Wild ramification and the cotangent bundle in mixed characteristic

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https://www.ms.u-tokyo.ac.jp/∼t-saito/talk/PRim.pdf

Analogy

3 Settings (chlonoligical order).

- Analytic: (Sato, Kashiwara, ...)
  $\mathcal{D}$-modules on complex manifolds.
- Algebraic: (Beilinson, S., ...)
  ℓ-adic sheaves on smooth varieties over perfect fields, e.g. $\mathbf{F}_p$.
- Arithmetic: (S. partial results)
  ℓ-adic sheaves on regular schemes of finite type over $\mathbf{Z}$, $\mathbf{Z}_p$,...

Analogy: 1. Analytic. 2. Algebraic. 3. Arithmetic.

1. Micro local analysis. To study $\mathcal{D}$-modules on X,
one need to work on the cotangent bundle T^*X.

Analogy:
Irregular singularities in 1. $\leftrightarrow$ Wild ramification in 2 and 3.

Mysterious relation: wild ramification and differential forms
e.g. explicit reciprocity law.

Contents

Algebraic case: Notation

- k: perfect field of characteristic $p \geqq 0$.
- X: smooth variety over k.
- Λ: finite field of characteristic $\ell \neq p$.
- $\mathcal{F}$: constructible complex of Λ-modules on X_{et}.
  (ℓ-adic sheaf modulo ℓ)

Notation: Cotangent bundle

$k, X, \Lambda, \mathcal{F},$

- T^*X: cotangent bundle = vector bundle associated to the locally free $\mathcal{O}_X$-module Ω^1_X of rank $n = \dim X$.
  $\dim T^*X = n + n = 2n$.
- C: closed conical subset of T^*X.
  conical = stable under multiplication on vector bundle
  = action of the multiplicative group $\mathbf{G}_m$.

Singular support and Characteristic cycle

$k, X, \Lambda, \mathcal{F}, T^*X.$

- $C = SS\mathcal{F} \subset T^*X$: Singular support of $\mathcal{F}$ (Beilinson)
  Closed conical (stable under $\mathbf{G}_m$) subset.
  $C = \bigcup_a C_a$ irreducible components.
  $\dim C_a = n = \dim X$.
- $CC\mathcal{F} = \sum_a m_a C_a$: Characteristic cycle of $\mathcal{F}$
  $\mathbf{Z}$-linear combination of irreducible components of $SS\mathcal{F}$.

Characteristic cycle: Example 1

- $\mathcal{F}$ locally constant on X, $n = \dim X$:

$$CC\mathcal{F} = (-1)^n rank\, \mathcal{F} \cdot T^*_X X.$$

$T^*_X X = X$: 0-section of T^*X.
Generalization to tamely ramified case.

Characteristic cycle: Example 2

- $\dim X = 1$. $\mathcal{F}|_U$ locally constant $U = X - D$:

$$CC\mathcal{F} = (-1)(rank\,\mathcal{F}|_U \cdot T^*_X X + \sum_{x \in D} a_x\mathcal{F} \cdot T^*_x X).$$

$T^*_X X = X$: 0-section of T^*X.
$a_x\mathcal{F} = rank\,\mathcal{F}|_U - rank\,\mathcal{F}_{\bar{x}} + Sw_x\mathcal{F}$: Artin conductor.
$Sw_x\mathcal{F}$: Swan conductor, measure of wild ramification.
$T^*_x X$: fiber at $x \in D \subset X$.

Characteristic cycle: Index formula

$$k, X, \Lambda, \mathcal{F}, SS\mathcal{F} = \bigcup_a C_a \subset T^*X, CC\mathcal{F} = \sum_a m_a C_a$$

Theorem: Index formula

If X is projective (and smooth),

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}.$$

Euler number: $\chi(X_{\bar{k}}, \mathcal{F}) = \sum_q (-1)^q \dim H^q(X_{\bar{k}}, \mathcal{F})$.
Intersection number: $(CC\mathcal{F}, T^*_X X)_{T^*X}$.
$T^*_X X = X$: 0-section of T^*X.
If $\dim X = 1$, recover the Grothendieck-Ogg-Shafarevich formula.

Characteristic cycle: Index formula and a variant

$$k, X, \Lambda, \mathcal{F}, SS\mathcal{F} = \bigcup_a C_a \subset T^*X, CC\mathcal{F} = \sum_a m_a C_a$$

Theorem: Index formula

If X is projective (and smooth),

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}.$$

Arithmetic refinement (Daichi Takeuchi)

k: finite, $\mathcal{F}$: $\bar{\mathbf{Q}}_\ell$-sheaf.

$$\det(Frob, H^*(X_{\bar{k}}, \mathcal{F})) = (\mathcal{E}\mathcal{F}, T^*_X X)_{T^*X}$$

in $\bar{\mathbf{Q}}_\ell^\times \otimes \mathbf{Q}$.

