Wild ramification:

Goal:

Grothendieck Ogg-Shafarevich formula:

$$\chi_c(U,\mathcal{F}) = \mathrm{rk}\,\mathcal{F}\,\chi_c(U) - \sum_{x\in X\setminus U} \mathrm{Sw}_x \mathcal{F} \qquad k=\bar{k}$$

ch $p>0$, U smooth curve, X = smooth compactification $\ell \neq p$ prime numbers. $\mathcal{F}$ smooth ℓ-adic sheaf. We have étale $H^i_c(U,\mathcal{F})$ finite dimensional, 0 if $i\neq 0,1,2$. $\chi_c(U,\mathcal{F}) = \sum_{i=0}^{2} \dim H^i_c(U,\mathcal{F})^{(-1)^i}$

$\chi_c(U) = \chi_c(U,\mathbb{Q}_\ell) = 2-2g-\#(X\setminus U)$

Generalizations and Variants: to higher dimension, $\chi_c(U,\mathcal{F})$ is an integer.. but we have refinements... characteristic class. Generalization of Euler number, etc...

Methods: 1) traditional: (you kill ramification the wild one, by taking a ramified covering)

$\mathcal{F}$ a smooth sheaf:

$\pi : V \longrightarrow U$ finite étale Galois covering such that $A_V = \pi^* \mathcal{F}$ has tame ramification along the boundary.

(This method works to define Swan class = generalisation of Swan conductor to higher dimension).

Ramification groups: they define it on the Galois group.

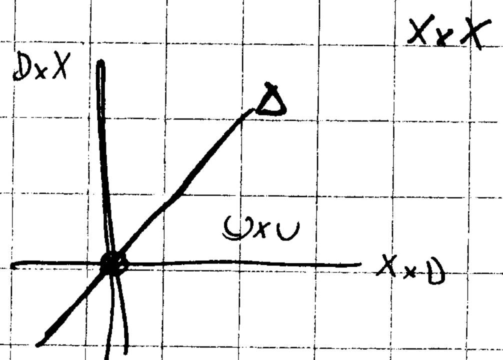
- lower numbering filtration.

2) New method (*). We kill ramification by blow-ups.

How does it works? We work with: $\Delta_U \subset U \times U \hookrightarrow X \times X$

$\mathcal{H} = \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^* \mathcal{F}) \Rightarrow \mathcal{H}_{|\Delta_U} = \mathcal{H}_{|U} = \mathcal{E}nd(\mathcal{F})$.

Here we have global sections: $id_{\mathcal{F}}$



$id_{\mathcal{F}}$ has ramification only at the origin! we can go close to 0 to kill ramification!

3) 3 method.

Characteristic class of cycles. This is based on upper numbering filtration.

Objects: I) arithmetic (= mixed characteristic). Base field number field or local fields, variety over it

II) Geometric case = equal characteristic case. Fix a perfect field k and study varieties over k. We focus on II.

We start with upper numbering filtration and the graded pieces of the filtration and Euler-Numbers ~~...~~ using upper numbering

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K local field, (complete DVR) $F = \mathcal{O}_K / \mathfrak{m}_K$ residue field.

May be not perfect.

Example of geometric case: k alg closed field, $K = k(T_2, \dots, T_d)((T_1))$

L/K finite Galois extension, $G = \mathrm{Gal}(L/K)$

Filtration by ramification groups, $G = Gal(L/K)$. We have

1) lower numbering filtration $\in \mathbb{Z}$: easy to define, compatible with the subgroups.

2) upper numbering filtration $\in \mathbb{Q}$: More difficult to define. But it is compatible with quotient

|| they are logarithm-variant!!

In classical case (residue field perfect) lower and upper: = upper and lower are equal up to renumbering : Herbrandt functions. In general case.. they have completely $\neq$ way of working!

1) $G_i = \mathrm{Ker}\,(G \longrightarrow \mathrm{Aut}\,(\mathcal{O}_L/\mathfrak{m}_L^i))$ $\quad i \geq 0$

The logarithmic version

$$G_{i,\log} := \mathrm{Ker}\,(G \longrightarrow \mathrm{Aut}\,(L^\times/1+\mathfrak{m}_L^i))$$

$i =$

$$G_i = \ker\left(G \longrightarrow \mathrm{Aut}\left(\mathcal{O}_L/\mathfrak{m}_L^i\right)\right)$$

$$G_{i,\mathrm{lg}} = \ker\left(G \longrightarrow \mathrm{Aut}\left(\mathcal{O}_L^\times/1+\mathfrak{m}_L^i\right)\right)$$

$$G_1 = \ker\left(G \longrightarrow \mathrm{Aut}\left(\mathcal{O}_L/\mathfrak{m}_L\right)\right) = I$$

Ex:

$$\begin{array}{ccccccc} G_1 & \supset & G_2 & \supset & G_3 & \supset \dots & = 1 \\ \cup & \shortparallel & \cup & \shortparallel & & & \\ G_{1,\mathrm{lg}} & \supset & G_{2,\mathrm{lg}} & \supset & G_{3,\mathrm{lg}} & & = 1 \end{array}$$

More precisely:

$$G_{1,\mathrm{lg}} = \ker\left(G_1 \longrightarrow \left(\mathcal{O}_L/\mathfrak{m}_L\right)^\times\right) \qquad i=1$$

$$\sigma \longmapsto \sigma(\pi_L)/\pi_L$$

$$G_{i,\mathrm{lg}} = \ker\left(G_i \longrightarrow \mathfrak{m}_L^{i-1}/\mathfrak{m}_L^i\right) \qquad i>1$$

$$\sigma \longmapsto \sigma(\pi_L)/\pi_L - 1$$

$$G_{i+1} = \ker\left(G_{i,\mathrm{lg}} \to \mathrm{Hom}_{\mathcal{O}_L}\left(\Omega_{\mathcal{O}_L/\mathcal{O}_K}, \mathfrak{m}_L^i/\mathfrak{m}_L^{i+1}\right)\right)$$

vector space over $\mathcal{O}_L/\mathfrak{m}_L$

$$\sigma \longmapsto \quad da \longmapsto \sigma(a)-a$$

<u>Now:</u> $\overset{I}{\overset{\shortparallel}{G_1}}/\underset{P}{G_{1,\mathrm{lg}}}$ cyclic group of order <u>prime to p</u>.

