$, but it is expected that $m_a \in \mathbb{Z}$.
- The expectation is that the properties of $\mathcal{F}$ are well understood by using $SS(\mathcal{F})$ and $\operatorname{Char}(\mathcal{F})$. Slogan: To understand $\mathcal{F}$ on X, we study $SS(\mathcal{F})$ and $\operatorname{Char}(\mathcal{F})$ on T^*X. This is analogous to microlocal analysis of $\mathscr{D}_X$-modules on complex manifolds X, due to Sato, Kashiwara, etc.

[1] $SS(\mathcal{F})$ is an invariant of the complex $\mathcal{F}$, and $SS(\mathcal{F}) \subseteq \bigcup_q SS(\mathcal{H}^q(\mathcal{F}))$.

Example 1.1. X a curve, i.e., $n = 1$. Let D be a divisor on X and $j : U := X \setminus D \hookrightarrow X$ the associated open immersion. Let $\mathcal{F} := j_! \mathcal{G}$, where $\mathcal{G} \neq 0$ is a locally constant sheaf on U. In this case the irreducible components are:

$$T^*X \supseteq SS(\mathcal{F}) = \underbrace{T^*_X X}_{0-\text{section}} \cup \bigcup_{x \in D} \underbrace{T^*_x X}_{\text{fiber}}$$

In fact any conical closed subset of T^*X has this shape.

In this example,

$$\operatorname{Char}(\mathcal{F}) = (-1) \left(\operatorname{rank} \mathcal{G} \cdot [T^*_X X] + \sum_{x \in D} \operatorname{dimtot}_x \mathcal{F} \cdot [T^*_x X] \right)$$

where $\operatorname{dimtot}_x = \dim + \operatorname{Sw}_x$, with $\operatorname{Sw}_x \in \mathbb{Z}$ the Swan conductor at x, which is a measure of wild ramification.

On the other hand, if $\mathcal{F} = j_* \mathcal{G}$, then replace dimtot_x by Artin conductor of $\mathcal{G}$. If $\mathcal{F} = Rj_* \mathcal{G}$, then $\operatorname{Char}(j_! \mathcal{G}) = \operatorname{Char}(Rj_* \mathcal{G})$.

- If X projective and k algebraically closed, then
$$\chi(X, \mathcal{F}) = (\underbrace{\operatorname{Char}(\mathcal{F})}_{\dim n} . \underbrace{T^*_X X}_{\dim n})_{\underbrace{T^*X}_{\dim 2n}}$$
intersection number. Have this formula in general, but in the 1-dimensional example from above, this is a reformulation of Grothendieck-Ogg-Shafarevich's formula.
- Why is there a sign (-1)? If $\mathcal{F}$ is a perverse sheaf (complex), then the coefficients of $\operatorname{Char}(\mathcal{F})$ are ≥ 0[2]. In the example above, $\mathcal{F}[1]$ is perverse. In general, $\operatorname{Char}(\mathcal{F}[n]) = (-1)^n \operatorname{Char}(\mathcal{F})$.

1.2. **Singular Support (after Beilinson).** Want to formulate relations between $C \subseteq T^*X$ and $\mathcal{F}$ on X, where C is a conical subset and $\mathcal{F}$ a constructible complex on the smooth scheme X of dimension n.

1.2.1. *C-transversality.* Want two definitions of C-transversality: One for morphisms $h : W \to X$ into X and one for morphisms $f : X \to Y$ from X. Here W, Y are both smooth k-schemes, of arbitrary dimension.

Definition 1.2. We say $f : X \to Y$ is *C-transversal* if $df^{-1}(C) \subseteq \underbrace{X \times_Y T^*_Y Y}_{\text{0-section}}$.

$$\begin{array}{ccc} X \times_Y T^*Y & \xrightarrow{df} & T^*X \\ \uparrow & & \uparrow \\ df^{-1}(C) & \longrightarrow & C \end{array}$$

Example 1.3. (a) $C = T^*_X X$ the zero-section. Then C-transversal means that df is injective, i.e., that f is smooth.

(b) $Y = \operatorname{Spec} k$ is a point. Then $f : Y \to \operatorname{Spec} k$ is C-transversal for any C.

[2] This relies on a deep theorem of Gabber: If $X \to S$, S a trait $\mathcal{F}$ perverse sheaf on X then $\Phi\mathcal{F}[-1]$ is perverse.

Definition 1.4. We say $h : W \to X$ is *C-transversal* if
$$h^*C \cap dh^{-1}(T^*_W W) \subseteq \underbrace{W \times_X T^*_X X}_{\text{0-section}}$$
where we use the diagram
$$\begin{array}{ccc} h^*C = W \times_X C & & T^*_W W \\ \big\downarrow & & \big\downarrow \\ W \times_X T^*X & \xrightarrow{\ dh\ } & T^*W \end{array}$$

Moreover, define $h^0C = dh(h^*C) \subseteq T^*W$. Then C-transversality implies that h^0C is closed, and
$$\begin{array}{ccc} h^*C & \xrightarrow{\text{finite}} & h^0C \\ \big\downarrow & & \big\downarrow \\ W \times_X T^*X & \xrightarrow{\ dh\ } & T^*W \end{array}$$

The terminology used to be *non-characteristic* (Kashiwara-Schapira).

