not familiar
microlocal analysis in algebraic geometry
ℓ-adic, mod ℓ

Characteristic cycle

X smooth / k perfect. $\dim X = n$.
Λ finite field char ℓ inv. in k
$\mathcal{F}$ constructible complex of Λ-mod on X.

$\operatorname{supp} \mathcal{F} \subset X$ closed subset
Beilinson $SS\,\mathcal{F} \subset T^*X$ singular support $\dim SS\mathcal{F} = n$
closed conical subset of the cotangent b'dle $\dim T^*X = 2n$
stable under $\mathbb{G}_m$-action

$SS\mathcal{F} = \bigcup C_a$. C_a irred $\dim n$. $SS\mathcal{F} \cap T^*_X X = \operatorname{supp}\mathcal{F}$
$CC\mathcal{F} = \sum m_a C_a$. $m_a \in \mathbb{Z}$. characteristic cycle.
$\mathcal{F}$ perverse $\Rightarrow$ $m_a > 0$ $\forall a$.

1. Classical example.
2. Characteristic class.
3. Direct image.

Example. $\dim X = 1$. $D \subset X$ $\mathcal{F}|_{X-D}$ locally const $\neq 0$ (smallest)

$$SS\mathcal{F} = \underset{\text{0-section}}{T^*_X X} \cup \underset{\text{fiber}}{\bigcup_{x \in D} T^*_x X}$$

$$CC\mathcal{F} = -\left(\operatorname{rk}\mathcal{F}|_{X-D} \cdot T^*_X X + \sum_{x \in D} a_x\mathcal{F} \cdot T^*_x X\right)$$

$a_x\mathcal{F}$ Artin conductor
$= \operatorname{rk}\mathcal{F} - \operatorname{rk}\mathcal{F}_{\bar{x}} + \operatorname{Sw}_x\mathcal{F}$ Swan conductor.

not Lagrangean in higher dim.

2. Direct image

$f: X \to Y$ proper X, Y smooth $n = \dim X$, $m = \dim Y$

$\mathcal{F}$ on X $Rf_*\mathcal{F}$ on Y.

$SS\mathcal{F}, CC\mathcal{F}$ on T^*X. $SSRf_*\mathcal{F}, CCRf_*\mathcal{F}$ on T^*Y

$$T^*X \leftarrow X \times_Y T^*Y \xrightarrow{\text{proper}} T^*Y$$

$SS\mathcal{F}$ inv image $f_\circ SS\mathcal{F}$ conical closed subset

$CC\mathcal{F}$ " $f_! CC\mathcal{F}$

in the sense of intersection theory

$CH_m(f_\circ SS\mathcal{F})$ group of algebraic cycles / rational equiv.

$\ni f_! CC\mathcal{F}$.

Conjecture 1. Assume $f: X \to Y$ is proper. Then we have

$$CCRf_*\mathcal{F} = f_* CC\mathcal{F} \quad \text{in } CH_m(f_\circ SS\mathcal{F}).$$

Theorem 1. Assume X, Y proj. $f: X \to Y$ proj & $\dim f_\circ SS\mathcal{F} \leq m$

Then Conj 1 is true. i.e. we have

$$CCRf_*\mathcal{F} = f_* CC\mathcal{F}$$ as algebraic cycles (w/o rat. equiv.)

Examples 1. $Y = \operatorname{Spec} k$ $\dim = 0$ always

$$\chi(X_{\bar k}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$$ index formula

Further if $\dim X = 1$. Grothendieck-Ogg-Shafarevich.

$$\chi(X_{\bar k}, \mathcal{F}) = \mathrm{rk}\mathcal{F} \cdot \chi(X_{\bar k}) - \sum_{x \in D} a_x \mathcal{F} \cdot \deg x$$

2 $\dim Y = 1$. for each closed pt $y \in Y$.

$$-a_y Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, X_y}$$ cond. formula

$$\chi(X_{\bar\eta}, \mathcal{F}) - \chi(X_{\bar y}, \mathcal{F}) + \mathrm{Sw}_y H^*(X_{\bar\eta}, \mathcal{F})$$

$\mathcal{F} = \Lambda$. Conductor formula conjectured by Bloch in '80s

diagram ~~flow chart~~ of pf GOS $\Rightarrow$ index formula $\Rightarrow$ $\dim Y = 1$ $\Rightarrow$ Th.

3 Characteristic class

X may be singular. $i : X \to M$ closed imm. M smooth dim N

$CC\, i_* \mathcal{F} = \sum m_a C_a \quad C_a \subset X \times_M T^*M \quad \dim C_a = N.$

projective completion

$$\bar{C}_a \subset \mathbb{P}(X \times_M T^*M \oplus \mathbb{A}^1_X)$$

$$\sum m_a \bar{C}_a \in CH_N(\mathbb{P}(\text{———}))$$

$$cc_X \mathcal{F} = \bigoplus CH_q(X) = CH_\cdot(X)$$

characteristic class indep.t of M or i.

additive in $\mathcal{F}$

$K(X, \Lambda)$ Grothendieck gp of constr sheaves of Λ-mod $/X$

$$cc_X : K(X, \Lambda) \to CH_\cdot(X)$$

A question of Grothendieck in SGA5. Recolte et Semaille

For $f : X \to Y$, one has

$$\begin{array}{ccc} K(X, \Lambda) & \xrightarrow{cc_X} & CH_\cdot(X) \\ Rf_* \downarrow & \circlearrowleft & \downarrow f_! \\ K(Y, \Lambda) & \xrightarrow{cc_Y} & CH_\cdot(Y) \end{array} \quad ?$$

Chan ○ MacPherson.

easy counter example in Japanese translation)

Conjecture 2. Replace $CH_\cdot$ by CH_0.

Conj 1 $\Rightarrow$ Conj 2.

Umezaki, Yang, Zhao OK if X, Y proj smooth / k, finite