$G_{i,g}$ = p-Sylow group of I

Geometric interpretation of the definition of G_i:

$$\mathcal{O}_L = \mathcal{O}_K[T_1 \dots T_n] \big/ (f_1 \dots f_n)$$

$\#f = \#T$
complete intersection
algebra of rel. dim $= 0$
(regular)

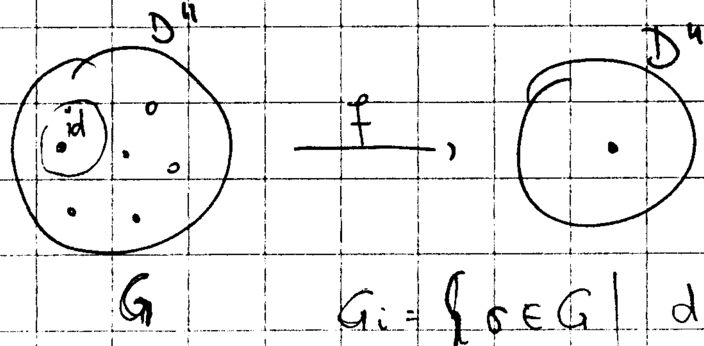
D^n = n-dimensional polydisk. The $\bar{K}$ valued points

$$\{(x_1 \dots x_n) \in \bar{K} \mid v(x_i) \geq 0\} = D^n = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}\left(\mathcal{O}_K[T_1 \dots T_n], \mathcal{O}_{\bar{K}}\right)$$

We define: $f : D^n \longrightarrow D^n$, $(x_1 \dots x_n) \longmapsto (f_1(\underline{x}) \dots f_n(x))$. We take $G = f^{-1}(0)$

$$G = f^{-1}(0) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_L, \mathcal{O}_{\bar{K}}) = G = \mathrm{Gal}(L/K)$$

f = finite, flat, locally fre



subgroup compatible
depends on (

$$G_i = \{\sigma \in G \mid d(\sigma, id) \leq |\pi_L^i|\}$$

Upper numbering? Small disks on the target!

$$V_r = f^{-1}(D(r,0)) = \{ x \in D^n \mid d(f(x),0) \leq |\pi_K|^r \}$$ compatible with quotient $\Rightarrow \pi_K$

$G^r = \{ \sigma \in G \mid \sigma$ lies on the same connected component of V_r as id $\}$

· Ramification parts of f are difficult.

Example: $(n=1)$ $\mathcal{O}_L$ = monogenetic (e.g. $\mathcal{O}_L/\mathfrak{m}_L$ is separable over $F = \mathcal{O}_K/\mathfrak{m}_K$)

Ex. V_r = disjoint union of finitely many subdisks

$\mathcal{O}_L = \mathcal{O}_K[T]/f(T)$ $\quad f(T) = \prod_{\sigma \in G} (T - \sigma(\alpha))$ in $\mathcal{O}_{\bar{K}}[T]$

And the radius (depending on r) and it is the Herbrandt functions.

(All the radius are equals... it is Galois...) (It is not necessarily of Galois!) (Interpolability is in Hasse-Arf??) (For all characteristic! For $\bar{K}$!!)

Rigid Geometry $\qquad\qquad$ Alg. Geometry

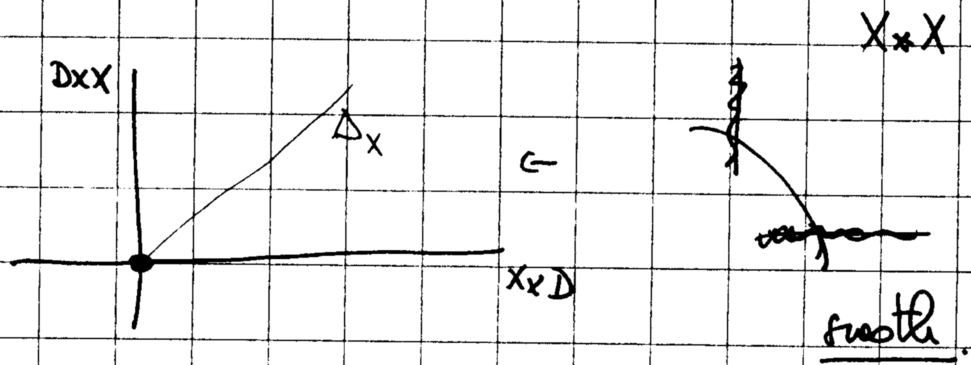
Shrinking Radius $\longleftrightarrow$ blow-ups.

We suppose K has geometric origin $\mathcal{O}_K = \hat{\mathcal{O}}_{X,\xi}$ $\quad$ X smooth $/k$ perfect

$D \subseteq X$ smooth irreducible divisor, $\xi \in D$ generic point. It is a DVR.

We take $\underset{\substack{\cup\\ D\times D}}{X\times X} \longleftarrow X * X$ $\quad$ blow-up $D\times D$, remove the

proper transform of $X\times D$ & $D\times X$



Log-product $X * X$ (by blow-ups)

$D\times D \cap \Delta_X = \Delta_D \subset \Delta_X$ is Cartier Divisor

So if we blow-up Δ_X along $\Delta_D \Rightarrow \Delta_X^{log} \cong X$ we don't

do anything.

$$\begin{tikzcd}
\pi^{-1}(D)_{red} = E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{finite flat}"] & Y \arrow[d] \arrow[dr, phantom, "\square"] & \supset V \arrow[d, "\text{finite étale } G\text{-torsor}"] \arrow[dr, phantom, "\square"] & \arrow[l, hook'] \operatorname{Spec} L \arrow[d] \\
D \arrow[r, hook] & X & \supset U = X \setminus D & \arrow[l, hook'] \operatorname{Spec} K
\end{tikzcd}$$

E is smooth over k, $G = \operatorname{Gal}(L/K)$ (equi characteristic case)

D = <u>smooth divisor</u>.

