1. Cotangent space
2. Results
3. Proofs

1 K discrete valuation ~~…~~ field F residue field char >0

$S = \mathrm{Spec}\, O_K$

Cotangent space of S.

Example k perfect field of ch $p>0$

X smooth $/k$ $D \subset X$ smooth divisor $\xi \in D$ gen pt

$O_K = O_{X,\xi}$

$(T^*X)_\xi = \Omega^1_{X,\xi} \otimes_{O_{X,\xi}} k(\xi)$

$0 \to T^*_D X \to (T^*X \times_X D)_\xi \to T^*D \to 0$ exact

$0 \to m_K/m_K^2 \to \Omega^1_{O_K} \otimes_{O_K} F \to \Omega^1_F \to 0$

works if p is not a unif

general $\bar F$ alg closure

$L_{\bar F/S}$ cotangent complex acyclic except at deg -1

$$0 \to m_K/m_K^2 \otimes_F \bar F \to H_1(L_{\bar F/S}) \to \Omega^1_F \otimes_F \bar F \to 0$$

$$\| $$

$$T^*_{\bar F} S$$

K discrete valuation field
F residue field char $F = p > 0$
not necessarily perfect
$\bar{F}$ alg closure
$S = \mathrm{Sp}\, O_K \leftarrow \mathrm{Sp}\, \bar{F}$
$L_{\bar{F}/S}$ cotangent complex
acyclic except at degree -1
$0 \to m_K/m_K^2 \otimes_F \bar{F} \to H_1(L_{\bar{F}/S}) \to \Omega^1_F \otimes_F \bar{F} \to 0$

cotangent space of S at $\bar{F}$ $T^*_{\bar{F}} S$

K'/K ext of d v f $\bar{F}' / \bar{F}$
We say K'/K is cotangentially unramified if

$$S^{\cdot}(T^*_{\bar{F}} S) \to S^{\cdot}(T^*_{\bar{F}'} S')$$

is an injection
For $\forall K$ $\exists K'$ cot. unr. F' perfect
formally smooth $\Rightarrow$ cotang. unr $\Rightarrow$ $e = 1$

finite
Thm 1 For any Galois extension of henselian discrete valuation fields L over K of Galois group G, there exists a unique way to define a decreasing filtration $(G^r)_{r \geq 1}$ by normal subgroups, satisfying the following conditions
(1) If F is perfect $G^r = G_{cl}^{r-1}$
(2) If K'/K is cotangentially ramified, for $L' = LK'$, the can. map $G' = \mathrm{Gal}(L'/K') \to G$ induces an ism $G'^r \to G^r$ for every $r \geq 1$

~~Cor 1. G^1 = I, G^{1+} =~~

$$G^{r+} = \bigcup_{s>r} G^s, \quad Gr_r G = G^r/G^{r+}$$

Cor 1. 1. $G^1 = I$, $G^{1+} = P$.

2. $\exists\ 1 = r_0 < r_1 < \cdots < r_n$ s.t.
G^r is constant on $(r_{i-1}, r_i]$ $i = 1, \ldots, n-1$
& (r_n, ∞) $s.t.$

3. For $r > 1$ $Gr^r G$ is an $\mathbb{F}_p$-v.sp

C field of ch $\neq p$
V rep of G on C-v.sp of fin dim
$\exists$ unique decomposition
$$V = \bigoplus V^{(r)} \qquad V^{G^r} = \bigoplus_{s<r} V^{(s)}$$

$$\dim\text{tot}\, V = \sum_r r \times \dim V^{(r)}$$

Cor 2. $\dim\text{tot}\, V \in \mathbb{N}$.

Theorem 2. For $r > 1$ known cases There exists a can. inj

$$\mathrm{Hom}(Gr^r G, \mathbb{F}_p) \to \mathrm{Hom}(m_K^r/m_K^{r+}, T^*_{\bar F} S)$$

characterization similar to Th 1.

