On the characteristic cycle of a constructible sheaf and the semi-continuity of the Swan conductor

Takeshi Saito

April 3, 2015

Abstract

We define the characteristic cycle of an étale sheaf as a cycle on the cotangent bundle of a smooth variety in positive characteristic assuming the existence of a singular support. We prove a formula à la Milnor for the total dimension of the space of vanishing cycles and an index formula computing the Euler-Poincaré characteristic.

An essential ingredient of the construction and the proof is a partial generalization to higher dimension of the semi-continuity of the Swan conductor due to Deligne-Laumon. Another geometric ingredient is a local version of the Radon transform. Both are based on a generalization of the formalism of vanishing cycles.

As is observed by Deligne in [6], a strong analogy between the wild ramification of an ℓ-adic sheaf in positive characteristic and the irregular singularity of partial differential equation on a complex manifold suggests to define the characteristic cycle of an $\bar{\mathbf{F}}_\ell$-sheaf as a cycle on the cotangent bundle of a smooth variety in positive characteristic $p \neq \ell$. It is expected to compute the Euler number and the total dimension of the space of vanishing cycles.

More primitively, we expect the existence of a singular support defined as a closed subset of the cotangent bundle, that controls the local acyclicity of morphisms to smooth curves. Assuming the existence of a singular support satisfying the condition (SS1) formulated in Section 2.3 in the text, we define the characteristic cycle and prove a formula (4.1) for the total dimension of the space of vanishing cycles. We also prove an index formula (4.28) computing the Euler-Poincaré characteristic assuming that the singular support satisfies a stronger condition (SSd) for the dimension d of the variety. Roughly speaking, we realize the program described in [6] assuming a stronger variant of the expectation [6, (?) p. 1]. Some of the key arguments in a previous article [18] where we studied sheaves on surfaces are generalized to higher dimension in an axiomatic way.

To define the characteristic cycle assuming the existence of a singular support, it suffices to determine the coefficient of each irreducible component of the singular support. To do this, we study the ramification of a local version of the Radon transform, constructed by choosing an immersion of the variety to a projective space.

To prove that the definition of the characteristic cycle is independent of the choice and that it satisfies the Milnor formula (4.1) in general, we show the stability Proposition 2.32 of the total dimension of the space of vanishing cycles under small deformation of morphisms to curves and the formula (4.1) for morphisms defined by pencils, Proposition 4.3.

The crucial ingredient in the proof of these two propositions is the continuity Proposition 1.17 of the total dimension of the space of vanishing cycles. This is a partial generalization to higher dimension of the semi-continuity of Swan conductor [14] and is stated using a generalization of the formalism of vanishing cycles with general base scheme [10], [15].

The index formula Theorem 4.13 computing the Euler-Poincaré characteristic is deduced from the compatibility Theorem 4.8 [6, 2e Conjecture, p. 10] of the construction of characteristic cycles with the non-characteristic immersion of a smooth divisor, by induction on dimension and the Milnor formula (4.1). The compatibility Theorem 4.8 implies the compatibility with smooth pull-back and a description Theorem 4.11 of the characteristic cycle in terms of ramification theory. In the tamely ramified case, the description has been proved by a different method in [19]. Using a variant of local Radon transform, the compatibility Theorem 4.8 is proved by reducing it to Theorem 4.11 for surfaces proved earlier in [18, Proposition 3.20].

We describe briefly the content of each section. In Section 1.1, we introduce and study flat functions on a scheme quasi-finite over a base scheme, used to formulate the partial generalization of the semi-continuity of Swan conductor to higher dimension. After briefly recalling the generalization of the formalism of vanishing cycles with general base scheme and its relation with local acyclicity, we recall from [15] basic properties Proposition 1.7 in the case where the locus of non local acyclicity is quasi-finite, in Section 1.2. We recall and reformulate the semi-continuity of Swan conductor from [14] using the formalism of vanishing cycles with general base scheme and give a partial generalization to higher dimension in Section 1.3.

We introduce singular support using local acyclicity of certain families of morphisms in Section 2.3, after some preliminaries on flat family of proper intersections in Section 2.1 and generalities on regular immersions with respect to a family of closed conic subsets of the cotangent bundle in Section 2.2. We recall in Section 2.4 from [17] that ramification theory implies the existence of a singular support on the complement of a closed subset of codimension $\geqq 2$. Assuming the existence of singular support, we prove the stability Proposition 2.32 of the total dimension of vanishing cycles at an isolated characteristic point in Section 2.5.

We introduce local Radon transform and study its local acyclicity in Section 3.3 using generalization of the formalism of vanishing cycles. We also study its variant in Section 3.4. Their definitions and properties are based on the preliminaries in Sections 3.1 and 3.2 on the universal families of hyperplane sections, of morphisms defined by pencils and of morphisms defined by pencils for hyperplane sections.