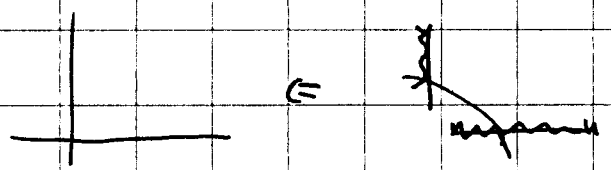
Saito $\boxed{\text{III}}$

$$\begin{tikzcd}
P \arrow[d] & \arrow[l, hook'] U \times K \arrow[d] \\
S = \operatorname{Spec} \mathcal{O}_K \arrow[u, bend left] & \arrow[l, hook'] \operatorname{Spec} K
\end{tikzcd}$$

$\mathcal{O}_K = \hat{\mathcal{O}}_{X,\xi}$

$U \subseteq X \supset D \ni \xi$

We have



We define <u>upper-numbering filtration</u>: L/K finite Galois, $G = \operatorname{Gal}(L/K)$

$G^r_{\log}$. We have to make operation in order to shrink the neighborhood ... How?

$r > 0$ rational,

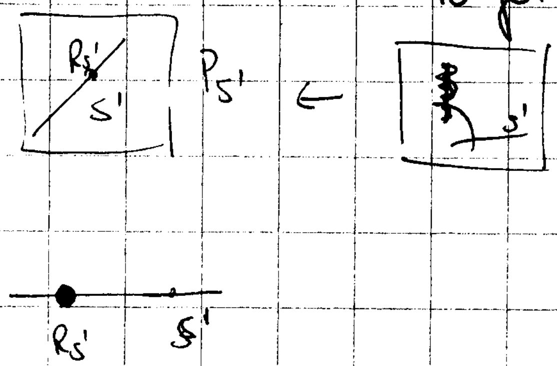
We need to kill denominators, K'/K finite separable extension

$e := e_{K'/K}$ ramification index, assume er = integer

$$\begin{tikzcd}
P \arrow[d, dash] & P_{S'} \arrow[l] \arrow[d] \\
S & S' = \mathrm{Spec}\, \mathcal{O}_{K'} \arrow[l] \supset R_{S'} = \mathrm{Spec}\, \mathcal{O}_{K'}/\mathfrak{m}_{K'}^{er}
\end{tikzcd}$$

Blow-ups $P_{S'}$ at $R_{S'}$ and remove the proper transform of the closed fiber to get $\underline{P}_{S'}^{(2)}$

S^1 is defined by
$U_1 = 1$
$S_2 = T_2$
$\vdots$
$S_d = T_d$



we don't touch generic fiber!

$$\begin{tikzcd}
\Theta^2_{F'} \arrow[d] & \underline{P}_{S'}^{(2)} \arrow[d] & U \times K' \arrow[l] \arrow[d] \\
\mathrm{Spec}\, F'_1 \arrow[r] & S' & \mathrm{Spec}\, K' \arrow[l]
\end{tikzcd}$$

$\mathrm{Spec}\, F'_1 = \mathrm{Spec}\, \mathcal{O}_{K'}/\mathfrak{m}_{K'}$

- $\Theta^2_{F'}$ is a vector bundle -- affine.. it is the projective space minus a hyperplane at ∞. Hence a vector space!

$$\Theta^{?}_{F'} = \mathrm{Spm}\, S^\circ\left(\mathrm{Hom}_{F'}\left(\mathfrak{m}_{k'}^{ez}/\mathfrak{m}_{k'}^{ez+1}, \Omega^1_X\langle \log D\rangle \otimes F'\right)\right) \simeq \mathbb{A}^d_{F'}$$

<u>Example</u>: $X = \mathbb{A}^d_k \supset D = (T_1 = 0)$

$\mathcal{O}_K = k[T_1 \dots T_d][[T_1]]$ $\qquad P = \mathrm{Spec}\, \mathcal{O}_K[U^{\pm 1}, S_2, \dots S_d]$

$$P_{S'} = \mathrm{Spm}\, \underbrace{\mathcal{O}_{K'}[U^{\pm 1}, S_2 \dots, S_d]} = A$$

$$P^{(2)}_S = \mathrm{Spm}\, A\left[\underset{\substack{\| \\ V_1}}{\frac{U_1 - 1}{\pi_{k'}^{ez}}}, \underset{\substack{\| \\ V_2}}{\frac{S_2 - T_2}{\pi_{k'}^{ez}}}, \dots \underset{\substack{\| \\ V_d}}{\frac{S_d - T_d}{\pi_{k'}^{ez}}}\right] \quad \pi_{k'} \text{ uniformizer of } k$$

$$\simeq \mathrm{Spm}\, A\left[V_1, \dots V_d, \frac{1}{1 - \pi_{k'}^{ez} V_1}\right]$$

//

L/k Galois extension, G $\qquad$ by base change ..

$$\begin{tikzcd} \mathrm{Spec}\, \mathcal{O}_L/\mathfrak{m}_L \arrow[r, hook] \arrow[d] & T = \mathrm{Spm}\, \mathcal{O}_L \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}\, L \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}\, F \arrow[r] & \mathrm{Spec}\, \mathcal{O}_K & \mathrm{Spec}\, K \arrow[l] \end{tikzcd} \Rightarrow \begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[l, hook'] \arrow[d] \\ D \arrow[r] & X & U \arrow[l, hook'] \end{tikzcd}$$