Example 1.5. (a) If h is smooth, then h is C-transversal for any C, because then dh is injective and $h^*C = h^0C$.

Remark 1.6. Being C-transversal is an open condition on the source of the morphism $f : X \to Y$.

Need one more definition.

Definition 1.7. Given $f : W \to Y$ and $h : W \to X$, we say that the pair (h, f) is *C-transversal* if h is C-transversal and $f : W \to Y$ is h^0C transversal.

Exercise 1.8. (a) Given $f : W \to Y$ and $h : W \to X$, then (h, f) is C-transversal if and only if

$(h^*C \times_W (W \times_Y T^*Y)) \cap (\text{inv. image of } T^*_W W) \subseteq \text{0-section}$

where we use the diagram
$$\begin{array}{ccc} h^*C \times_W (W \times_Y T^*Y) & & T^*_W W \\ \big\downarrow & & \big\downarrow \\ (W \times_X T^*X) \times_W (W \times_Y T^*Y) & \longrightarrow & T^*W \end{array}$$

(b) If $f : W \to Y$ is smooth,
$$\begin{array}{ccc} h^*C & & W \times_Y T^*Y \\ \big\downarrow & & \big\downarrow \text{injective because of smoothness} \\ W \times_X T^*X & \longrightarrow & T^*W \end{array}$$

Then (f, h) is C-transversal iff
$$h^*C \cap (\text{inv. image of } W \times_Y T^*Y) \subseteq \text{0-section}$$

1.3. **Local acyclicity.** Given $f : X \to Y$ we have the notion of the *Milnor fiber*. Let x be a geometric point of X and $y := f(x)$, a geometric point of Y. Let Y_y be the strict localization of Y at y, so $Y_y = \operatorname{Spec} \mathcal{O}^{sh}_{Y,y}$. Let z be a geometric point of Y_y. Notation: $x \mapsto y \leftarrow z$. The *Milnor fiber* is $X_x \times_{Y_y} z$. This is not interesting if z maps to y, but, e.g., if z maps to the generic point of Y_y.

Definition 1.9. Let $\mathcal{F}$ on X be as in the beginning. We say $f : X \to Y$ is *locally acyclic relatively to* $\mathcal{F}$ if for all situations $x \mapsto y \leftarrow z$ as above the canonical restriction morphism

$$\mathcal{F}_x = R\Gamma(X_x, \mathcal{F}) \to R\Gamma(X_x \times_{Y_y} z, \mathcal{F})$$

is an isomorphism.

Example 1.10. Let Y be a curve, and y a geometric point over a closed point. Then Y_y only has two points; let z be a geometric point above the generic point of Y_y. In this situation we have a distinguised triangle

$$\to \mathcal{F}_x \to R\Gamma(X_x \times_{Y_y} z, \mathcal{F}) \to \text{ vanishing cycles.}$$

So local acyclicity in this situation means that there are no nonzero vanishing cycles.

Definition 1.11. We say that $f : X \to Y$ is *universally locally acyclic relatively to* $\mathcal{F}$ if for every $g : Y' \to Y$, $X \times_Y Y' \to Y'$ is locally acyclic relatively to the pullback of $\mathcal{F}$.

Enough to just take every smooth $g : Y' \to Y$.

2. Lecture 2

Here are some facts about local acyclicity.

Facts.

(a) (*local acyclicity of smooth morphisms*, SGA 4) If $f : X \to Y$ is smooth and $\mathcal{F}$ on X locally constant (i.e. $\mathcal{H}^q(\mathcal{F})$ is locally constant for all q), then f is locally acyclic relatively to $\mathcal{F}$.

(b) (*generic local acyclicity*, SGA 4½) Let $\mathcal{F}$ be arbitrary and $f : X \to Y$. There exists a dense open subset $V \subseteq Y$, such that $f_V : X \times_Y V \to V$ is universally locally acyclic relatively to $\mathcal{F}|_{X \times_Y V}$.

(c) $f : X \to Y$, $g : Y \to Z$ and $\mathcal{F}$ on X. Suppose that f is (universally) locally acyclic relative to $\mathcal{F}$ and g smooth. Then the composition gf is (universally) locally acyclic with respect to $\mathcal{F}$. (This is a consequence of (a)).

(d) $f, g, \mathcal{F}$ as in (c). Suppose that gf is (universally) locally acyclic relative to $\mathcal{F}$ and that f is proper. Then g is (universally) locally acyclic with respect to $Rf_*\mathcal{F}$. (This follows from the proper base change theorem).