Pf. uniqueness

cot. unr ext'n of

Lemma K h.d.v.f. $\exists K'/K$ h.d.v.f

s.t. F' perfect &

existence. Abbes - S. (1), (2)

Prep K'/K cot. unr $\Rightarrow$ $G'^{r} \to G^{r}$ ~~[illegible]~~ surj

Lemma k'/k ext of alg closed fd of char $p>0$

E, E' k v-sp (resp k' v-sp) of f-dim

$E' \to E \otimes_k k'$ k'-linear TFAE

(1) $\pi_1(E', 0)_{\text{pro-}p} \to \pi_1(E, 0)_{\text{pro-}p}$ is surj

(2) $S^\circ(E^\vee) \to S^\circ(E'^\vee)$ is inj

L/K Galois

$Q \xleftarrow[\text{imm}]{} T = \mathrm{Spf}\, \mathcal{O}_L \qquad Q = \mathrm{Spf}\, A \qquad \mathcal{O}_L = A/I.$

smooth

$S = \mathrm{Spf}\, \mathcal{O}_K$

$r>0$ rational. $e \geq 1$ integer er integer

K'/K finite ramified ext. $e = e_{K'/K}$.

$$Q_{S'} = \mathrm{Spf}\, A \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{K'} = \mathrm{Spf}\, A' \leftarrow Q_{S'}^{[er]} = \mathrm{Spf}\, A'\left[\frac{I}{\pi'^{er}}\right]$$

$$\leftarrow Q_{S'}^{(er)}$$

normalization

RFT $Q_{S'}^{(er)} \otimes_{S'} \bar{F}$ reduced for K' suff large

indep't of such K'

$Q_{\bar{F}}^{(r)} \to Q_{\bar{F}}^{[r]}$ $= (Q_{S'}^{[er]} \times_{S'} \mathrm{Spf}\, \bar{F})_{\text{red}}$.

$G^{v+} = 1 \iff Q_{\bar{F}}^{(v)} \to Q_{\bar{F}}^{[v]}$ finite étale

$\Rightarrow Q_{\bar{F}}^{(v)0} \to Q_{\bar{F}}^{[v]}$ $G^v = G_v G$-torsor

$$\overset{\text{can.}}{\cong} \operatorname{Hom}_{\bar{F}}(m_{\bar{K}}^{v}/m_{\bar{K}}^{v+}, N_{T/Q} \otimes_{O_L} \bar{F})^{\vee}$$

$0 \to N_{T/Q} \to \Omega^1_{Q/S} \otimes O_T \to \Omega^1_{T/S} \to 0$

$\cdot \operatorname{Tor}_1^{O_L}(\Omega^1_{T/S}, \bar{F}) \to N_{T/Q} \otimes_{O_L} \bar{F}$ inj (isom if $T \to Q$ minimal

$\downarrow$ inj

$T^*_{\bar{F}} S$

Lemma

cot. ann $\Rightarrow \pi_1(Q_{\bar{F}}^{\prime (v)}) \xrightarrow[\text{prop}]{} \pi_1(Q_{\bar{F}}^{[v]})$ surj

$$\begin{array}{ccc} \pi_1(Q_{\bar{F}}^{\prime (v)}) & \to & \pi_1(Q_{\bar{F}}^{[v]}) \\ \downarrow & & \downarrow \\ G'^v & \twoheadrightarrow & G^v \end{array}$$

Pf of Thm 2

Lemma Let V be a k v.sp of f.d., k alg closed ch $p>0$.
G finite gp. $1 \to G \to W \to V \to 1$.
exact seq of smooth comutative gp schm, W conn.
$\Rightarrow G$ is an $\mathbb{F}_p$-v.sp. and
$[W]: \operatorname{Hom}(G, \mathbb{F}_p) \to \operatorname{Ext}(V, \mathbb{F}_p) \cong \operatorname{Hom}(V, k)$
is an inj

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{F}_p & \to & * & \to & V & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{F}_p & \to & k & \to & k & \to & 0 \end{array}$$

Suffices to shw $Q_{\bar{F}}^{(v)0}$ has a str of gp sch
s.t. $Q_{\bar{F}}^{(v)0} \to Q_{\bar{F}}^{[v]}$ is a morphism of gp sch