$\times S$

finite étale Galois with Galois group G

$$\begin{array}{ccccc} T & \longrightarrow & Q & \hookleftarrow & V_K \\ \downarrow & \square & \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{P} & \hookleftarrow & U_K \end{array}$$

Q resolution

Algebraic version of

$$\begin{array}{ccc} f^{-1}(0) & \longrightarrow & D^n \\ \downarrow & \square & \downarrow f \\ 0 & \longrightarrow & D^n \end{array}$$

We have:

$$\begin{array}{ccccc} \boxtimes & \longrightarrow & \overset{(2)(\text{resolution})}{Q_{S'}} & \longleftarrow & V_{K'} \\ \downarrow & \square & \downarrow & & \downarrow \\ S' & \longrightarrow & \mathcal{P}^{(2)}_{K'} & \longleftarrow & U_{K'} \end{array}$$

condition on K' i) $e_L \in \mathbb{N}$ $e_{K'}$

ii) $L \subset K'$ (arbitrary large!)

because of this, we have: $\boxtimes = \bigsqcup_{\sigma \in G} S'$

iii) $Q^{(2)}_{\overline{F}} \overset{def}{=} Q^{(2)}_{S'} \times_{S'} \operatorname{Spec} \overline{F}$ is reduced!

Epp's theorem $\Rightarrow \exists K'$ (large enough) to get (iii) is ok.

If we enlarge K' we get more. Grauert-Remmert:

As translated by Lut-Bosch-Raynaud, after enlarging $Q^{(2)}_{\bar{F}}$ is
satisfactory on K'. (And it is reduced...)

In this way we get reduction map

$$G \longrightarrow Q^{(2)}_{\bar{F}}$$

$$G^{2}_{\log} = \{ \sigma \in G \mid \text{the image of } \sigma \text{ in } Q^{(2)}_{\bar{F}} \text{ lies in the same connected component of id} \}$$

__Example__: L/K cyclic __extension__ of deg p, $t^p - t = a$, $a \in K$

$$H^1(G, \mathbb{F}_p) \simeq K/\wp(K) \qquad \wp(b) = b^p - b$$

$$\chi_a \leftarrow a$$

$$\mathrm{Fil}^n = \text{Image of } \mathfrak{m}_K^{-n}$$

$\chi_a \in \mathrm{Fil}^n \notin \mathrm{Fil}^{n-1}$ $\quad \mathrm{ord}(a) = -n \qquad G^r_{\log} = 0 \iff \underline{r > n}$

$/\!/ \quad /\!/$

We defined filtration.. we want to study graded parts.. on the
last step of filtration -- it is compatible __with quotient..__
so it is equivalent (?)

Last jump is the "conductor" : r such that

$G^r_{log} \neq 0$ and $G^s_{log} = 0 \quad \forall s > r$. This last piece

is elementary abelian p-group. ($\simeq \mathbb{F}_p^n$). How to describe?

By character group. it has a local definition.

$$\mathrm{Hom}(G^r_{log}, \mathbb{F}_p) \xrightarrow[\text{injection}]{\text{"refinement"}} \mathrm{Hom}_{\bar{F}}\Big(\mathfrak{m}^r_{\bar{K}} / \mathfrak{m}^{r+}_{\bar{K}}, \underbrace{\Omega^1_X \langle \log D \rangle_s \otimes_{\mathcal{O}_{X,s}} \bar{F}}_{\circledast}\Big)$$

$\bar{K}$ = separable closure, $\quad \mathfrak{m}^r_{\bar{K}} = \{ a \in \bar{K} \mid \mathrm{ord}(a) \geq r \}$

$\mathfrak{m}^{r+}_{\bar{K}} = \{ a \in \bar{K} \mid \mathrm{ord}(a) > r \} \quad \mathrm{ord}\, \pi_K = 1$

$\bar{F}$ residue field of $\bar{K}$

$\mathfrak{m}^r_{\bar{K}} / \mathfrak{m}^{r+}_{\bar{K}}$ = has dim $= 1$ over $\bar{F}$

$\circledast$ It is of dim $= d$.

<u>AS</u> : Artin-Schreier case. ~~χ_a~~ $\chi_a \in \mathrm{Hom}(G^r_{log}, \mathbb{F}_p)$

a^{-1} basis of $\mathfrak{m}^{-n}_{\bar{K}} / \mathfrak{m}^{-n+}_{\bar{K}} \qquad \mathrm{rsw}(\chi_a) : a^{-1} \longrightarrow d\, a$

Blow ups and differential forms:

$$\begin{array}{ccc} D^n & & P = X *_k S \\ f \downarrow & \rightsquigarrow & \downarrow \\ D^n & & S \end{array}$$

this is not symmetric. It is better to work with $X * X$. and we will love differential forms...

eg

alg. geometry over S, now over k, here

r = ramification $\in \mathbb{Q}$

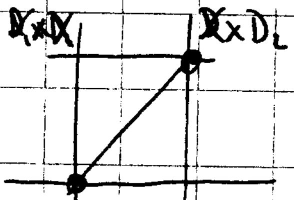
we love more than one component! X smooth over k perfect

$D = \bigcup D_i$ divisor with simple normal crossing.

$R = \sum r_i D_i$ $r_i \geq$ rationals. Simplifying the assumption $r_i > 0$ and integers.

$X \times X \leftarrow X * X$ log products. We blow up $D_i \times D_i$ and we remove proper transform of $D \times X \cup X \times D$

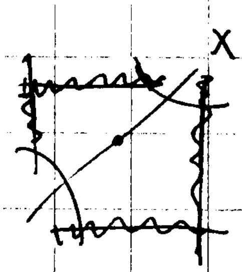
$i = 1, \dots, n$



$\Delta^{\log} : X \hookrightarrow X * X$ log diagonal exact

$R = \sum r_i D_i$

$X*X$:= "?"