(e) $\mathcal{F}$ is locally constant if and only if $\mathrm{id}_X : X \to X$ is locally acyclic relatively to $\mathcal{F}$. In fact, id_X is locally acyclic with respect to $\mathcal{F}$ iff for every specialization $x \leftarrow y$, $\mathcal{F}_x \xrightarrow{\cong} \mathcal{F}_y$ iff (exercise!) $\mathcal{F}$ is locally constant.

2.1. **Micro support.** We combine the notions introduced above. Let $C \subseteq T^*X$ be a closed conical subset and $\mathcal{F}$ a constructible complex of Λ-modules on X.

Definition 2.1. (a) We say $\mathcal{F}$ *is micro supported on* C if for every C-transversal pair

$$X \xleftarrow{h} W \xrightarrow{f} Y,$$

the map $f : W \to Y$ is universally locally acyclic relative to $h^*\mathcal{F}$.

(b) We say that $\mathcal{F}$ *is weakly micro supported on* C if the above holds true for pairs

$$X \xleftarrow{h} W \xrightarrow{f} Y$$

where h is an open immersion and Y is a curve ($= \mathbb{A}^1_k$).

Example 2.2. $\mathcal{F}$ is locally constant $\Leftrightarrow$ $\mathcal{F}$ is micro supported on the 0-section $C = T^*_X X$.
$\Rightarrow$ is Fact (a).
$\Leftarrow$ is Fact (e): id_X is transversal to $C = T^*_X X$.

What's the difference between micro supported and *weakly* micro supported?

Lemma 2.3. *Suppose $\mathcal{F}$ is weakly micro supported on C and C'. Then $\mathcal{F}$ is weakly micro supported on $C \cap C'$.*

Remark 2.4. (a) Suppose $\mathcal{F}$ is (weakly) micro supported on C and let C' be conical closed, such that $C \subseteq C'$. Then $\mathcal{F}$ is (weakly) micro supported on C'. The question is: How small can we make C?

(b) The statement of the lemma also holds true for *micro supported* instead of *weakly micro supported*, but to see this we first have to prove the main theorem.

(c) If C is a minimal (with respect to $\subseteq$) among the conical closed subsets of T^*X on which $\mathcal{F}$ is micro supported, then we say that $\mathcal{F}$ is *tightly supported* on C (a priori there could be many minimal C).

(d) On the other hand, for the notion of *weakly micro suppoted*, the lemma shows that there is a unique minimal C on which $\mathcal{F}$ is weakly micro supported.

Definition 2.5. The smallest conical closed $C \subseteq T^*X$ on which $\mathcal{F}$ is weakly micro supported is called the *singular support of* $\mathcal{F}$ and denoted $SS(\mathcal{F})$.

Theorem 2.6 (Beilinson)**.** *Every irreducible component of $SS(\mathcal{F})$ is of dimension* $\dim X$ *and $\mathcal{F}$ is micro supported on $SS(\mathcal{F})$.*

This follows from two intermediate theorems.

Theorem A (Beilinson, Thm. 1.2)**.** *There exists $C \subseteq T^*X$ such that $\mathcal{F}$ is micro supported on C and* $\dim C \leq n = \dim X$.

Theorem B (Beilinson, Thm. 1.3)**.** *Assume that k is perfect and that $\mathcal{F}$ is tightly micro supported on C. Then every irreducible component of C is of dimension $n = \dim X$ and $C = SS(\mathcal{F})$.*

- Theorem 2.6 follows from Theorem A and Theorem B.

- To prove Theorems A and B we reduce to $X = \mathbb{P}^n$. To do this one roughly proceeds like this: For Theorem A, take $X \to \mathbb{P}^n$ étale. For Theorem B, take $X \xrightarrow{i} U \xrightarrow{j} \mathbb{P}^n$, i =closed, j =open.
- From now on we assume $X = \mathbb{P}^n$. Here an important tool will be the *Radon Transform*.

2.2. **Radon Transform.** Standard reference is Brylinski (Asterisque), and SGA7, Exp. XVII. Let V be an $(n+1)$-dimensional k-vector space and denote by

$$\mathbb{P} := \mathbb{P}(V) = \{\text{lines in } V\}$$

the associated projective space. The dual projective space is

$$\mathbb{P}^\vee = \mathbb{P}(V^\vee) = \{\text{hyperplanes in } V\} = \{\text{hyperplanes in } \mathbb{P}\}.$$

Let $Q \subseteq \mathbb{P} \times \mathbb{P}^\vee$ be the universal family of hyperplanes, i.e.,

$$Q = \{(x, x^\vee) \in \mathbb{P} \times \mathbb{P}^\vee | x \in x^\vee\}.$$

We have two projections:

$$\begin{array}{ccc} Q & \xrightarrow{p^\vee} & \mathbb{P}^\vee \\ \downarrow{\scriptstyle p} & & \\ \mathbb{P} & & \end{array}$$