We define the characteristic cycle using the local Radon transform in Section 4.1. We prove that the definition is independent of the choice of immersion to a projective space and prove the Milnor formula (4.1). In Section 4.2, first we prove the compatibility Theorem 4.8 of the construction of characteristic cycles with non-characteristic immersion of smooth divisor. From this, we deduce the compatibility Corollary 4.10 with smooth pull-back and a description of the characteristic cycle Theorem 4.11 in terms of ramification theory. Finally, we deduce an index formula Theorem 4.13 computing the Euler number from Theorem 4.8 using a good pencil whose existence is proved in Section 3.5. The proof in Sections 4.1 and 4.2 are based on the constructions of Sections 3.3 and 3.4 respectively.

The author thanks Pierre Deligne for sending him an unpublished notes [6]. This article is the result of an attempt to understand the content of [6]. The author thanks Luc Illusie for sending him an unpublished preprint [11] and for the introduction to the formalism of generalized vanishing cycles. The author thanks Alexander Beilinson for the suggestion of use of Radon transform [2]. The author greatly acknowledges Ahmed Abbes for pointing out the similarity of the theory of étale sheaves and the content of the book [12], which was the starting point of the whole project, and also for suggesting a link between the vanishing topos and the semi-continuity of Swan conductor. The author would also like to thank Luc Illusie and Ahmed Abbes for inspiring discussion. The research was partially supported by JSPS Grants-in-Aid for Scientific Research (A) 26247002.

Contents

1 Vanishing topos and the semi-continuity of the Swan conductor

1.1 Calculus on vanishing topos

Let $f\colon X \to S$ be a morphism of schemes. For the definition of the *vanishing topos* $X \overleftarrow{\times}_S S$ and the morphisms

$$\begin{array}{ccccc} X & \xrightarrow{\Psi_f} & X \overleftarrow{\times}_S S & \xrightarrow{p_2} & S \\ & & {\scriptstyle p_1}\big\downarrow & & \\ & & X & & \end{array}$$

of toposes, we refer to [10, 1.1, 4.1, 4.3] and [11, 1.1]. For a geometric point x of a scheme X, we assume in this article that the residue field of x is a separable closure of the residue field at the image of x in X, if we do not say otherwise explicitly. For a geometric point x of X, the fiber of $p_1\colon X \overleftarrow{\times}_S S \to X$ at x is the vanishing topos $x \overleftarrow{\times}_S S$ and is canonically identified with the strict localization $S_{(s)}$ at the geometric point $s = f(x)$ of S defined by the image of x (cf. [11, (1.8.2)]).

A point on the topos $X \overleftarrow{\times}_S S$ is defined by a triple denoted $x \leftarrow t$ consisting of a geometric point x of X, a geometric point t of S and a specialization $s = f(x) \leftarrow t$ namely a geometric point $S_{(s)} \leftarrow t$ of the strict localization lifting $S \leftarrow t$. The fiber of the canonical morphism $X \overleftarrow{\times}_S S \to S \overleftarrow{\times}_S S$ at a point $s \leftarrow t$ is canonically identified with the geometric fiber X_s. The fiber products $X_{(x)} \times_{S_{(s)}} S_{(t)}$ and $X_{(x)} \times_{S_{(s)}} t$ are called the *Milnor tube* and the *Milnor fiber* respectively.

For a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

of morphisms of schemes, the morphism $\overleftarrow{g}\colon X \overleftarrow{\times}_Y Y \to X \overleftarrow{\times}_S S$ is defined by functoriality and we have a canonical isomorphism $\Psi_p \to \overleftarrow{g} \circ \Psi_f$. On the fibers of a geometric point x of X, the morphism $\overleftarrow{g}$ induces a morphism

$$g_{(x)}\colon Y_{(y)} = x \overleftarrow{\times}_Y Y \to S_{(s)} = x \overleftarrow{\times}_S S \tag{1.1}$$

on the strict localizations at $y = f(x)$ and $s = p(x)$ [11, (1.7.3)]. In particular for $Y = X$, we have a canonical isomorphism $\Psi_p \to \overleftarrow{g} \circ \Psi_{\mathrm{id}}$.

Let Λ be a finite field of characteristic ℓ invertible on S. Let $D^+(-)$ denote the derived category of complexes of Λ-modules bounded below and let $D^b(-)$ denote the subcategory consisting of complexes with bounded cohomology. In the following, we assume that S and X are quasi-compact and quasi-separated. We say that an object of $D^b(X \overleftarrow{\times}_S S)$ is constructible if there exist locally finite partitions $X = \coprod_\alpha X_\alpha$ and $S = \coprod_\beta S_\beta$ by locally closed subschemes such that the restrictions to $X_\alpha \overleftarrow{\times}_S S_\beta$ of cohomology sheaves are locally constant and constructible [11, 1.3]. Let $D^b_c(-)$ denote the subcategory of $D^b(-)$ consisting of constructible objects.