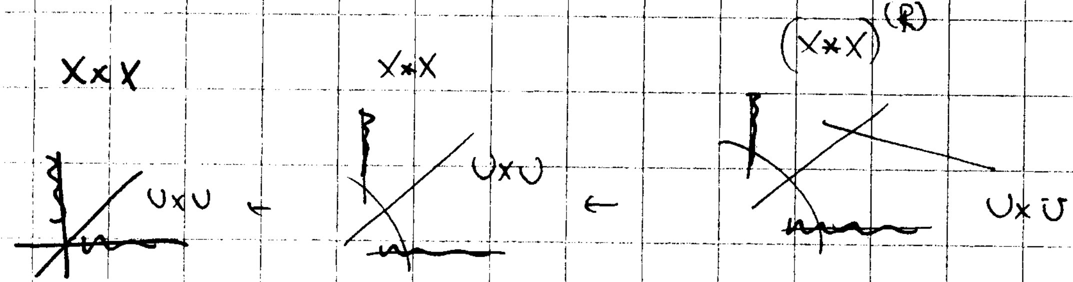


$X \hookrightarrow X*X$ closed since we have $\overset{R}{\sum r_i D_i}$ closed. Blow up at R and we remove the proper transform of $(X*X) \underset{pr_2}{\times_X} D$

$$\begin{array}{ccc} X\times X & \longleftarrow & X*X \\ {\scriptstyle pr_1}\downarrow\downarrow{\scriptstyle pr_2} & & \\ X & & \end{array}$$

. We then get

$$\begin{array}{c} (X*X)^{(R)} \\ \downarrow \\ X*X \end{array}$$



$U \subset X \supset D$

$\boxed{U\times U \text{ is always the same...}}$

$$\begin{tikzcd} \Theta_D^{(R)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (X*X)^{(R)} \arrow[d, "pr_2"] \arrow[dr, phantom, "\square"] & U\times U \arrow[l] \arrow[d] \\ D \arrow[r] & X & U \arrow[l] \end{tikzcd}$$

$\leftarrow$ because we removed strict transforms!!!

$$\Theta_D^{(R)} = \mathbb{V}(\Omega^1_X\langle \log D\rangle \otimes \mathcal{O}_X(R) \otimes_{\mathcal{O}_X} \mathcal{O}_D) \qquad \mathbb{V}(\mathcal{E}) = \operatorname{Spec} S^\bullet \mathcal{E}$$

$$N_{X/X \times X} = \mathbb{R}^2 \qquad N_{X/X * X} = \mathbb{R}^1 \langle \log \rangle \qquad N_{X/(X*X)^{(R)}} \cong \mathbb{R}^1_X \langle \log D \rangle (R)$$

<u>Example</u>: $X = \mathbb{A}^d_k \quad D = (T_1 T_2 \ldots T_n = 0) \quad 1 \leq n \leq d$

$$X * X = \operatorname{Spec} \overbrace{k[U_1^{\pm 1}, \; U_n^{\pm 1}, \; S_{n+1} \ldots S_d, \; T_1 \ldots \; T_d]}^{=A} \qquad \text{log-diagonal}$$

$X \hookrightarrow X * X$ is given by $U_1 = 1, \ldots U_n = 1 \quad S_{n+1} = T_{n+1}$ $\vdots$ $S_d = T_d$

$$(X * X)^{(R)} = \operatorname{Spec} A \left[\frac{U_1 - 1}{T^R}, \; \frac{U_n - 1}{T^R}, \; \frac{S_{n+1} - T_{n+1}}{T^R}, \ldots, \frac{S_d - T_d}{T^R} \right] \quad R \circlearrowleft U \times U$$

$T^R = T_1^{r_1} \ldots T_n^{r_n} \qquad R = \sum r_i D_i$

$$= \operatorname{Spec} k[V_1 \ldots V_d, T_1 \ldots T_d] \left[\frac{1}{1 - T^R V_1}, \ldots \frac{1}{1 - T^R V_n} \right]$$

$$\Theta_D^{(R)} = (T_1 \cdot \ldots \cdot T_n = 0) = \operatorname{Spec} k[V_1 \ldots V_d, T_1 \ldots T_d] / (T_1 \cdots T_n)$$

$\Theta_D^{(R)}$ is a vector bundle ~~over~~ with coordinates $V_1 \ldots V_d$

$T_i = 1 \otimes T_i \quad , \quad V_i = \frac{1}{T^R} dT_i \quad n+1 \leq i \leq d \quad V_i = \frac{1}{T^R} d\log T_i \quad 1 \leq i \leq n$

$X * X \longleftarrow (X * X)^{(R)}$ we love ~~sy~~ symmetry now! We love a groupoid structure.

$$(X \times X) \underset{\substack{pr_2 \searrow \; X \; \swarrow pr_1}}{\times} (X \times X) \simeq X \times X \times X \qquad \text{"looks like"}$$

$$\begin{array}{ccc} X \times X \times X & & G \times G \\ \downarrow p_{13} & & \downarrow \\ X \times X & & G \end{array}$$

but we love pr_2 and pr_1 .. so $\neq$ map hence it is "groupoid structure". Now this groupoid structure p_{13} induces a

$$\mu : (X * X)^{(R)} \underset{\substack{pr_2 \searrow \; X \; \swarrow pr_1}}{\times} (X * X)^{(R)} \longrightarrow (X * X)^{(R)}$$ in fact on

and then we extend!!

$$U \times U \underset{\otimes}{\times} U \times U \longrightarrow U \times U$$ we love this! by restriction.

vector bundle

$$\begin{tikzcd} \Theta_D^{(R)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (X \times X)^{(R)} \arrow[r, hook] \arrow[d, shift right, "pr_1"'] \arrow[d, shift left, "pr_2"] & U \times U \\ D \arrow[r] & X & \end{tikzcd}$$

$$\boxed{pr_1 \equiv pr_2 \text{ on } \Theta_D^{(R)}!}$$

So μ restricted to $\Theta_D^{(R)}$ gives the <u>addition</u> on the <u>vector bundle</u>!