Definition 2.7.
- For $\mathcal{F}$ on $\mathbb{P}$, the *Radon transform of $\mathcal{F}$* is

$$R(\mathcal{F}) := Rp^\vee_* p^* \mathcal{F}[n-1].$$

- Given $\mathcal{G}$ on $\mathbb{P}^\vee$, we get the inverse (dual) Radon transform

$$R^\vee(\mathcal{G}) := Rp_* p^{\vee,*} \mathcal{G}[n-1]$$

  These two constructions are *almost* inverse to each other (i.e., up to a geometrically constant object, but we will not make this precise).

We recall the *Legendre transform* on C. We have the identification

$$Q = \mathbb{P}(T^*\mathbb{P}) = (T^*\mathbb{P} - T^*_{\mathbb{P}}\mathbb{P})/\mathbb{G}_m.$$

This identification works as follows: From

$$0 \longrightarrow \Omega^1_{\mathbb{P}} \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

we obtain

$$\mathbb{P}(T^*\mathbb{P}) \subseteq \mathbb{P}(V^\vee \otimes \mathcal{O}_{\mathbb{P}}(-1)) = \mathbb{P}(V^\vee) \times \mathbb{P} = \mathbb{P}^\vee \times \mathbb{P}.$$

Similarly, we also have an identification $Q = \mathbb{P}(T^*\mathbb{P}^\vee)$. Let $C \subseteq \mathbb{P}(T^*\mathbb{P})$ be a conical closed subset and consider its projectivization $\mathbb{P}(C)$. We get the diagram

$$\begin{array}{ccc} \mathbb{P}(C) & \hookrightarrow & \mathbb{P}(T^*\mathbb{P}) \\ \| & & \| \\ & & Q \\ \| & & \| \\ \mathbb{P}(C^\vee) & \hookrightarrow & \mathbb{P}(T^*\mathbb{P}^\vee) \end{array}$$

and $C^\vee \subseteq T^*\mathbb{P}^\vee$ is closed and conical and called the *Legendre transform of* C.

2.3. Reformulation of Theorems A and B.

Definition 2.8. Let $f : X \to Y$ be a morphism and $\mathcal{F}$ a constructible complex on X. Define $E_f(\mathcal{F}) \subseteq X$ to be the closed subset such that its complement U is the largest open subscheme where $f_U : U \to Y$ is universally locally acyclic relative to $\mathcal{F}|_U$ ($U = \emptyset$ possible).

Theorem A' (Thm. 1.4, equivalent to Theorem A)**.** *For $\mathcal{G}$ on $\mathbb{P}$, $E_{p^\vee}(p^*\mathcal{G})$ is of dimension $\leq n-1$.*

Theorem A' is equivalent to Theorem A. For $\Leftarrow$, one uses that if $\mathcal{G}$ is micro supported on C with $\dim C \leq n$, then $E_{p^\vee}(p^*\mathcal{G}) \subseteq \mathbb{P}(C)$, with $\dim \mathbb{P}(C) \leq n-1$.

Let $d \geq 1$ and let

$$i_d : \mathbb{P} \to \widetilde{\mathbb{P}} = \mathbb{P}(\Gamma(\mathbb{P}, \mathcal{O}(d))^\vee)$$

be the d-th Veronese embedding. We get a diagram

$$\begin{array}{ccccc} & & & \widetilde{Q} & & \\ & & \tilde{p}^\vee \swarrow & & \searrow \tilde{p} & \\ \mathbb{P} & \xhookrightarrow{i_d} & \widetilde{\mathbb{P}} & & & \widetilde{\mathbb{P}}^\vee \end{array}$$

Theorem B' (Thm. 1.6, implies Theorem B)**.** *Fix $\mathcal{G}$ on $\mathbb{P}$. Assume $d \geq 3$ and let $D \subseteq \widetilde{\mathbb{P}}^\vee$ be the complement of the largest open subset $U \subseteq \widetilde{\mathbb{P}}$ where $\widetilde{R}(i_{d,*}\mathcal{G})$ is locally constant. (Here $\widetilde{R}$ is the Radon transform on $\widetilde{\mathbb{P}}$.)*

(a) *D is a divisor, i.e., purely of codimension 1.*

(b) *For each irreducible component D_a of D, there is a unique irreducible closed conical subset $C_a \subseteq T^*\mathbb{P}$ such that $D_a = \tilde{p}^\vee(\mathbb{P}(i_0 C_a))$ and $\dim C_a = \dim X$. For the definition of $i_0 C_a$, see below. The surjection*

$$\tilde{p}^\vee : \mathbb{P}(i_0 C_a) \to D_a$$

is generically radicial, i.e., the associated extension of function fields is purely inseparable.