We canonically identify a function on the underlying set of a scheme X with the function on the set of isomorphism classes of geometric points x of X. Similarly, we call a function on the set of isomorphism classes of points $x \leftarrow t$ of $X \overleftarrow{\times}_S S$ a function on $X \overleftarrow{\times}_S S$.

We say that a function on $X \overleftarrow{\times}_S S$ is a *constructible function* if there exist locally finite partitions $X = \coprod_\alpha X_\alpha$ and $S = \coprod_\beta S_\beta$ as above such that the restrictions to $X_\alpha \overleftarrow{\times}_S S_\beta$ are locally constant. For an object $\mathcal{K}$ of $D^b_c(X \overleftarrow{\times}_S S)$, the function $\dim \mathcal{K}_{x\leftarrow t} = \sum_q (-1)^q \dim \mathcal{H}^q \mathcal{K}_{x\leftarrow t}$ is a constructible function on $X \overleftarrow{\times}_S S$.

Definition 1.1. *Let Z be a quasi-finite scheme of finite type over S and let $\varphi\colon Z \to \mathbf{Q}$ be a function. We define the* derivative $\delta(\varphi)$ *of φ as a function on $Z \overleftarrow{\times}_S S$ by*

$$\delta(\varphi)(x \leftarrow t) = \varphi(x) - \sum_{z \in Z_{(x)} \times_{S_{(s)}} t} \varphi(z) \tag{1.2}$$

where $s = f(x)$. If the derivative $\delta(\varphi)$ is 0 (resp. $\delta(\varphi) \geqq 0$), we say that the function φ is flat *(resp.* increasing*) over S. If the morphism $f\colon Z \to S$ is finite, we define a function $f_*\varphi$ on S by*

$$f_*\varphi(s) = \sum_{x \in Z_s} \varphi(x). \tag{1.3}$$

If φ is constructible, the function $f_*\varphi$ is also constructible.

Lemma 1.2. *Let S be a noetherian scheme, Z be a quasi-finite scheme of finite type over S and $\varphi\colon Z \to \mathbf{Q}$ be a function.*

1. *If the* derivative *$\delta(\varphi)\colon Z \overset{\leftarrow}{\times}_S S \to \mathbf{Q}$ defined in* (1.2) *is constructible, the function φ is also constructible.*

2. *Assume that φ is* flat *over S and constructible. Then, $\varphi = 0$ if and only if $\varphi(x) = 0$ for the generic point x of every irreducible component of Z.*

3. *Assume that Z is étale over S and that φ is constructible. Then, φ is flat (resp. increasing) over S if and only if it is locally constant (resp. upper semi-continuous).*

4. *Assume that the morphism $f\colon Z \to S$ is finite and that the derivative $\delta(\varphi)$ is constructible and satisfies $\delta(\varphi) \geqq 0$. Then, the function $f_*\varphi$ on S is upper semi-continuous. The function φ is flat over S if and only if $f_*\varphi$ is locally constant.*

Proof. 1. By noetherian induction, it suffices to show the following; for an open subscheme U of S and a geometric point x of Z dominating an irreducible component of $S - U$, the function φ is constant on a neighborhood of x. By replacing S by $S - U$ and further replacing S by an étale neighborhood of the image of x, we reduce the assertion to the case where Z is a split finite étale covering of S. Then, the assertion is clear.

2. By noetherian induction, a function flat over S is uniquely determined at the values of the generic points of irreducible components.

3. Since the question is étale local on Z, we may assume that $Z \to S$ is an isomorphism. Then the assertion is clear.

4. If $f\colon Z \to S$ is finite, for a specialization $s \leftarrow t$, we have $\sum_{x \in Z_s} \delta(\varphi)(x \leftarrow t) = f_*\varphi(s) - f_*\varphi(t) = \delta(f_*\varphi)(s \leftarrow t)$. Hence we may assume $Z = S$ and then the assertion is clear. □

We give an example of flat function. Let S be a locally noetherian scheme, X be a scheme of finite type over S and $Z \subset X$ be a closed subscheme quasi-finite over S. Let A be a complex of $\mathcal{O}_X$-modules such that the cohomology sheave $\mathcal{H}^q(A)$ are coherent $\mathcal{O}_X$-modules supported on Z for all q and that A is of finite tor-dimension as a complex of $\mathcal{O}_S$-modules. For a geometric point z of Z and its image s in S, let $\mathcal{O}_{X,z}$ and $\mathcal{O}_{S,s}$ denote the *strict* localizations and $k(s)$ the separably closed residue field of $\mathcal{O}_{S,s}$. Then, the $\mathcal{O}_{X,z}$-modules $Tor_q^{\mathcal{O}_{S,s}}(A_z, k(s))$ are of finite length and are 0 except for finitely many q. We define a function $\varphi_A\colon Z \to \mathbf{Z}$ by

$$\varphi_A(z) = \sum_q (-1)^q \dim_{k(s)} Tor_q^{\mathcal{O}_{S,s}}(A_z, k(s)). \tag{1.4}$$