$X \supset U$; $U = X \setminus D$ $\qquad G = $

$V \longrightarrow U$ finite étale galois covering.. How to see its ramification at the boundary? $\qquad G \times G \hookleftarrow \Delta_G$ diagonal subgroup

$$\begin{tikzcd} & V \times V \arrow[dl] \arrow[d] \\ W = {}^{V\times V}/_{\Delta_G} \arrow[r] & U \times U \end{tikzcd}$$

we have

$$\begin{tikzcd} W \times_U W \arrow[r] \arrow[d] & W \arrow[d] \\ U \times U \underset{pr_2 \times pr_1}{\times} U \times U \arrow[r] & U \times U \end{tikzcd}$$

we can extend the groupoid structure on $U \times U$ to W. It is possible

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & Z \arrow[d] & \text{modification} \arrow[l] \\ U \times U \arrow[r, hook] & (X * X)^{(R)} & X \arrow[l, "\Delta^{(R)}"'] \arrow[ul, dashed, "?"'] \end{tikzcd}$$

we would like to have groupoid structure on Z .. this is not possible!! We need conditions! $\qquad X \hookrightarrow (X * X)^{(R)} \qquad \Delta_X^{(R)}$ = diagonal

$$\begin{tikzcd} W = & {}^{V\times V}/_{\Delta G} \\ U \arrow[u, "\Delta"] = & {}^{V}/_{G} \arrow[u, "\Delta"'] \end{tikzcd}$$

we can extend to $X \longrightarrow Z$

Def: We say that the ramification of V over U along the boundary is bounded by R if the map $Z \longrightarrow (X * X)^{(R)}$ is etale in the neighborhood of X in Z

$$Z = \text{" normalization of } (X * X)^{(R)} \text{ in } V \times V \text{ "} / \Delta G$$

$$X = V/G$$

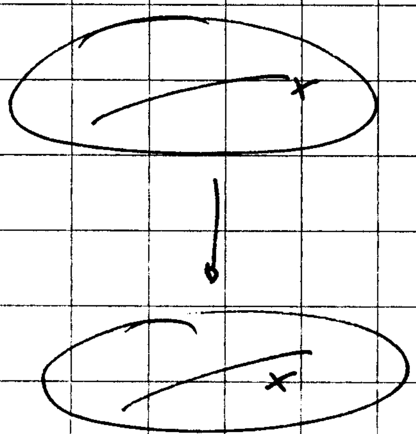
$V \longrightarrow U \subset X \setminus D$ G galois covering

$$\begin{tikzcd} & Z \arrow[d] & \supset W = V \times V / \Delta G \arrow[dr] & \\ X \arrow[ur] \arrow[r, hook] & (X * X)^{(R)} \arrow[rr] & & U \times U \end{tikzcd}$$

resolution

Ramification of $V \to U$ is bounded by $R+$ $(R = \sum r_i R_i)$

if $f : \underset{(*)}{Z} \longrightarrow (X*X)^{(R)}$ is étale in a neigh of $X \hookrightarrow Z$



Z

$(X*X)^{(R)}$

of course we may have $\neq$ components in the inverse image of X, but we have $(*)$, hence ~~one~~.

they are disjoint

$\boxed{V/U \text{ bounded by } R\underset{(\cdot)}{+} \text{ (global)}}$

$\overset{(1)}{\Rightarrow}$ $Z \longrightarrow (X*X)^{(R)}$ is étale at $\xi_i \in D_i \subset X$ $\forall D_i \subset D$

ξ_i generic point of D_i $\overset{(2)}{\Leftrightarrow}$ $\forall i$ L_i / K_i $K_i = \mathrm{Frac}(\mathcal{O}_{X,\xi_i})$

$G_i := \mathrm{Gal}(L_i / K_i) \subseteq G$ decomposition groups ~~we'll have~~

$G^{s}_{i,\log} = 1$ $\forall s \underset{(\cdot)}{\geq} r_i$ $R = \sum_i r_i D_i$

(2)

⇒ "base change"
"⇐" descent.

(1): We have "⇐" if the p-Sylow subgroup of G is a normal subgroup (In particular if G is nilpotent or abelian). It is a consequence of Zariski–Nagata purity. (If unramified in codim $= 1$ then unramified everywhere)

$=$

How to get diff. forms from condition V/U bounded by $R+$ (global

We had:

$$(X * X)^{(R)} \times_X (X * X)^{(R)} \longrightarrow (X * X)^{(R)}$$

from

$$\begin{array}{ccc} Z_0 \times_X Z_0 \longrightarrow Z_0 & \qquad & W \times W \to W \\ \downarrow & & \downarrow \\ (X * X)^{(R)} \times (X * X)^{(R)} \longrightarrow (X * X)^{(R)} & \longrightarrow & (U \times U) \times (U \times U) \end{array}$$

V/U

↑ amply extension

Z_0 = normal open subscheme étale over

$\left(W = \dfrac{V \times V}{\Delta G}\right)$

$(X*X)^{(R)}$ (inside Z). We get a groupoid in Z_0

because pr_2, pr_1 are equal in $\Theta_D^{(R)}$

$$\begin{array}{ccc} Z_{0,D} & \longrightarrow & Z_0 \\ \downarrow & \square & \downarrow \\ \Theta_D^{(R)} & \longrightarrow & (X*X)^{(R)} \\ \downarrow & & \downarrow pr_2 \\ D & \hookrightarrow & X \end{array}$$

Now $Z_{0,D}$ is smooth - group-scheme over D : $Z_{0,D} \to \Theta_D^{(R)}$ is étale morphism of smooth group scheme. ($\Theta_D^{(R)}$ is étale)