Arithmetic case: Notation

- K: complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$.
  E.g. $\mathbf{Q}_p$ or its finite extension.
  (cf. k: perfect field of characteristic $p \geqq 0$. e.g. $k = \mathbf{F}_p$.)
- X: regular flat scheme of finite type over $\mathcal{O}_K$.
  (cf. smooth over k.)
- Λ: finite field of characteristic $\ell \neq p$.
- $\mathcal{F}$: constructible complex of Λ-modules on $X_{\rm et}$.

Notation (Algebraic case)

$k, X, \Lambda, \mathcal{F},$

- T^*X: cotangent bundle = vector bundle associated to the locally free $\mathcal{O}_X$-module Ω^1_X of rank $n = \dim X$.
  $\dim T^*X = n + n = 2n$.
- C: closed conical subset of T^*X.
  conical = stable under multiplication on vector bundle
  = action of the multiplicative group $\mathbf{G}_m$.

Cotangent bundle in arithmetic case?

$X/\mathcal{O}_K$ regular flat of finite type.

Problem: Cotangent bundle T^*X?

Ω^1_{X/O_K} is not a locally free sheaf of rank $n = \dim X$.

Solution: Modify Ω^1_{X/O_K} so that "dp" $\neq 0$.

Cotangent bundle T^*X? FW-derivation

Definition: Frobenius-Witt derivation
cf. total p-derivation by Dupuy, Katz, Rabinoff, Zureick-Brown

p: prime number. A: ring flat over $\mathbf{Z}_{(p)}$.

- A mapping $w\colon A \to M$ to an A-module is an FW-derivation if

$$w(a+b) = w(a) + w(b) + \frac{a^p + b^p - (a+b)^p}{p} \cdot w(p),$$

$$w(ab) = b^p \cdot w(a) + a^p \cdot w(b).$$

(modified Leibniz' rule)

- $(F\Omega^1_A, w\colon A \to F\Omega^1_A)$: universal pair of A-module and FW-derivation.

cf. δ-structure by Bhatt, Scholze $=$ p-derivation by Buium

Cotangent bundle T^*X? Construction

X regular flat scheme of finite type over $\mathcal{O}_K$.
$(F\Omega^1_X, w)$: sheafification of universal FW-differentials.

$$w(a+b) = w(a) + w(b) + \frac{a^p + b^p - (a+b)^p}{p} \cdot w(p),$$
$$w(ab) = b^p \cdot w(a) + a^p \cdot w(b).$$
(modified Leibniz' rule)

Theorem

X: regular, $X_{\mathbf{F}_p} = X \times_{Spec\,\mathbf{Z}} Spec\,\mathbf{F}_p$.

- $F\Omega^1_X$ is a locally free $\mathcal{O}_{X_{\mathbf{F}_p}}$-module of rank $\dim X$.

Cotangent bundle T^*X? Definition of $FT^*X|_{X_k}$

$(F\Omega^1_X, w)$: sheafification of universal FW-derivation.
$F\Omega^1_X$ is a locally free $\mathcal{O}_{X_{\mathbf{F}_p}}$-module of rank $\dim X$.

Definition: $FT^*X|_{X_k}$

X: regular:

- $FT^*X|_{X_k}$: vector bundle on X_k associated to $F\Omega^1_X \otimes_{\mathcal{O}_{X_{\mathbf{F}_p}}} \mathcal{O}_{X_k}$.

E.g. X is smooth over $\mathcal{O}_K$ and $F\colon X_k \to X_k$ Frobenius:

$$0 \to F^*T^*_{X_k}X \to FT^*X|_{X_k} \to F^*T^*X_k \to 0 \qquad \textit{exact}$$

Micro support: support and singular support

$X/\mathcal{O}_K, \Lambda, \mathcal{F}, FT^*X|_{X_k}$.

How to define $SS\mathcal{F}$ as a closed conical subset C of $FT^*X|_{X_k}$?

Support of $\mathcal{F}$

$A \subset X$ closed subset. $\mathcal{F}$ is supported on $A \Leftrightarrow \mathcal{F}|_{X-A} = 0$.
Supp $\mathcal{F}$: Smallest A such that $\mathcal{F}$ is supported on A.

Singular support $SS\mathcal{F}$ MOST involved part of the talk!

Smallest closed conical subset C of $FT^*X|_{X_k}$ such that $\mathcal{F}$ is micro-supported on C.

Micro support: Definition MOST involved part of the talk!

$X/\mathcal{O}_K, \Lambda, \mathcal{F}, FT^*X|_{X_k}$.

