Singular support & characteristic cycles in mixed characteristic

Steps/obstruction

✓ 1. Cotangent bundles — rk = dim. $\mathbb{Z}_p$? log Kato

✓ 2. Def'n of sing. supp.

? 3. Existence of sing supp. — Radon?

? 4. Def of C.C — Milnor formula?

Today 1, 2.

1. "Frobenius pull-back" of the restriction on the closed fiber.

2. geom. case. $\mathcal{F}$, $C \subset T^*X$ closed conical

2-1. $\mathcal{F}$ micro supp'd on C

2-2. $SS\mathcal{F}$: smallest C on which $\mathcal{F}$ is micro supp'd

Micro supported

Relation between $\mathcal{F}$ on X and $C \subset T^*X$

Use morphisms from X (def'd et loc.)

$f: X \to Y$ C-acyclic (C-trans.)

$\Rightarrow$ $f: X \to Y$ loc. acyclic rel to $\mathcal{F}$

This def'n works well because there are suff. many morphisms from X

E.g. if $X = \mathrm{Spec}\, \mathbb{Z}_p$ not enough.

Instead. equiv. condition

Using morphisms to X

$f: W \to X$ C-transv

$\Rightarrow$ $f: W \to X$ $\mathcal{F}$-transversal

Contents.

1. Frobenius-Witt differentials.
2. C-trans v-s Z-trans
3. Example of existence of SSZ.

1. p prime number

$$P(X,Y) = \frac{(X+Y)^p - X^p - Y^p}{p} = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} X^{p-i} Y^i \in \mathbb{Z}[X,Y]$$

A ring. M A-module

Def'n A mapping $w: A \to M$ is an Frob-Witt derivation if (Witt.)

- $w(x+y) = w(x) + w(y) - P(x,y)w(p)$
- $w(xy) = x^p w(y) + y^p w(x)$ (Frob)

Eg If $\varphi: A \to A$ s.t $\varphi(x) \equiv x^p \bmod p$, A flat $/\mathbb{Z}_{(p)}$

$$w: A \to A/pA \qquad w(x) = \frac{\varphi(x) - x^p}{p}$$

$$pA \overset{\shortparallel}{\simeq} M$$

cf. p-derivation Buium

δ-ring Bhatt-Scholze. Prism

if

A ring $/\mathbb{Z}_{(p)}$. w. FW-deriv $\Rightarrow$ $p.w = 0$.

$$\left(w(na) = a^p w(n) + n \cdot w(a) \right. \quad w$$
$$\left. = a^p w(n) + n^p w(a) \right) \quad F$$

D, K, R, ZB

$(F\Omega^1_A,\ w: A \to F\Omega^1_A)$
univ pair of FW-deriv

FW-der'n $\overset{1:1}{\longleftrightarrow}$ A-linear
$w: A \to M$ $\longleftrightarrow$ $F\Omega^1_A \to M$.

$F\Omega^1_A$ module of FW-diff'ls.

$A/\mathbb{Z}_{(p)} \Rightarrow F\Omega^1_A$ A/pA-mod.

Regularity

A noether. local $d = \dim A$

F res fd $[F:F^p] = p^r$

A flat $/\mathbb{Z}_{(p)}$ or $pA = 0$

$F\Omega^1_A$ free A/pA-mod of rk $d+r$

$\Rightarrow$ A regular

conversely under a certain finiteness condition

$\Leftarrow$ holds.

O_K disc. val ring with perf. res. fd k. char $k = p > 0$.

X/O_K reg sch of finite type

$F\Omega^1_X$ locally free O_X/pO_X-mod of rk. $\dim X$.

E.g. X smooth X_k closed fiber

$$0 \to F^*N_{X_k/X} \to F\Omega^1_X \otimes_{O_X} O_{X_k} \to F^*\Omega^1_{X_k/k} \to 0$$

exact

Ext constructed by Deligne-Illusie

studied by PKRZB $(\Omega_X^{1\,\mathrm{tot}})$

$FT^*X|_{X_k}$ vect b'le on X_k

ass to $F\Omega^1_X \otimes_{O_X} O_{X_k}$

$\mathrm{rk} = \dim X = \dim X_k + 1.$

Functoriality

$h : W \to X$ morphism of reg sch$/O_K$

$$FT^*X|_{X_k} \leftarrow h_k^*(FT^*X|_{X_k}) \to FT^*W_{W_k}$$

2 C-transversality.

$C \subset T^*X|_{X_e}$ closed conical

$h: W \to X$. C-trans. if

$h^* C \cap \ker(h^*_e T^*X|_{X_e} \to T^*W|_{W_e})$

is a subset of the 0-section

E.g. • If $C \subset$ 0-section or h smooth.

• $Z \subset X$ reg closed, codim c.