We fix $D_i \subset D$ irreducible component, $\xi_i \in D_i$ generic point

$Z_{0,\xi_i} = Z_0 \times_X \xi_i$ smooth group scheme over ξ_i

$\cup$ | commutative

Z^0_{0,ξ_i} connected component of the zero section → smooth group scheme

$$\begin{array}{ccc} & & Z \\ & \nearrow & \downarrow \\ X & \longrightarrow & (X*X)^{(R)} \end{array}$$

We have

commutative — finite étale

$$0 \to G^{r_i}_{i,\log} \to Z^0_{0,\xi_i} \longrightarrow \Theta_D^{(R)} \longrightarrow 0$$

By V/U bounded by $R+$ $\Rightarrow \begin{cases} G^s_{i,\log} = 1 & \text{for } s > r_i \\ G^{r_i}_{i,\log} & \text{abelian} \end{cases}$

($G^{r_i}_{i,\log} \subseteq Z^0_{0,\xi_i}$ because it contains the identity !)

So the extension (drop i = of the component of the divisor)

$$[Z^0_{0,\xi}] \in \mathrm{Ext}(\Theta^{(R)}_{\xi}, G^2)$$

$$\mathrm{Ext}(\Theta^{(R)}_{\xi}, \mathbb{F}_p) \otimes G^2$$

(V vect space of ~~dim~~ $= n$ over $\mathbb{F}_p$

linear fun.

$$\mathrm{Ext}(V, \mathbb{F}_p) \longleftarrow \mathrm{Hom}(V, G_a)$$

$$f^*(\text{Artin-Schreier}) \longleftarrow\!\!\!| \ f$$

$$0 \to \mathbb{F}_p \to G_a \xrightarrow{x \mapsto x^p - x} G_a \to 0 \quad)$$

And for us, linear fun on $\Theta^{(R)}_{\xi}$'s :

$$\Theta^{(R)}_{\xi} = \mathbb{V}\left(\Omega^1_X(\log D)(R) \otimes F_{k(\xi)}\right)$$

$$\mathrm{Ext}(\Theta^{(R)}_{\xi}, G^2) = \Omega^1_X(\log D)(R) \otimes F \otimes_{\mathbb{F}_p} G^2$$

The refined Swan conductor

$$\mathrm{Hom}(G^2, \mathbb{F}_p) \longrightarrow \Omega^1_X(\log D)(R) \otimes F$$

$$\cup\!\!\!\shortmid$$

$$\mathrm{Hom}_{k}\left(\mathfrak{m}^{2}/\mathfrak{m}_{k}^{2+1},\ \Omega^{1}_{X}(\log D)\otimes F\right)$$

dim = 1 $d = \dim X$

<u>Example</u>: $X = \mathbb{A}^1 \supset D = (T_1 = 0)$

$U = \mathbb{G}_m = X \setminus D$, V. $t^p - t = \frac{1}{T_1^n}$ $\quad (n,p) = 1$

$R = nD$

$$(X * X)^{(R)} \supseteq U \times U = \operatorname{Spec} k[S_1^{\pm 1}, T^{\pm 1}]$$

$$\| $$

$$\operatorname{Spec} k[U_1^{\pm 1}, T_1^{\pm 1}, V_1] \Big/ (U_1 - (1 + V_1 T_1^{n}))$$

just log-product. Here we make another blow-up!

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & W = V \times V / \Delta G \arrow[d] \\ (X * X)^{(R)} \arrow[r] & U \times U \end{tikzcd}$$

$G = \mathbb{F}_p$

$$W \ni \quad t^p - t = \frac{1}{S_1^n} - \frac{1}{T_1^n} \qquad S_1 = U_1 T_1 = (1 + V_1 T_1^n)\, T_1$$

$$t^p - t = \frac{1}{T_1^n}\left(\left(\frac{1}{1 + V_1 T_1^n}\right)^n - 1\right)$$

We develop it $\otimes$

$$n V_1 T_1^n + (T_1^n)^2$$ Hence we divide by T_1^n

$$t^P - t^P = \frac{\otimes}{T_1^n}$$ does not have poles... $\Rightarrow$ Z étale covering

$\Downarrow$

the ramification on $\mathcal{U}$ is bounded by $D+$. $(Z_0 = Z)$

<u>Now</u>: $\Theta_D^{(R)} \subseteq (X \times X)^{(R)}$ (defined by $T_1 = 0$)

$$Z_{0,D} \longrightarrow \Theta_D^{(R)}$$ étale covering

$$t^P - t = -n V_1$$ <u>linear form</u> on $\Theta_D^{(R)}$

$\hookrightarrow$ $\frac{1}{T^n} d\log T \Rightarrow d(T^{-n})$ refined Swan conductor

//

$$V_1 = \frac{1}{T_1^n} \frac{\delta_1 - T_1}{T_1}$$

$$\frac{T_1 \otimes 1 - 1 \otimes T_1}{T_1 \otimes 1}$$

<u>Pau</u> 2 pm

~~Abbe~~ Saito

References: ICM 2010, Arizona Winter School $\Rightarrow$ wij'd

Ramification on local fields I AJM, II Kato Volume

The Characteristic class Inv. Math, Tokyo Math Journal .. Jussieu Math. Journal (Here any references...)

Sheaf ramification: (In Inv. Math.)

Ramification of l-adic sheaves: X smooth over $/k$ perfect field $U = X \setminus D$ SNC, $l \neq \operatorname{char} k$, $\mathcal{F}$ smooth sheaf l-adic on $\mathcal{F}$. on U.

$$\mathcal{H} = \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^* \mathcal{F}) \quad \text{on} \quad U \times U \qquad \text{rank } \mathcal{H} = (\text{rank } \mathcal{F})^2$$

$\mathcal{H}_{|\Delta_U} = \mathcal{E}nd(\mathcal{F})$ $\quad id_{\mathcal{F}}$: is not ramified along the divisor...

it is a global section.. we try to reach this section.