(c) *$C = \bigcup C_a \subseteq T^*\mathbb{P}$ is $SS(\mathcal{G})$.*

How to define i_0C? If $i : X \hookrightarrow Y$ is a closed immersion with X, Y smooth, and $C \subseteq T^*X$, then $i_0C \subseteq T^*Y$ is defined using the following diagram:

$$\begin{array}{ccccc} C & & & & \\ \downarrow & & & & \\ T^*X & \longleftarrow & X \times_Y T^*Y & \longrightarrow & T^*Y \end{array}$$

Then i_0C is defined to be the image in T^*Y of the pullback of C along $X \times_Y T^*Y \to T^*X$. Thus in the situation of Theorem B', (b), $\mathbb{P}(i_0C_a) \subseteq \mathbb{P}(T^*\widetilde{\mathbb{P}}) = \widetilde{Q}$.

Remark 2.9. The fact that $\tilde{p}^\vee : \mathbb{P}(i_0C_a) \to D_a$ is generically purely inseparable gives rise to the problem that the coefficients of $\mathrm{Char}(\mathcal{F})$ can (at the moment) only be shown to lie in $\mathbb{Z}[1/p]$ (although they are expected to be integers).

3. Lecture 3

3.1. The Characteristic Cycle.

- k is a field of characteristic $p > 0$, perfect or even algebraically closed.
- X/k is smooth, $n = \dim X$.
- $\mathcal{F}$ a constructible complex on X.
- Last time we defined $C = SS(\mathcal{F}) \subseteq T^*X$, a closed conical subset, $C = \bigcup_a C_a$, the C_a the irreducible components, $\dim C_a = n$.
- Recall that $\mathcal{F}$ is *micro supported* on C if for every pair of maps $X \xleftarrow{h} W \xrightarrow{f} Y$, where h is C-transversal and f is h^0C-transversal, $f : W \to Y$ is universally locally acyclic relative to $h^*\mathcal{F}$.
- The characteritic cycle will have the form $\mathrm{Char}(\mathcal{F}) = \sum_a m_a[C_a]$, $m_a \in \mathbb{Z}[1/p]$.

3.1.1. *Definition of characteristic cycle — Milnor formula.* We slightly generalize the notion of *weakly micro supported*: Instead of putting a condition on all pairs $X \xleftarrow{j} U \xrightarrow{f} Y$ with Y a curve and j open, we just require j to be étale and Y to be a curve.

Definition 3.1. For a fixed closed conical subset $C \subseteq T^*X$, we say that a closed point $u \in U$ is an *isolated characteristic point with respect to C*, if $X \leftarrow U \setminus \{u\} \to Y$ is C-transversal.

Example 3.2. Let $X \xleftarrow{j} U \xrightarrow{f} Y$ be such that Y is a curve and j is étale. Let $C = T^*_X X$. Then u is an isolated characteristic point if and only if u is an isolated singular point of $f : U \to Y$.

Now assume that $C = SS(\mathcal{F})$. Let u be an isolated characteristic point. We define two invariants. On the "$\mathcal{F}$-side": $f : U \to Y$ is universally locally acyclic relative to $j^*\mathcal{F}$ outside u. If $k = \bar{k}$ and $v = f(u) \in Y$ (closed point), write $Y_v = \mathrm{Spec}(\mathcal{O}^{sh}_{Y,v})$, which is the spectrum of a strictly henselian discrete valuation ring. Let $\bar{\eta}$ denote a generic geometric point of Y_v.

Recall the definition of universally locally acyclic relative to $j^*\mathcal{F}$: There is a distinguished triangle

$$\mathcal{F}_u \to R\Gamma(X_u \times_{Y_v} \bar{\eta}) \to \Phi_u(j^*\mathcal{F}, f) \to$$

and locally acyclic means that the first arrow is an isomorphism. $\Phi_u(j^*\mathcal{F}, f)$ is the stalk of the complex (of Λ-modules) of vanishing cycles. We may assume without loss of generality that Λ is a finite field extension of $\mathbb{F}_\ell$[3]. Its q-th cohomology

$$\Phi_u^q(j^*\mathcal{F}, u)$$

is a Λ-vector space of finite dimension and which is zero except for finitely many q. It carries a natural continuous action of $\mathrm{Gal}(\overline{K}_v/K_v)$, where $K_v = \mathrm{Frac}(\mathcal{O}_{Y,v})$.

Define

$$\operatorname{dimtot}\Phi := \sum_q (-1)^q \operatorname{dimtot}\Phi^q = \sum_q (-1)^q(\dim(\Phi^q) + \mathrm{Sw}(\Phi^q))$$

which is an integer by the theorem of Hasse-Arf.