Lemma 1.3. *Let schemes $Z \subset X \to S$ and a complex A be as above.*

1. *The function $\varphi_A\colon Z \to \mathbf{Z}$ defined by* (1.4) *is flat over S and constructible.*

2. *Suppose that S and Z are integral and that the image of the generic point ξ of Z is the generic point η of S. If $A = \mathcal{O}_Z$, the value of the function φ_A at a geometric point of Z above ξ is the inseparable degree $[k(\xi) : k(\eta)]$.*

Proof. 1. Since the assertion is étale local on Z, we may assume that Z is finite over S, that X and S are affine and that z is the unique point in the geometric fiber $Z \times_S \mathrm{Spec}\, k(s)$. Then, the complex Rf_*A is a perfect complex of $\mathcal{O}_S$-modules and $\varphi_A(z)$ equal the rank of Rf_*A. Hence, the assertion follows.

2. By the same argument as in the proof of 1., it is reduced to the case where $S = \mathrm{Spec}\, k(\eta)$ and $Z = \mathrm{Spec}\, k(\xi)$ and the assertion follows. □

We generalize the definition of derivative to functions on vanishing topos.

Definition 1.4. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array} \tag{1.5}$$

be a commutative diagram of morphisms of schemes such that Z is quasi-finite over S. Let $\psi\colon Z \overset{\leftarrow}{\times}_Y Y \to \mathbf{Q}$ be a function such that $\psi(x \leftarrow w) = 0$ unless w is not supported on the image of $f_{(x)}\colon Z_{(x)} \to Y_{(y)}$ where $y = f(x)$. We define the derivative $\delta(\psi)$ as a function on $Z \overset{\leftarrow}{\times}_S S \to \mathbf{Z}$ by

$$\delta(\psi)(x \leftarrow t) = \psi(x \leftarrow y) - \sum_{w \in Y_{(y)} \times_{S_{(s)}} t} \psi(x \leftarrow w) \tag{1.6}$$

where $s = p(x)$. The sum on the right hand side is a finite sum by the assumption that Z is quasi-finite over S and the assumption on the support of ψ. We say that ψ is flat *over S if $\delta(\psi) = 0$.*

If $Z = Y$, we recover the definition (1.2) by applying (1.6) to the pull-back $p_2^*\varphi\colon Z \overset{\leftarrow}{\times}_Z Z \to \mathbf{Z}$ by $p_2\colon Z \overset{\leftarrow}{\times}_Z Z \to Z$.

The following elementary Lemma will be used in the proof of a generalization of the continuity of the Swan conductor.

Lemma 1.5. *Let the assumption on the diagram* (1.5) *be as in Definition* 1.4 *and let φ be a function on Z. We define a function ψ on $Z \overset{\leftarrow}{\times}_Y Y$ by*

$$\psi(x \leftarrow w) = \sum_{z \in Z_{(x)} \times_{Y_{(y)}} w} \varphi(z) \tag{1.7}$$

where $y = f(x)$. Then the derivative $\delta(\varphi)$ on $Z \overset{\leftarrow}{\times}_S S$ defined by (1.2) *equals $\delta(\psi)$ defined by* (1.6).

Proof. It follows from $\psi(x \leftarrow y) = \varphi(x)$ and $Z_{(x)} \times_{S_{(s)}} t = \coprod_{w \in Y_{(y)} \times_{S_{(s)}} t} (Z_{(x)} \times_{Y_{(y)}} w)$. □

1.2 Nearby cycles and the local acyclicity

For a morphism $f\colon X \to S$, the morphism $\Psi_f\colon X \to X \overset{\leftarrow}{\times}_S S$ defines the nearby cycles functor $R\Psi_f : D^+(X) \to D^+(X \overset{\leftarrow}{\times}_S S)$. The canonical morphism $p_1^* \to R\Psi_f$ of functors is defined by adjunction and by the isomorphism $\mathrm{id} \to p_1 \circ \Psi_f$. The cone of the morphism

$p_1^* \to R\Psi_f$ defines the vanishing cycles functor $R\Phi_f : D^+(X) \to D^+(X \overset{\leftarrow}{\times}_S S)$. If S is the spectrum of a henselian discrete valuation ring and if s, η denote its closed and generic points, we recover the classical construction of complexes ψ, ϕ of nearby cycles and vanishing cycles as the restrictions to $X_s \overset{\leftarrow}{\times}_S \eta$ of $R\Psi_f$ and $R\Phi_f$ respectively.