$(X * X)^R \qquad R = \sum r_i D_i$

$$\begin{tikzcd} & & \mathcal{H} \\ X \arrow[r, "\delta^{(R)}"] & (X * X)^{(R)} & U \times U \arrow[l, hook', "j^{(R)}"'] \\ & X \arrow[u, "\delta^{(R)}"] & U \arrow[l, hook', "j"'] \arrow[u, "\delta_U"'] \end{tikzcd}$$

$$\delta^{(R)*} j^{(R)}_* \mathcal{H} \longrightarrow j_* \underbrace{\delta_U^* \mathcal{H}}_{= \mathcal{E}nd(\mathcal{F})} \qquad / X$$

$id_{\mathcal{F}}$

We say the ramification of $\mathcal{F}$ along D is bounded by $R+$ if

$\mathrm{id}_{\mathcal{F}}$ is in the image of

$$\Gamma(X, \delta^{(R)*} j_*^{(R)} \mathcal{H}) \longrightarrow \Gamma(X, j_* \mathcal{E}nd(\mathcal{F}_1))$$

How much we have to blow up to extend such a section.

Ramification is bounded by $0+ \Leftrightarrow \mathcal{F}$ is tamely ramified along D

(i.e. the log blow-up is enough)
i.e. the exactification ... For $(X*X)^{(R)}$ we need to

blow-up more...

· Assume ramification of $\mathcal{F}$ is bounded by $R+$.

$$j_*^{(R)} \mathcal{H} \text{ on } (X*X)^{(R)}, \quad j_*^{(R)} \mathcal{H} |_{\Theta_D^{(R)}}$$

$$\Theta_D^{(R)} \hookrightarrow (X*X)^{(R)} \hookleftarrow U \times U$$

complement

$D = \bigcup D_i \qquad \xi_i \in D_i$ generic point of such a irreducible

component. ($\Theta_{\bar{\xi}_i}^{(R)}$ vector space over $\bar{\xi}_i$)
↳ geometric point over ξ_i

$$j_* \mathcal{H} |_{\Theta_{\bar{\xi}_i}^{(R)}} \qquad *$$

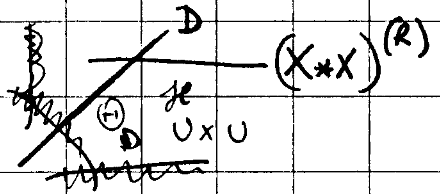
For Galois covering, we get isogeny / $\Theta^{(R)}_{\bar{\xi}_i}$ (abelian), what about $\#$?

On $\#$, this is isomorphic to the direct sum of rk $=1$ sheaves defined by Artin-Schreier equation : $t^p - t = f_{iJ} \rightsquigarrow \mathcal{L}_J$

$$\# = \bigoplus_J \mathcal{L}_J$$

(then for isogeny $\Rightarrow$ abelian ...)

f_{iJ} is a linear form on the vector space $\Theta^{(R)}_{\bar{\xi}_i}$



Drop the index i.

$$\mathrm{Hom}(G^{?}_{ab}, \mathbb{F}_p) \xrightarrow{\mathrm{rsw}} \mathrm{Hom}(\Theta^{(R)}_{\xi}, \mathbb{G}_a)$$

elementary ab. group

$\hookrightarrow \mathcal{G}|_{\mathrm{spec}\, K_i} \cong \bigoplus \chi_J$ ~~best~~ each character

$K_i = \mathrm{Frac}(\hat{\mathcal{O}}_{X,\xi_i})$

$$\chi_J \longmapsto \mathrm{rsw}(\chi_J) = f_J$$

up to multiplicity

(Some of χ_J can be constant !!).

Formula for the <u>Euler-Number</u>:

Assume $\operatorname{rk} \mathcal{F} = 1$ for each D_i r_i $G^s_{r, \log}$

χ_i $\pi_1(U)^{ab}$ $G^{ab}_{K_i} \longrightarrow \pi_1(U)^{ab}$

χ_i induced a character of G_{K_i} $\exists!\, r_i$ such that $\chi_i(G^{r_i}_{K_i}) \neq 1$

<u>integer</u>

$$\chi_i(G^s_{\log}) = 1 \quad \forall s > r_i$$

$$\operatorname{rsw}(\chi_i) \in \Omega^1_X(\log D)(R) \underset{\mathcal{O}_{X, \xi_i}}{\otimes} F_i \qquad F_i = k(\xi_i)$$

$$\operatorname{rsw}(\chi) \in \Gamma(D, \Omega_X(\log D)(R)) \qquad \left(\underset{\text{USING}}{\underline{\text{Kato}}} \Rightarrow \text{Zariski-Nagata Purity}\right)$$

<u>Assume</u> (C) cleanness condition

$$\operatorname{rsw}(\chi) \text{ is no-where vanishing on } D$$

(i.e. knowing at codimension 1 points then we know all... no <u>other</u> problem we will <u>have</u>)

(Analogous for <u>Mochizuki</u> - <u>Kedlaya</u>)

$\mathcal{D}$-modules in $\operatorname{ch} = 0$

$j_*^{(R)}(\mathcal{H})$ smooth on $(X * X)^{(R)}$

$\hookrightarrow$ rk $= 1$

The condition (C) gives : $j_*^{(R)}(\mathcal{H})\big|_{\Theta_D^{(R)}}$ $\forall \bar{x} \in D$

<u>generic point</u> : $j_*^{(R)}\mathcal{H}\big|_{\Theta_{\bar{x}}^{(R)}}$ is <u>non-constant</u>

<u>Theorem</u> : Assume rk $\mathcal{F} = 1$, (C) is satisfied, X proper

$\Rightarrow \quad \chi_c(U_{\bar{k}}, \mathcal{F}) = (X, X)_{(X*X)^{(R)}}$

How to prove this? Using <u>characteristic classes</u>.

One has hypothesis of "<u>resolvers</u>"