

Singular support & characteristic cycles in mixed characteristic

Steps/obstruction

- ✓ 1. Cotangent bundles $rk = \dim. \mathbb{Z}_p?$
by Kato
- ✓ 2. Def'n of sing. supp.
- ? 3. Existence of sing supp. Reidem?
- ? 4. Def of C.C Milnor form?

Today 1, 2.

1. "Frobenius pull-back" of the restriction on the closed fiber.
2. geom. case. $\mathbb{Z}, \subset \subset TX$ closed conical
- 2-1. \mathbb{Z} micro support on C
- 2-2. SSZ: smallest C on which \mathbb{Z} is micro supp'd

Micro supported

Relation between

\mathbb{Z} on X and $C \subset TX$

Use morphisms from X (def'd by loc.)

$f: X \rightarrow Y$ C-acyclic (C-trans.)

$\Rightarrow f: X \rightarrow Y$ loc. acyclic rel to \mathbb{Z}

This def'n works well because there are suff. many morphisms from X

E.g. if $X = S_n \mathbb{Z}_p$ not enough.

Instead. equiv. condition

Using morphisms to X

$f: W \rightarrow X$ C-transu

$\Rightarrow f: W \rightarrow X$ \mathbb{Z} -transversal

Contents

1. Frobenius - Witt differentials.
2. C-trans vs F-trans
3. Example of existence of SSZ.

1. p prime number

$$P(X, Y) = \frac{(X+Y)^p - X^p - Y^p}{p}$$

$$= \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X^{p-i} Y^i + Z[X, Y]$$

A ring. M A -module

Def'n A mapping $w: A \rightarrow M$ is an Frob-Witt derivation if Witt.

- $w(x+y) = w(x) + w(y) - P(x, y) v(p)$
- $w(xy) = x^p w(y) + y^p w(x)$ Frob

Eg. $\exists \varphi: A \rightarrow A$ s.t. $\varphi(x) = x^p$ mod p
 A flat / $\mathbb{Z}(p)$
 $w: A \rightarrow A/pA$ $w(x) = \frac{\varphi(x) - x^p}{p}$

cf. p -derivation Buium
 δ -ring Bhatt - Scholze. Prim

cf A ring / $\mathbb{Z}(p)$. w . FW-deriv
 $\Rightarrow p \cdot w = 0$
 $(w(na) = a^p w(n) + n \cdot w(a) \quad w)$
 $= a^p w(n) + n^p w(a) \quad F)$

$(F\Omega_A^1, w: A \rightarrow F\Omega_A^1)$ $D, K, R, \mathbb{Z}B$
 univ pair of FW-deriv

FW-deriv $1:1$ A -lin
 $w: A \rightarrow M \iff F\Omega_A^1 \rightarrow M$

$F\Omega_A^1$ module of FW-diff's.

$A/\mathbb{Z}_{(p)} \Rightarrow F\Omega^1_A \quad A/pA\text{-mod.}$

Regularity

A noether. local $d = \dim A$
 F res fld $[F:F^p] = p^d$

A flat $\mathbb{Z}_{(p)}$ or $pA=0$

$F\Omega^1_A$ free A/pA -mod of rk d then

$\Rightarrow A$ regular

conversely. Under a certain finiteness condition

\Leftarrow holds.

\mathbb{O}_K disc. val ring with prof. res. fld
 k . $\dim k = p > 0$.

X/\mathbb{O}_K reg ~~sch~~ sch of finite type

$F\Omega^1_X$ locally free $\mathbb{O}_X/p\mathbb{O}_X$ -mod
of rk. $\dim X$.

E.g. X smooth Y_x closed fiber

$0 \rightarrow F^b N_{Y_x/X} \rightarrow F\Omega^1_X \otimes_{\mathbb{O}_X} \mathbb{O}_{Y_x} \rightarrow F\Omega^1_{Y_x/k} \rightarrow 0$

Ext controlled by Deligne-Illusie
studied by PKRZB ($\Omega^1_{X/k}$)

$F T^* X|_{Y_x}$ vect + b'le on Y_x

ass to $F\Omega^1_X \otimes_{\mathbb{O}_X} \mathbb{O}_{Y_x}$

$\text{rk} = \dim X = \dim Y_x + 1$.

Functoriality

$h: W \rightarrow X$

morphism of var sch/ \mathbb{O}_K

$F T^* X|_{Y_x} \leftarrow h^* (F T^* X|_{Y_x}) \rightarrow F T^* W|_{W_x}$

2 C-transversality.

$C = FT^*X|_{Z_c}$ closed curve
 $h: W \rightarrow X$. C-trans. if

$h^*C \cap \ker(h^*FT^*X|_{Z_c} \rightarrow FT^*W|_{Z_c})$
 is a subset of the 0-section

E.g. • If $C \subset 0$ -section or h smooth.