We consider a commutative diagram

(1.8)
$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

of schemes. The canonical isomorphism $\overleftarrow{g} \circ \Psi_f \to \Psi_p$ induces an isomorphism of functors

(1.9)
$$\overleftarrow{g} \circ \Psi_f \to \Psi_p$$

For an object $\mathcal{K}$ of $D^+(X \overset{\leftarrow}{\times}_Y Y)$ and a geometric point x of X, the restriction of $R\overleftarrow{g}_*\mathcal{K}$ on $x \overset{\leftarrow}{\times}_S S = S_{(s)}$ for $s = f(x)$ is canonically identified with $Rg_{(x)*}(\mathcal{K}|_{Y_{(y)}})$ for $y = f(x)$ in the notation of (1.1) by [11, (1.9.2)]. For the stalk at a point $x \leftarrow t$ of $X \overset{\leftarrow}{\times}_S S$, this identification gives a canonical isomorphism

(1.10)
$$R\overleftarrow{g}_*\mathcal{K}_{x \leftarrow t} \to R\Gamma(Y_{(y)} \times_{S_{(s)}} S_{(t)}, \mathcal{K}|_{Y_{(y)} \times_{S_{(s)}} S_{(t)}}).$$

For an object $\mathcal{K}$ of $D^+(X)$, (1.10) applied to $Y = X$ gives a canonical identification

(1.11)
$$R\Psi_f\mathcal{K}_{x \leftarrow t} \to R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, \mathcal{K}|_{X_{(x)} \times_{S_{(s)}} S_{(t)}})$$

with the cohomology of the Milnor tube [11, (1.1.15)].

A cartesian diagram

$$\begin{array}{ccc} X & \longleftarrow & X_T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_T} \\ S & \xleftarrow{\ i\ } & T \end{array}$$

of schemes defines a 2-commutative diagram

$$\begin{array}{ccccc} X_T & \xleftarrow{\ p_1\ } & X_T \overset{\leftarrow}{\times}_T T & \xleftarrow{\ \Psi_{f_T}\ } & X_T \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \overleftarrow{i}} & & \downarrow{\scriptstyle i} \\ X & \xleftarrow{\ p_1\ } & X \overset{\leftarrow}{\times}_S S & \xleftarrow{\ \Psi_f\ } & X \end{array}$$

and the base change morphisms define a morphism

(1.12)
$$\begin{array}{ccccccc} \longrightarrow & \overleftarrow{i}^* p_1^* & \longrightarrow & \overleftarrow{i}^* R\Psi_f & \longrightarrow & \overleftarrow{i}^* R\Phi_f & \longrightarrow \\ & \simeq\downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & p_1^* i^* & \longrightarrow & R\Psi_{f_T} i^* & \longrightarrow & R\Phi_{f_T} i^* & \longrightarrow \end{array}$$

of distinguished triangles of functors. For an object $\mathcal{K}$ of $D^+(X)$, we say that the formation of $R\Psi_f\mathcal{K}$ commutes with the base change $T \to S$ if the middle vertical arrow defines an isomorphism $\overleftarrow{i}^* R\Psi_f\mathcal{K} \to R\Psi_{f_T} i^*\mathcal{K}$.

For a point $x \leftarrow t$ of $X \overset{\leftarrow}{\times}_S S$, if $T \subset S$ denotes the closure of the image of t in S, the left square of (1.12) induces a commutative diagram

$$\begin{array}{lccc} (1.13) & (p_1^*\mathcal{K})_{x\leftarrow t} = \mathcal{K}_x & \longrightarrow & R\Psi_f\mathcal{K}_{x\leftarrow t} = R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, \mathcal{K}|_{X_{(x)}\times_{S_{(s)}} S_{(t)}}) \\ & & \searrow & \downarrow \\ & & & R\Psi_{f_T}(\mathcal{K}|_{X_T})_{x\leftarrow t} = R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{K}|_{X_{(x)}\times_{S_{(s)}} t}) \end{array}$$

where the vertical arrow is the canonical morphism from the cohomology of the Milnor tube to that of the Milnor fiber. Recall that we assume that the residue field of t is a separable closure of the residue field at the image of $S_{(s)}$.

We interpret the local acyclicity in terms of vanishing topos.

