**Galois representation in arithmetic geometry.**

1. Local-global in arithmetic.

"An $\ell$-adic representation is described by its $L$-function".

An analogy between algebraic curve and Spec $\mathbb{Z}$.

| | | |
|---|---|---|
| function field $K$ | : | $\mathbb{Q}$ |
| closed points | : | prime numbers and infinite places |
| $K \subset$ local field $K_x$ | : | $\mathbb{Q} \subset$ $p$-adic field $\mathbb{Q}_p$ and $\mathbb{R} = \mathbb{Q}_\infty$ |

Two features:

Each point $p$ is a "circle" since the fundamental group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ of Spec $\mathbb{F}_p$ is a pro-cyclic group generated by the Frobenius.

A global representation is uniquely determined by the local data by the Cebotarev density theorem: The Frobenius conjugacy classes form a dense open subset of the global Galois group.

$\ell$-adic representations.

A continuous representation $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C})$ has open kernel and finite image. Not useful to study arithmetic geometry. An $\ell$-adic representation is a continuous representation $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_\ell)$ unramified outside finite set of prime numbers, where $\ell$ denotes a prime number. Except finitely many primes, eigenpolynomials

$$\det(1 - \varphi_p t : V) \in \mathbb{Q}_\ell[t]$$

is defined. Consequence of the Cebotarev density theorem: upto semi-simplification, an $\ell$-adic representation $V$ is determined by the local $L$-factors $L_p(V, t) = \det(1 - \varphi_p t : V) \in \mathbb{Q}_\ell[t]$ at primes $p$ where $V$ is unramified. In most cases, $\det(1 - \varphi_p t : V)$ is in $\mathbb{Q}[t]$ and is independent of $\ell$, i.e. $\ell$-adic representation is a member of a compatible system. $L$-function of $V$:

$$L(V, s) = \prod_p L_p(V, p^{-s})^{-1}.$$

Example 1. $E$ elliptic curve over $\mathbb{Q}$ e.g. $E = X_0(11)$ defined by the equation $y^2 = 4x^3 - 4x^2 - 40x - 79$. $T_\ell E = \varprojlim_n E[\ell^n](\bar{\mathbb{Q}})$. $T_\ell E$ is an $\ell$-adic representation of $G_\mathbb{Q}$. If one forget the $G_\mathbb{Q}$-action, it is isomorphic to $\mathbb{Z}_\ell^2$ as a module. For a prime number $p$ prime to the discriminant of $E$,

$$\det(1 - \varphi_p t : T_\ell E) = 1 - a_p(E)t + pt^2$$

where $a_p(E)$ is an integer defined by $\sharp E(\mathbb{F}_p) = 1 - a_p(E) + p$.

$$L(E, s) = \prod_p (1 - a_p(E)p^{-s} + p^{1-2s})^{-1}.$$

Example 2. $f(\tau) = \sum_{n=1}^\infty a_n(f) q^n$ ($q = \exp 2\pi\sqrt{-1}\tau$) normalized eigen cusp form of weight 2 with trivial character that is an eigenvector for every Hecke operator e.g. $f_{11}(\tau) = q \prod_{n=1}^\infty (1 - q^n)^2 (1 - q^{11n})^2$. $V_f$ $\ell$-adic representation associated to $f$ For $p$ prime to the level of $f$,

$$\det(1 - \varphi_p t : V_f) = 1 - a_p(f)t + pt^2.$$

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p(f)p^{-s} + p^{1-2s})^{-1}.$$

Taniyama-Shimura Conjecture. (proved by Wiles-Taylor-Diamond-Conrad-Breuil)

An $\ell$-adic representation of type in Example 1 is necessarily of type in Example 2. Or, equivalently, for an elliptic curve $E$, there exists a cusp form $f$ such that

$$L(E,s) = L(f,s).$$

The other implication was established by Eichler-Shimura.

E.g. For $E$ in Example 1, $f_{11}$ in Example 2 works.

2. Etale cohomology as an $\ell$-adic representation.

"The Weil conjecture implies that the $L$-function of the etale cohomology is the Hasse-Weil $L$-function."

$X$ projective smooth algebraic variety over $\mathbb{Q}$. Etale cohomology $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is defined. As a vector space, simply $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = H^m(X^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. The $\ell$-adic representation $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is unramified at a prime $p$ where $X$ has good reduction. The $L$-function of the $\ell$-adic representation $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is the Hasse-Weil $L$-function $L(H^m(X), s)$.