On the "C-side": $j^*C \subseteq T^*U = U \times_X T^*X$. After shrinking Y we obtain from $f : U \to Y$ a map

$$f : U \to Y \xrightarrow{\text{étale}} \mathbb{A}^1_k = \operatorname{Spec} k[t].$$

This defines $df := f^*dt$, which is a section of the projection $T^*U = U \times_X T^*X \to U$. The assumption that u is an isolated characteristic point means that the intersection of j^*C and $df(U)$ consists of at most one isolated closed point (which is essentially independent of the choice of t, as C is a conic subset). We can take the intersection, because $\dim j_*C = n$, $\dim df(U) = n$ and $\dim T^*U = 2n$. It follows that the intersection number

$$(j^* \sum_a m_a[C_a], df)_{T^*U,u}$$

is defined.

Theorem 3.3 (Milnor Formula). *There exists a unique $\mathbb{Z}[1/p]$-linear combination*

$$\mathrm{Char}(\mathcal{F}) = \sum_a m_a C_a$$

of irreducible components C_a of $C = SS(\mathcal{F}) = \bigcup_a C_a$ such that for every pair $X \xleftarrow{j} U \xrightarrow{f} Y$ as above, with isolated characteristic point $u \in U$, we have

$$-\operatorname{dimtot}\Phi_u(j^*\mathcal{F}, f) = (j^*\,\mathrm{Char}(\mathcal{F}), df)_{T^*U,u} \qquad (\star)$$

Example 3.4. (a) $\mathcal{F} = \Lambda$. Then the right hand side of ($\star$) is length $\Omega^n_{U/Y,u}$ (Deligne, SGA7 Exp. XVI), and $\mathrm{Char}(\Lambda) = (-1)^n T^*_X X$.

(b) $X = \mathbb{A}^2$,

$$j : V = \mathbb{A}^2 - D \hookrightarrow \mathbb{A}^2 = \operatorname{Spec} k[x,y],$$

where D the x-axis of $\mathbb{A}^2$. Consider the Artin-Schreier equation $t^p - t = \frac{y}{x^d}$, $p \neq 2$, $p|d$. It defines a cyclic covering $W \to V$ of degree

[3] To compute the Swan conductor or coefficients of the characteristic cycle we can work on the residue field of Λ.

p. Fix a character $\mathrm{Gal}(W/V) \hookrightarrow \Lambda^\times$, which corresponds to a locally constant sheaf $\mathcal{G}$ of rank 1 on V. Define $\mathcal{F} = j_!\mathcal{G}$. Then

$$SS(\mathcal{F}) = T^*_X X \cup \langle dy/D \rangle \subseteq T^*X.$$

and

$$\mathrm{Char}(F) = [T^*_X X] + d[\langle dy/D \rangle].$$

Idea of the proof of Theorem 3.3: Follow Deligne! We use a local version of the Radon transform, using vanishing cycles over a general base scheme (Deligne, Laumon, Illusie, Orgogozo). We need to define the multiplicities m_a. In the notations from last lecture, we defined divisors $D_a \subseteq \widetilde{\mathbb{P}}^\vee$, and cut with a general pencil L. This directly gives the coefficients m_a locally[4]. Main point: Show that they are independent of all choices. To this end we use 'stability of vanishing cycles'. Given

$$\begin{array}{ccc} & U & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y & & Y' \end{array}$$

If g, f are 'sufficiently close', continuity of the Swan conductor (Deligne, Laumon) implies that in this situation $\mathrm{dimtot}(-, f) = \mathrm{dimtot}(-, g)$,

3.1.2. *Functoriality of* $\mathrm{Char}(\mathcal{F})$ — *Index formula.* We would like to have functoriality for maps $h : W \to X$, and $f : X \to Y$.

Definition 3.5. $h : W \to X$ is *strongly C-transversal* if it is C-transversal and if $h^*C := W \times_X C \subseteq W \times_X T^*X$ is equidimensional of dimension $\dim W$, i.e., every irreducible component of h^*C has dimension $\dim W$.

Write $C = \bigcup_a C_a$, and assume that h is strongly C-transversal. Then we can define

$$h^!\left(\sum_a m_a[C_a]\right) := (-1)^{\dim W - \dim X}\left(\sum_a m_a h^0([C_a])\right),$$

because we have the diagram

$$\begin{array}{ccccc} \underbrace{C_a}_{\dim X} & \longleftarrow & h^*C_a & \xrightarrow{\text{finite}} & \underbrace{h^!C_a}_{\dim W} \\ \downarrow & & \downarrow & & \downarrow \\ T^*X & \longleftarrow & W \times_X T^*X & \longrightarrow & T^*W. \end{array}$$

Theorem 3.6. *If $h : W \to X$ is strongly C-transversal for $C = SS(\mathcal{F})$, then*

$$\mathrm{Char}(h^*\mathcal{F}) = h^!(\mathrm{Char}(\mathcal{F})).$$

[4]The denominators come from the fact that $p(i_0 C_a) \to D_a$ is purely inseparable, but it is expected that the denominators always cancel

Idea of the proof: We can assume that $W \subseteq X$ is a divisor of X, $\dim X = 2$. Then use a global argument originally due to Deligne (and resolution of singularities in dimension 2) and some ramification theory.