Lemma 1.6. *Let $f\colon X \to S$ be a morphism of schemes. Then, for an object $\mathcal{K}$ of $D^+(X)$, the conditions* (1) *and* (2) *in* 1. *and* 2. *below are equivalent to each other respectively.*

1. (1) *For every point $x \leftarrow t$ of $X \overset{\leftarrow}{\times}_S S$, the vertical arrow in* (1.13) *is an isomorphism.*

(2) *The formation of $R\Psi_f\mathcal{K}$ commutes with finite base change $T \to S$.*

2. ([8, Corollaire 2.6]) (1) *The morphism $f\colon X \to S$ is (resp. universally) locally acyclic relatively to $\mathcal{K}$.*

(2) *The canonical morphism $p_1^*\mathcal{K} \to R\Psi_f\mathcal{K}$ is an isomorphism and the formation of $R\Psi_f\mathcal{K}$ commutes with finite (resp. arbitrary) base change $T \to S$.*

Proof. 1. Since the vertical arrow in (1.13) is induced by the base change morphism for a closed immersion $T \to S$, the condition (2) implies the condition (1).

Conversely, let $T \to S$ be a finite morphism and $x \leftarrow t$ be a point of $X \overset{\leftarrow}{\times}_T T$. Let $T' \subset T$ be the closure of the image of t. Then the vertical arrow in (1.13) for $x \leftarrow t$ regarded as a point of $X \overset{\leftarrow}{\times}_S S$ is the stalk of the base change morphism for a finite morphism $T' \to T \to S$. Hence, the condition (1) implies the condition (2).

2. The condition (1) is equivalent to that the slant arrow in (1.13) is an isomorphism for every geometric point $x \leftarrow t$ of $X \overset{\leftarrow}{\times}_S S$ (resp. after any base change $T \to S$). Hence the condition (2) implies the condition (1) by (2)⇒(1) in 1. and the commutativity of the diagram (1.13).

Conversely, by [8, Corollaire 2.6], if the condition (1) is satisfied, the formation of $Rf_{(x)*}(\mathcal{K}|_{X_{(x)}})$ commutes with finite base change for every geometric point x of X where $f_{(x)}\colon X_{(x)} \to S_{(s)}$ is the morphism on the strict localizations induced by f. Since the vertical arrow in (1.13) is the stalk $Rf_{(x)*}(\mathcal{K}|_{X_{(x)}})_t \to R(f_T)_{(x)*}(\mathcal{K}|_{(X_T)_{(x)}})_t$ of the base change morphism, the condition (1) implies the condition (2) by (1)⇒(2) in 1. and the commutativity of the diagram (1.13). □

Proposition 1.7. *Let $f\colon X \to S$ be a morphism of finite type of schemes and let $Z \subset X$ be a closed subscheme quasi-finite over S. Let $\mathcal{K}$ be an object of $D_c^b(X)$ such that the restriction of $f\colon X \to S$ to the complement $X - Z$ is (resp. universally) locally acyclic relatively to the restriction of $\mathcal{K}$.*

1. (cf. [15, Proposition 6.1]) *$R\Psi_f\mathcal{K}$ and $R\Phi_f\mathcal{K}$ are constructible and their formations commute with finite (resp. arbitrary) base change. The constructible object $R\Phi_f\mathcal{K}$ is supported on $Z \overset{\leftarrow}{\times}_S S$.*

2. *Let x be a geometric point of X and $s = f(x)$ be the geometric point of S defined by the image of x by f. Let t and u be geometric points of $S_{(s)}$ and $S_{(t)} \leftarrow u$ be a specialization. Then, there exists a distinguished triangle*

$$\longrightarrow R\Psi_f\mathcal{K}_{x\leftarrow t} \longrightarrow R\Psi_f\mathcal{K}_{x\leftarrow u} \longrightarrow \bigoplus_{(Z\times_X X_{(x)})\times_{S_{(s)}} t} R\Phi_f\mathcal{K}_{z\leftarrow u} \longrightarrow \tag{1.14}$$

where $R\Psi_f\mathcal{K}_{x\leftarrow t} \to R\Psi_f\mathcal{K}_{x\leftarrow u}$ is the cospecialization.

The commutativity of the formation of $R\Psi_f\mathcal{K}$ with any base change implies its constructibility by [15, 8.1, 10.5] as noted after [11, Theorem 1.3.1].

Proof. 1. The constructibility is proved by taking a compactification in [15, Proposition 6.1]. The commutativity with base change is proved similarly by taking a compactification and applying the proper base change theorem.

The assertion on the support of $R\Phi_f\mathcal{K}$ follows from Lemma 1.6.2 (1)⇒(2).

2. Let t and u be geometric points of $S_{(s)}$ and $t \leftarrow u$ be a specialization. By replacing S by the strict localization $S_{(s)}$ and shrinking X, we may assume that $S = S_{(s)}$, that X is affine and that $Z = Z \times_X X_{(x)}$ is finite over S.