Definition: micro support

1. $\mathcal{F}$ is micro-supported on a closed conical subset $C \subset FT^*X|_{X_k}$ if
(1) For any morphism $h\colon W \to X$ of regular schemes of f. t. / $\mathcal{O}_K$,
C-transversality implies $\mathcal{F}$-transversality.
(2) $C \cap FT^*_X X|_{X_k} \supset supp\mathcal{F} \cap X_k$.

2. $SS\mathcal{F}$: smallest C on which $\mathcal{F}$ is micro-supported.

Transversality: Definition MOST involved part of the talk!

$X/\mathcal{O}_K, \Lambda, \mathcal{F}, C \subset FT^*X|_{X_k}$.

Definition: transversality

1. $h\colon W \to X$ morphism of regular schemes of finite type over $\mathcal{O}_K$.
   - h is C-transversal if the intersection

     $$(C \times_{X_k} W_k) \cap Ker(h^*\colon FT^*X|_{X_k} \times_{X_k} W_k \to FT^*W|_{W_k})$$

     is a subset of the 0-section.
   - h is $\mathcal{F}$-transversal if $h^*\mathcal{F} \otimes Rh^!\Lambda \to Rh^!\mathcal{F}$ is an isomorphism.

2. $\mathcal{F}$ is micro-supported on C:
(1) h C-transversal $\Rightarrow$ $\mathcal{F}$-transversal + (2) a condition on support.

3. $SS\mathcal{F}$: smallest C on which $\mathcal{F}$ is micro-supported.

Transversality: Examples

$X/\mathcal{O}_K, \Lambda, \mathcal{F}$. $C \subset FT^*X|_{X_k}$.

$h\colon W \to X$: morphism of regular schemes of finite type over $\mathcal{O}_K$.

- $Z \subset X$ regular closed subscheme, $C = F^*T^*_ZX|_{Z_k} \subset FT^*X|_{X_k}$ Frobenius pull-back of conormal bundle:

  h is C-transversal $\Leftrightarrow$

  h is transversal with $Z \subset X$ on a nbd of W_k.
- h smooth $\Rightarrow$ $\mathcal{F}$-transversal for any $\mathcal{F}$. (Poincaré duality)
- $\mathcal{F}$ locally constant $\Rightarrow$ any h is $\mathcal{F}$-transversal.

Singular support: Existence?

$X/\mathcal{O}_K, \Lambda, \mathcal{F}$.
$SS\mathcal{F}$: smallest closed conical subset $C \subset FT^*X|_{X_k}$ on which $\mathcal{F}$ is micro-supported.

Proposition

Suppose $supp\mathcal{F} = X$.
$SS\mathcal{F} = $ 0-section $\Leftrightarrow$ $\mathcal{F}$ is locally constant on a nbd of X_k.

Question

Does $SS\mathcal{F}$ exist?

We don't know yet in general. Example with wild ramification.

Example: Kummer covering

K: finite extension of $\mathbf{Q}_p$ containing a primitive p-th root ζ_p of 1.
π: uniformizer of K. $e = e_{K/\mathbf{Q}_p}$: ramification index.
$i \geqq 1$ integer.

$$X = Spec\, \mathcal{O}_K[T^{\pm 1}, (1+\pi^i T)^{-1}]$$
$$\supset U = X_K = Spec\, K[T^{\pm 1}, (1+\pi^i T)^{-1}].$$

$V \to U$: Kummer covering defined by $t^p = 1 + \pi^i T$.
$\mathcal{F}$: locally constant sheaf of Λ-modules of rank 1 on U
defined by a non-trivial character $\mu_p = Gal(V/U) \to \Lambda^\times$.

Example: Kummer covering

$K/\mathbf{Q}_p$: finite, $\zeta_p \in K$. π: unif. $e = e_{K/\mathbf{Q}_p}$. k: residue field.
$\mathcal{F}$: rank 1 on $U = X_K \subset X = \mathit{Spec}\,\mathcal{O}_K[T^{\pm 1}, (1+\pi^i T)^{-1}]$ defined by $t^p = 1 + \pi^i T$.
$X_k = X - U = \mathit{Spec}\, k[T^{\pm 1}]$.
$FT^*X|_{X_k}$: vector bundle of rank 2, basis $w(\pi), w(T)$.
$\mathcal{F}$: unramified along X_k if $i \geqq ep/(p-1)$.

Proposition

Assume $1 \leqq i < ep/(p-1)$. $j\colon U \to X$ open immersion.

- $SS\, j_!\mathcal{F}$ exists.
- $SS\, j_!\mathcal{F} = F^*T^*_{X_k}X = \langle w(\pi)\rangle$ if $p \nmid i$,
  $= \langle w(T)\rangle$ if $p | i$ unless $p = 2,\ i = 2(e-1)$,
  $= \langle w(T) - T \cdot w(\frac{2}{\pi^{e-1}})\rangle$ if $p = 2,\ i = 2(e-1)$.