Example 1. If $E$ is an elliptic curve over $\mathbb{Q}$, we have $H^1(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = Hom(T_\ell E, \mathbb{Q}_\ell)$. In other words,

$$L(H^1(E), s) = L(E,s).$$

Example 2. For an integer $N \geq 1$, let $X_0(N)$ be the modular curve of level $N$. ($X_0(N)^{\mathrm{an}}$ is a compactification of $\Gamma_0(N)\backslash H$ where $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 \bmod N\}$.) Then,

$$H^1(X_0(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = \bigoplus_{f:N_f|N} Hom(V_f, \mathbb{Q}_\ell)^{\oplus \sharp\{d|N/N_f\}}.$$

Decomposition is given by Hecke operators. In other words,

$$L(H^1(X_0(N), s) = \prod_{f:N_f|N} L(f,s)^{\sharp\{d|N/N_f\}}.$$

Hasse-Weil $L$-function. Let $X \bmod p$ be the reduction modulo a good prime $p$, that is a projective smooth variety over $\mathbb{F}_p$. Let

$$Z(X \bmod p, t) = \exp\left(\sum_{n=1}^{\infty} \frac{\sharp X(\mathbb{F}_{p^n})}{n} t^n\right)$$

denote the congruence $\zeta$-function. By the Weil conjecture proved by Deligne, we have

$$Z(X \bmod p, t) = \frac{P_1(X \bmod p, t) \cdots P_{2d-1}(X \bmod p, t)}{P_0(X \bmod p, t) \cdot P_2(X \bmod p, t) \cdots P_{2d}(X \bmod p, t)}$$

where $d = \dim X$, $P_m(X \bmod p, t) \in \mathbb{Z}[t]$. The decomposition is characterized by the property that, if we put $P_m(X \bmod p, t) = \prod_i (1 - \alpha_{i,p,m} t)$, the complex eigenvalue of $\alpha_{i,p,m}$ is $p^{\frac{m}{2}}$. This is an analogue of the Riemann hypothesis.

$$L(H^m(X), s) = \prod_p P_m(X \bmod p, p^{-s})^{-1}.$$

Note: Bad factors are missing.

Further, we have

$$\det(1 - Fr_p t : H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)) = P_m(X \bmod p, t)$$

and consequently,

$$L(H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell), s) = L(H^m(X), s).$$

The local factor at a prime of good reduction is determined by the Weil conjecture, upto semi-simplicification.

Semi-simplicity conjecture:(Tate) The action of $Fr_p$ on $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$ is semi-simple.

The semi-simplicity conjecture implies that, the $\ell$-adic representation $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$ of $G_{\mathbb{F}_p}$ is determined by $\det(1 - Fr_p t : H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$.

The Hasse-Weil functions are conjectured to have analytic continuation and to satisfy a functional equation. To formulate a function equation, we need to include the bad primes and to introduce the $\Gamma$-factor that is a contribution of the infinite place.

$\Gamma$-factor:(Serre) $V = H^m(X^{\rm an}, \mathbb{Q})$ is a pure Hodge structure of weight $m$ with $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$-action. Put $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Define

$$\Gamma_{\mathbb{R}}(H^m(X), s) = \prod_{p<m/2} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \Gamma_{\mathbb{R}}(s - m/2)^{h^+} \Gamma_{\mathbb{R}}(s - m/2 + 1)^{h^-}.$$

If $m$ is odd, we have only the first term. If $m$ is even $h^{\pm}$ is the dimension of the subspace of $V^{\frac{m}{2},\frac{m}{2}}$ where $\sigma$ acts as $(-1)^{n/2}$.

3. Primes of bad reduction.

Bad factors of the Hasse-Weil $L$-function.(Serre)

$P_p(H^m(X), t) = \det(1 - Fr_p t : H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_p^I)$ where ${}^{I^p}$ indicates the inertia fixed part.

Functional equation.(Serre)

Put $\Lambda(H^m(X), s) = L(H^m(X), s) \cdot \Gamma_{\mathbb{R}}(H^m(X), s)$. Define $N = \prod_{\text{bad } p} p^{f_p}$ where $f_p$ is the Artin conductor of $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ at $p$. Then we expect to have a function equation

$$\Lambda(H^m(X), s) = \pm N^{\frac{m+1}{2} - s} \Lambda(H^m(X), m + 1 - s).$$

Question.(Serre) Are $P_p(H^m(X), t)$ and $f_p$ well-defined?

This question fits in more general problems.

(i) Description of local Galois representation.

"The monodromy-weight conjecture together with a part of the Tate conjecture implies an affirmative answer to Question."

(ii) Invariants of ramification.

"We have a geometric formula computing the conductor."