We consider the diagram

$$\begin{array}{ccccc} s & & t & & \\ {\scriptstyle i_s}\downarrow & & \downarrow{\scriptstyle i_t} & & \\ S & \xleftarrow{\ j\ } & S_{(t)} & \xleftarrow{\ k\ } & u \end{array}$$

and let the morphisms obtained by the base change $X \to S$ denoted by the same letters, by abuse of notation. Similarly as the sliced vanishing cycles in the proof of [15, Proposition 6.1], we consider an object $\Phi_{t\leftarrow u}\mathcal{K}$ on $X \times_S S_{(t)}$ fitting in the distinguished triangle $\to j^*\mathcal{K} \to Rk_*(j\circ k)^*\mathcal{K} \to \Phi_{t\leftarrow u}\mathcal{K} \to$. Since the formation of $R\Phi_f\mathcal{K}$ commutes with finite base change by 1., we have a distinguished triangle (1.14) with the third term replaced by $\Delta_x = (Rj_*\Phi_{t\leftarrow u}\mathcal{K})_x$. Further, the third term itself is canonically isomorphic to the direct sum of $\Delta_z = (\Phi_{t\leftarrow u}\mathcal{K})_z$ for $z \in Z_t$.

Since $R\Phi_f\mathcal{K}$ is acyclic outside $Z \overleftarrow{\times}_S S$ by 1., the canonical morphisms $i_s^*\mathcal{K} \to i_s^*Rj_*j^*\mathcal{K}$ and $i_s^*\mathcal{K} \to i_s^*R(j\circ k)_*(j\circ k)^*\mathcal{K}$ are isomorphisms on $X_s - Z_s$. Hence, the restriction $i_s^*Rj_*\Phi_{t\leftarrow u}\mathcal{K}$ is acyclic on $X_s - Z_s$. Similalry, the restriction $i_t^*\Phi_{t\leftarrow u}\mathcal{K}$ is acyclic on $X_t - Z_t$.

We take a compactification $\bar{X}$ of X and an extension $\bar{\mathcal{K}}$ of $\mathcal{K}$ to $\bar{X}$. Define $\Phi_{t\leftarrow u}\bar{\mathcal{K}}$ on $\bar{X} \times_S S_{(t)}$ similarly as $\Phi_{t\leftarrow u}\mathcal{K}$ and set $Y = \bar{X} - X$. By the proper base change theorem, the canonical morphisms $R\Gamma(\bar{X}_s, i_s^*Rj_*\Phi_{t\leftarrow u}\bar{\mathcal{K}}) \leftarrow R\Gamma(\bar{X}, Rj_*\Phi_{t\leftarrow u}\bar{\mathcal{K}}) \to R\Gamma(\bar{X} \times_S S_{(t)}, \Phi_{t\leftarrow u}\bar{\mathcal{K}}) \to R\Gamma(\bar{X}_t, i_t^*\Phi_{t\leftarrow u}\bar{\mathcal{K}})$ are isomorphisms and similarly for the restrictions to Y. Hence, we obtain a commutative diagram

$$\begin{array}{rccc} \Delta_x = & R\Gamma(Z_s, i_s^*Rj_*\Phi_{t\leftarrow u}\mathcal{K}) & \longrightarrow & R\Gamma_c(X_s, i_s^*Rj_*\Phi_{t\leftarrow u}\mathcal{K}) \\ & \downarrow & & \downarrow \\ \bigoplus_{z\in Z_t}\Delta_z = & R\Gamma(Z_t, i_t^*\Phi_{t\leftarrow u}\mathcal{K}) & \longrightarrow & R\Gamma_c(X_t, i_t^*\Phi_{t\leftarrow u}\mathcal{K}) \end{array}$$

of isomorphisms and the assertion follows. □

Corollary 1.8. *We keep the assumptions in Proposition 1.7 and let x be a geometric point of X and $s = f(x)$ be the geometric point of S defined by the image of x by f as* *in Proposition* 1.7.2*. Then, the restriction of* $R^q\Psi_f\mathcal{K}$ *on* $x \overset{\leftarrow}{\times}_S S = S_{(s)}$ *is locally constant and constructible outside the image of the finite scheme* $Z \times_X X_{(x)}$ *for every* q*.*

Proof. Let t and u be geometric points of $S_{(s)}$ not in the image of $Z \times_X X_{(x)}$ and $t \leftarrow u$ be a specialization. Since $R\Psi_f\mathcal{K}$ is constructible, it suffices to show that the cospecialization morphism $R\Psi_f\mathcal{K}_{x\leftarrow t} \to R\Psi_f\mathcal{K}_{x\leftarrow u}$ is an isomorphism. Then by the assumption on the local acyclicity, the complex $\Phi_{t\leftarrow u}\mathcal{K}$ in the proof of Proposition 1.7.2 is acyclic. Hence the assertion follows from (1.14). □