Lemma 3.7. *If $f : X \to Y$ is proper and C-transversal for $C = SS(\mathcal{F})$, then $Rf_*\mathcal{F}$ is locally constant, i.e., every $R^q f_*\mathcal{F}$ is locally constant.*

Theorem 3.8 (Index formula)**.** *Assume that X is projective and $k = \bar{k}$. Then*

$$\chi(X, \mathcal{F}) = (\mathrm{Char}(\mathcal{F}), T^*_X X)_{T^*X}$$

Idea of proof: Induction on $\dim X$. Let $X \leftarrow X' \xrightarrow{p} L$ be a pencil. We compute

$$\chi(X, \mathcal{F}) = \chi(X', \mathcal{F}') - \chi(Z, \mathcal{F}|_Z)$$

where Z is the center of the blow-up $X' \to X$. Use induction hypothesis and Theorem 3.6 to compute $\chi(Z, \mathcal{F}|_Z)$.

Using Grothendieck-Ogg-Shafarevich formula, we compute

$$\chi(X', \mathcal{F}') = \underbrace{\chi(L)}_{2} \mathrm{rank}(Rp_*\mathcal{F}') - \sum \underbrace{\mathrm{dimtot}_x \Phi}_{\text{Milnor formula Theorem 3.3}}$$

where $\mathrm{rank}(Rp_*\mathcal{F}') = \chi(Y, \mathcal{F}|_Y)$ where Y is a generic hyperplane section. Then use induction hypothesis plus Theorem 3.6.

4. Lecture 4

4.1. Equivalent characterization of singular support.

In this section, we define the notion of $\mathcal{F}$-transversality. The following table shows how it fits into the story:

	quantitive/Char($\mathcal{F}$)	qualitative/$SS(\mathcal{F})$	
$\mathcal{F}$-side	Euler number, Milnor formula	locally acyclic	$\mathcal{F}$-transversal
C-side	Intersection number	$f : X \to Y$	$h : W \to X$
		C-transversal	

Definition 4.1. $h : W \to X$ a morphism of smooth schemes over k, $\mathcal{F}$ a constructible complex on X. We say that h *is $\mathcal{F}$-transversal* if the canonical morphism

$$h^*\mathcal{F} \otimes^L \underbrace{Rh^!\Lambda}_{\Lambda(a)[2a]} \to Rh^!\mathcal{F}$$

is an isomorphism. Here a is an integer depending on h.

Example 4.2.
(a) If h is smooth, then h is $\mathcal{F}$-transversal for any $\mathcal{F}$ (Poincaré duality).
(b) If $\mathcal{F}$ is locally constant, then any h is $\mathcal{F}$-transversal.

Theorem 4.3. *Let $\mathcal{F}$ be constructible on X, $C \subseteq T^*X$ conical and closed. Then the following conditions are equivalent.*

(a) *$\mathcal{F}$ is micro supported on C.*
(b) *Every C-transversal $h : W \to X$ is $\mathcal{F}$-transversal.*

Points of the proof: (a)$\Rightarrow$(b): Easier. Uses smooth base change theorem. (b)$\Rightarrow$(a): Harder. Uses local acyclicity of smooth morphism.

4.2. **Ramification.** We can always find a dense open $U \subseteq X$ such that $\mathcal{F}|_U$ is locally constant. Then

$$SS(\mathcal{F})|_U \subseteq T^*_U U$$

(equality if $\mathcal{F}|_U \neq 0$) and

$$\mathrm{Char}(\mathcal{F})|_U = (-1)^n \,\mathrm{rank}(\mathcal{F})[T^*_U U].$$

Let $D := X \setminus U$ and assume that it is an irreducible divisor. Let ξ be the generic point of D, $F = k(\xi)$ the function field of D and $K = \mathrm{Frac}(\mathcal{O}^h_{X,\xi})$. K is a henselian discrete valuation field. Let $G_K := \mathrm{Gal}(\overline{K}/K)$. On G_K we have a decreasing filtration G^r_K, indexed by $r \in \mathbb{Q}_{>0}$, the ramification filtration of G_K. The group G^1_K is the inertia group. For $r \in \mathbb{R}_{>0}$ we define

$$G^{r+}_K := \overline{\bigcup_{s>r} G^s_K} \subseteq G^{r-}_K := \bigcap_{s<r} G^s_K$$

If $r \notin \mathbb{Q}$, then $G^{r+}_K = G^{r-}_K$. If $r \in \mathbb{Q}$ then $G^r_K = G^{r-}_K$. The group G^{1+}_K is also denoted P; it is the unique pro-p-Sylow subgroup and it is called the *wild inertia group.*