$C = T^*_Z X|_{Z_e} \to T^*X|_{X_e}$.

h C-tran $\Leftrightarrow$ $Z \times_X W \subset W$

reg codim c

on a nbd of W_e

C-transversality

condition on a nbd of W_e

$\mathcal{F}$-transversality

$\mathcal{F}$ const. shf on X_{et} of

Λ-modules Λ. finite ext of $\mathbb{F}_\ell$

$\ell \neq p$.

$h: W \to X$

$c_{\mathcal{F}}: h^*\mathcal{F} \otimes Rh^! \Lambda \to Rh^! \mathcal{F}$

adjoint of

$Rh_!(h^*\mathcal{F} \otimes Rh^!\Lambda) \to \mathcal{F}$

$\downarrow$ proj formula

$\mathcal{F} \otimes Rh_! Rh^! \Lambda \xrightarrow{1 \otimes \mathrm{adj}} \mathcal{F}$

Def h: $\mathcal{F}$-trans

if $c_{\mathcal{F}}$ is an isom.

E.g. • $\mathcal{F}$ loc const or h smooth

• $\mathcal{F} = i_* \Lambda$ $i: Z \to X$ as before

C-trans $\Rightarrow$ $\mathcal{F}$-trans

purity.

Def $\mathcal{F}$ micro supported on C. if

① $\operatorname{supp}\mathcal{F} \cap X_k \subset C$ base of C
(= $C \cap 0$-section)

② $\forall h: W \to X$

C-trans $\Rightarrow$ $\mathcal{F}$-trans on a nbd of W_k.

Rmk cond on a nbd of $X_{\bar{k}}$.

E.g. $\mathcal{F}$ loc. const on a nbd of $X_{\bar{k}}$
$\Leftrightarrow$ micro supported on 0-section

Def If $\exists$ smallest C, $C = SS\mathcal{F}$

Conj $\exists$ $SS\mathcal{F}$. Ex $\mathcal{F} = \Lambda$, $SS\mathcal{F} =$ 0-section

$h: W \to X$ proper

& $\mathcal{F}$ on W m.s on C

$\Rightarrow$ $Rh_*\mathcal{F}$ m.s on $h_\circ C$

$$\begin{tikzcd}
C \arrow[d, phantom, "\cap"] & \arrow[l] \Box \arrow[d] \arrow[r] & h_\circ C \arrow[d, phantom, "\cap"] \\
FT^*W|_{W_k} & \arrow[l] W_k \times_{X_k} FT^*X|_{X_k} \arrow[r, "\text{proper}"'] & FT^*X|_{X_k}
\end{tikzcd}$$

3.

Example

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "h"'] & V \arrow[d] \\
X & \supset U = X - D
\end{tikzcd}$$

G-torsor G finite p-gp

$D \subset X_k$ smooth/k

W normalization

assume: W regular

$U = W - E$ E smooth/k

, $E \to D$ purely insep

$$\begin{tikzcd}
W \arrow[d] & E \arrow[l] \arrow[d] \arrow[dl, phantom, "\Box"] \\
X & D \arrow[l]
\end{tikzcd}$$

& $\mathcal{O}_E$ locally gen by 1. elt / $\mathcal{O}_D$

$$\Rightarrow \Omega^1_D \otimes_{\mathcal{O}_D} \mathcal{O}_E \to \Omega^1_E$$

rk $= \dim E - 1$

$\mathcal{G}$ loc. const on U

corr. to irred rep of G / Λ

$\mathcal{F} = j_!\mathcal{G}$ $\quad j : U \to X$

$\Rightarrow$ • $SS\mathcal{F} = h_0$ (0-section of $FT^*W|_{W_k}$)

for every pt of D. fiber is a line

• $j_!\mathcal{G} \to Rj_*\mathcal{G}$ isom.

$$FT^*X|_{X_k} \xleftarrow{\text{flat. homeo}} W \times_X FT^*X|_{X_k} \to FT^*W|_{W_k}$$

$$\cup \qquad \cup \qquad \cup$$

$$\leftarrow \text{rk } 1 \longrightarrow 0$$

Concrete example

Kummer covering.

$X = \mathbb{G}_{m,\mathcal{O}_K} \supset U = \mathbb{G}_{m,K}$

$= \mathrm{Spec}\, \mathcal{O}_K[T, T^{-1}]$ $\quad K \ni \zeta_p$

e_K $\quad i < \frac{e_K}{p-1}$

$V \to U$ $\quad x^p = 1 + \pi^{pi} T$

W regular.

$SS\mathcal{F} \subset FT^*X|_{X_k}$

spanned by the section $w(T)$

unless $p = 2$ $\quad e_K = i+1$

$(1 + \pi^i x)^p =$

is the line b'dle