• $Z \subset X$ reg closed, codim c .

$$C = F^b T^*X|_{Z_c} \rightarrow FT^*X|_{Z_c}.$$

h C-trans $\Leftrightarrow Z \times_W C \subset W$
 reg codim c
 on a nbd of W_c

C-transversality
 condition on a nbd of W_c

\mathbb{Z} -transversality

\mathbb{Z} const. sheaf on X et of
 Λ -modules Λ . finite ext of \mathbb{F}_2
 $\mathbb{Z} \neq \mathbb{P}$.

$$h: W \rightarrow X$$

$$C_{\mathbb{Z}}: h^* \mathbb{Z} \otimes Rh^* \Lambda \rightarrow Rh^* \mathbb{Z}$$

adjoint of

$$\begin{array}{ccc} Rh_! (h^* \mathbb{Z} \otimes Rh^* \Lambda) & \rightarrow & \mathbb{Z} \\ \downarrow \text{proj. fun.} & & \nearrow 1 \otimes \text{adj.} \\ \mathbb{Z} \otimes Rh_! Rh^* \Lambda & & \end{array}$$

Def h : \mathbb{Z} -trans

if $C_{\mathbb{Z}}$ is an isom.

E.g. • \mathbb{Z} loc const or h smooth
 • $\mathbb{Z} = i_* \Lambda$ $i: Z \rightarrow X$ as before

C -trans \Rightarrow \mathcal{F} -trans
 purity.

Def \mathcal{F} micro supported on C if

① $\text{supp } \mathcal{F} \cap X_e \subset \text{base of } C$
 $(= C \cap O\text{-section})$

② $\forall h: W \rightarrow X$

C -trans \Rightarrow \mathcal{F} -trans on a
 nbd of W_e .

Rank cond on a nbd of X_e .

E.g. \mathcal{F} loc. const on a nbd of X_e

\Leftrightarrow micro supported on O -section

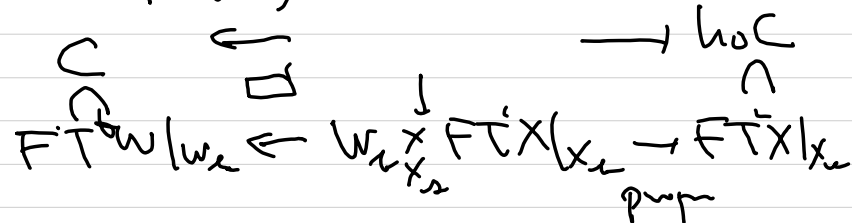
Def If \exists smallest C , $C = \text{SS}\mathcal{F}$

Conj \exists $\text{SS}\mathcal{F}$. Ex $\mathcal{F} = \bigwedge \text{SS}\mathcal{F} =$
 O -section

$h: W \rightarrow X$ prop

& \mathcal{F} on W m.s on C

\Rightarrow $Rho \mathcal{F}$ m.s on $h_* C$



3.

Example

$W \rightarrow V$

$h \downarrow \downarrow$ G -torsor G finite p-gp

$X \supset U = X - D$

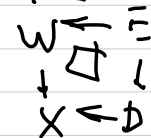
DCX_k
 smooth/ k

W normalization

assume W regular

$U = W - E$ E smooth/ k

$E \rightarrow D$ purely inseparable



& \mathcal{O}_E locally gen by 1. elt / \mathcal{O}_D

$$\Rightarrow \Sigma_D \otimes_{\mathcal{O}_D} \mathcal{O}_E \rightarrow \mathcal{O}_E$$

$$rk = \det E - 1$$

g loc. cond on U
 cov. to inversed rep of G

$$\exists = j_i g \quad j_i: U \rightarrow X$$

$$\Rightarrow \text{SSZ} = h_0(\mathcal{O}\text{-section of } FT^*(W|_{X_D}))$$

for every pt of D . fiber is a line

$$j_i g \rightarrow R_{j_i} g \text{ isom.}$$

$$FT^*(X|_{X_D}) \xleftarrow{\text{flat. lines}} W \otimes FT^*(X|_{X_D}) \rightarrow FT^*(W|_{X_D})$$

$$\cup \quad \cup \quad \cup$$

$$\leftarrow \quad rk 1 \quad \rightarrow 0$$

Concrete example

Kummer covering.

$$X = \mathbb{A}^n_{\mathbb{C}} \supset U = \mathbb{A}^n_{\mathbb{C}} \setminus \{0\}$$

$$= \text{Spec } \mathbb{C}[T, T^{-1}] \quad k \supset \mathbb{F}_p$$

$$e_k \quad i < \frac{p-1}{p-1}$$

$$U \rightarrow U \quad x^p = 1 + \pi^{p^i} T$$

W regular.

$$\text{SSZ} \subset FT^*(X|_{X_D})$$

Spanned by the section $W(T)$

$$\text{unless } p=2 \quad e_k = i+1$$

$$(1 + \pi^i x)^p =$$

is the line bundle