For $r > 1$, $\mathrm{Gr}^r(G_K) := G^r_K/G^{r+}_K$ is abelian and annihilated by p. There is a canonical injection

$$\mathrm{Hom}_{\mathbb{F}_p}(\mathrm{Gr}^r(G_K), \mathbb{F}_p) \xrightarrow{\mathrm{char}} \mathrm{Hom}_{\overline{F}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_{X,\xi} \otimes \overline{F}),$$

which is called *the characteristic form.* We define

$$\overline{K} \supseteq \mathfrak{m}^r_{\overline{K}} = \{a \in \overline{K} |\, \mathrm{ord}\, a \geq r\}$$

and

$$\overline{K} \supseteq \mathfrak{m}^{r+}_{\overline{K}} = \{a \in \overline{K} |\, \mathrm{ord}\, a > r\}$$

where ord is the normalized discrete valuation of $\overline{K}$. The characteristic form links the ramification filtration to the tangent bundle of X.

Let $j : U = X \setminus D \hookrightarrow X$ be the open immersion and define $\mathcal{F} = j_!\mathcal{G}$, where $\mathcal{G}$ is locally constant on U, hence corresponds to a Λ-representation V of $\pi_1(U)$. The map $G_K \to \pi_1(U)$ gives rise to the *slope decomposition*

$$V = \bigoplus_{r \geq 1, r \in \mathbb{Q}} V^{(r)}$$

characterized by

$$V^{G^{r+}_K} = \bigoplus_{s \leq r} V^{(s)}.$$

For example, $V^{(1)}$ is the maximal tame sub-G_K-module. In this situation, we define

$$\mathrm{dimtot}\, V = \sum_{r \in \mathbb{Q}_{\geq 1}} r \dim V^{(r)} \in \mathbb{N}.$$

This number lies in $\mathbb{N}$: If X is a curve, this is the integrality of the Swan conductor, which follows from Hasse-Arf. In general, we can reduce to the curve case by cutting with curves.

For $r > 1$, and $\zeta_p \in \Lambda^\times$, we compute the character of $V^{(r)}$:

$$V^{(r)} = \bigoplus_{\chi : \mathrm{Gr}^r G_K \to \Lambda^\times, \chi \neq 1} \chi^{\oplus m(\chi)}.$$

Let $L_\chi := \operatorname{im}(\operatorname{Char}(\chi)) \subseteq T^*X \times_X \bar{\xi}$. This is a line defined over a finite extension of F_χ over F.

Now assume $V \neq 0$. The singular support is given by

$$SS(\mathcal{F})|_{\operatorname{Spec}\mathcal{O}_{X,\xi}} = T^*_X X \cup \underbrace{T^*_D X}_{\text{if } V^{(1)} \neq 0} \cup \bigcup_{r>1} \bigcup_{m(\chi)\neq 0} \text{Image of } L_\chi$$

Similarly, the characteristic cycle is given by

$$\operatorname{Char}(\mathcal{F})|_{\operatorname{Spec}\mathcal{O}_{X,\xi}} = (-1)^d \left(\operatorname{rank}(\mathcal{G})[T^*_X X] + \dim V^{(1)} \underbrace{[T^*_D X]}_{\text{conormal bundle}} + \sum_{r>1} r \sum_{m(\chi)\neq 0} \frac{\pi_{\chi,*}[L_\chi]}{[F_\chi : F]} \cdot m(\chi) \right)$$

Here $\pi_\chi : L_\chi \to T^*X \times_X F \subseteq T^*X \times_X \operatorname{Spec}\mathcal{O}_{X,\xi}$ is the map above $\operatorname{Spec} F_\chi \to \operatorname{Spec} F$:

$$\begin{array}{ccc} L_\chi & \xrightarrow{\pi_\chi} & T^*X \times_X F \\ \downarrow & & \downarrow \\ \operatorname{Spec} F_\chi & \longrightarrow & \operatorname{Spec} F \end{array}$$

Example 4.4. $X = \mathbb{A}^2$, $D = y$-axis. $U = X \setminus D$, $j : U \hookrightarrow X$, $\mathcal{G}$ given by $t^p - t = y/x^d$ and $\mathcal{F} = j_!\mathcal{G}$, $p|d$. Choose a nontrivial character $\mathbb{Z}/p\mathbb{Z} \to \Lambda^\times$. The case $p = 2, d = 2$ is exceptional. Otherwise, we have, $r = d$, $\operatorname{Char}(\chi)$: $x^d \mapsto dy$ and get

$$SS(\mathcal{F}) = T^*X \cup \langle dy/D \rangle$$

and

$$\operatorname{Char}(\mathcal{F}) = [T^*_X X] + d\,\langle dy/D \rangle\,.$$