(i) Absolute Galois group of a local field. To

$$\mathbb{Q}_p \subset \mathbb{Q}_p^{\mathrm{ur}} = \mathbb{Q}_p(\zeta_n\ (p \nmid n)) \subset \mathbb{Q}_p^{\mathrm{tr}} = \mathbb{Q}_p^{\mathrm{ur}}(p^{\frac{1}{n}}\ (p \nmid n)) \subset \bar{\mathbb{Q}}_p,$$

corresponds

$$G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \supset I = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}}) \supset P = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{tr}}) \supset 1.$$

$I$ is called the inertia and $P$ is called the wild inertia. The quotients $G_{\mathbb{Q}_p}/I = G_{\mathbb{F}_p}$ and $I/P = \varprojlim_{p\nmid n}\mu_n(\bar{\mathbb{F}}_p)$ are pro-cyclic and $P$ is a huge pro-$p$ group. Take an isomorphism $\varprojlim_n \mu_{\ell^n} \to \mathbb{Z}_\ell$ and let $t_\ell : I \to \mathbb{Z}_\ell$ denote the composition. Also take a lifting $F \in G_{\mathbb{Q}_p}$ of $Fr_p$. The inverse image $W_{\mathbb{Q}_p} = \langle F, I\rangle$ of $\langle Fr_p\rangle \subset G_{\mathbb{F}_p}$ is called the Weil group.

We assume $\ell \neq p$. The $p$-adic Hodge theory deals with the case $\ell = p$ (Faltings's Kuwait lecture on 28 October 2003, Fontaine's Kuwait lecture on 26 February 2003)

Semi-simplicity conjecture.(Tate) The action of $F$ on $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is semi-simple.

Monodromy theorem.(Grothendieck) Let $\ell$ be a prime number different from $p$ and $\rho : G_{\mathbb{Q}_p} \to GL_n(\mathbb{Q}_\ell)$ be a continuous representation. Then, there exists a pair of representation $\rho' : W_{\mathbb{Q}_p} \to GL_n(\mathbb{Q}_\ell)$ and a nilpotent endomorphism $N \in M_n(\mathbb{Q}_\ell)$ such that $\rho(F^n\sigma) = \rho'(F^n\sigma)\exp(t_\ell(\sigma)N)$.

$\rho$ is uniquely determined by $(\rho', N)$. $\rho'$ is determined by Tr $\rho'$ upto semi-simplification.

Monodromy filtration: For $N$ an nilpotent endomorphism of $V$ $(N^{n+1} = 0)$, the filtration $W_rV = \sum_{p-q=r} \mathrm{Ker}N^{p+1} \cap \mathrm{Im}N^q$ is the unique increasing filtration satisfying the following property:

$N(W_rV) \subset W_{r-2}V$ for all $r \in \mathbb{Z}$, $W_nV = V, W_{-n-1}V = 0$, and the induced map $N^r : Gr_r^W V \to Gr_{-r}^W V$ is an isomorphism for $r \geq 0$.

Monodromy-weight conjecture:(Deligne) The eigenvalues of $F$ on $Gr_r^W H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ are of weight $m+r$. Namely are an algebraic integer and their complex absolute values are $p^{\frac{m+r}{2}}$.

MWC is an analogue of the Weil conjecture for a variety over a local field. MWC is know if $m \leq 2$. MWC implies that $N$ is determined by $\rho'$. Further Semi-simplicity conjecture implies that $\rho'$ on $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$ is determined by Tr $\rho =$ Tr $\rho'$ on $H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$.

Theorem 1. Assume WMC and further assume that the projectors to the Künneth components are algebraic. Then, $P_p(H^m(X), t)$ and $f_p$ are well-defined. More precisely, the function Tr $(\sigma, (\mathrm{Ker}N :)H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$ on $\sigma \in W_{\mathbb{Q}_p}$ is $\mathbb{Q}$-valued and is independent of $\ell$.

Proof. Alteration and the weight spectral sequence (Steenbrink-Rapoport-Zink).

(ii) Conductor.

$$f_p = \dim H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) - \dim H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} + \mathrm{Sw}_p H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell).$$

Take a regular proper model $X_{\mathbb{Z}}$ and put

$$\mathrm{Art}_p(X) = \chi(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) - \chi(X_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) + \sum_{m=0}^{2d}(-1)^m \mathrm{Sw}_p H^m(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell).$$

$\mathrm{Sw}V = \sum_v v \times \dim V^{G^{v+}}/V^{G_v}$ $G_v$ filtration by ramification groups. $\mathrm{Sw}V = 0$ if and only if $P$ acts trivially on $V$.

Theorem 2 (Kato-T). If the closed fiber $X_{\mathbb{F}_p}$ has normal crossings as a divisor of $X$, we have

$$\mathrm{Art}_p(X) = \deg(-1)^d c_{d+1}{}^X_{X_{\mathbb{F}_p}}(\Omega_{X/\mathbb{Z}}).$$

The right hand side is the degree of a 0-cycle class supported on the closed fiber. Theorem 2 is conjectured by S. Bloch without the extra assumption. A generalization of the conductor-discriminant formula in algebraic number theory. The Tate-Ogg formula for an elliptic curve is a special case.

Tomorrow: A related formula in a more geometric setting.