Corollary 1.9. *We keep the assumptions in Proposition* 1.7*. Define a constructible function* $\delta_\mathcal{K}$ *on* $X \overset{\leftarrow}{\times}_S S$ *supported on* $Z \overset{\leftarrow}{\times}_S S$ *by* $\delta_\mathcal{K}(x \leftarrow t) = \dim R\Phi_f\mathcal{K}_{x\leftarrow t}$*. Assume that* $R\Phi_f\mathcal{K}$ *is acyclic except at degree* 0*.*

Then, we have $\delta_\mathcal{K} \geqq 0$ *and the equality* $\delta_\mathcal{K} = 0$ *is equivalent to the condition that the morphism* f *is (resp. universally) locally acyclic relatively to* $\mathcal{K}$*.*

Proof. The positivity $\delta_\mathcal{K} \geqq 0$ follows from the assumption that $R\Phi_f\mathcal{K}$ is acyclic except at degree 0. Further the equality $\delta_\mathcal{K} = 0$ is equivalent to $R\Phi_f\mathcal{K} = 0$. Since the formation of $R\Psi_f\mathcal{K}$ commutes with finite (resp. arbitrary) base change by Proposition 1.7.1, it is further equivalent to the condition that the morphism f is (resp. universally) locally acyclic relatively to $\mathcal{K}$ by Lemma 1.6.2. □

Lemma 1.10. *The assumption that* $R\Phi_f\mathcal{K}$ *is acyclic except at degree* 0 *in Corollary* 1.9 *is satisfied if the following conditions are satisfied: The scheme* S *is noetherian, the restriction of* $f\colon X \to S$ *to* $X - Z$ *is universally locally acyclic relatively to the restriction of* $\mathcal{K}$ *and the following condition* (P) *is satisfied.*

(P) *For every morphism* $T \to S$ *from the spectrum* T *of a discrete valuation ring, the pull-back of* $\mathcal{K}[1]$ *to* X_T *is perverse.*

Proof. Let $x \leftarrow t$ be a point of $X \overset{\leftarrow}{\times}_S S$ and let $T \to S$ be a morphism from the spectrum T of a discrete valuation ring such that the image of $T \to S$ is the same as that of $\{f(x), t\}$. Since the formation of $R\Phi_f\mathcal{K}$ commutes with arbitrary base change by Proposition 1.7.1, the base change morphism $R\Phi_f\mathcal{K}_{x\leftarrow t} \to R\Phi_{f_T}(\mathcal{K}|_{X_T})_{x\leftarrow t}$ is an isomorphism. The complex $R\Phi_{f_T}(\mathcal{K}|_{X_T})$ is a perverse sheaf by the assumption (P) and by the theorem of Gabber [9, Corollaire 4.6]. Since $R\Phi_{f_T}(\mathcal{K}|_{X_T})$ vanishes outside the closed fiber Z_s, this implies that the complex $R\Phi_f\mathcal{K}$ is acyclic except at degree 0. □

The condition (P) is satisfied if $f\colon X \to S$ is smooth of relative dimension d and $\mathcal{K} = j_!\mathcal{F}[d]$ for the open immersion $j\colon U \to X$ of the complement $U = X - D$ of a Cartier divisor D and a locally constant sheaf $\mathcal{F}$ on U.

We study the local acyclicity of a complex on the vanishing topos.

Definition 1.11. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

be a commutative diagram of schemes and $\mathcal{K}$ *be a constructible complex of* Λ*-modules on* $X \overset{\leftarrow}{\times}_Y Y$*. We say that* $g\colon Y \to S$ *is locally acyclic relatively to* $\mathcal{K}$ *if for every point* $x \leftarrow t$

of $X \overset{\leftarrow}{\times}_S S$ and for $y = f(x)$ and $s = p(x)$, the canonical morphism

$$\mathcal{K}_{x \leftarrow y} = R\Gamma(Y_{(y)}, \mathcal{K}) \to R\Gamma(Y_{(y)} \times_{S_{(s)}} t, \mathcal{K})$$

is an isomorphism where the fiber $x \overset{\leftarrow}{\times}_Y Y$ of $p_1 \colon X \overset{\leftarrow}{\times}_S S \to X$ is identified with $Y_{(y)}$.

Lemma 1.12. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

be a commutative diagram of schemes and $\mathcal{K}$ be a constructible complex of Λ-modules on X. Assume that the formation of $R\Psi_f \mathcal{K}$ commutes with finite base change. Then, the following conditions are equivalent:

(1) *$p \colon X \to S$ is locally acyclic relatively to $\mathcal{K}$.*

(2) *$g \colon Y \to S$ is locally acyclic relatively to $R\Psi_f \mathcal{K}$.*

Proof. For a point $x \leftarrow t$ of $X \overset{\leftarrow}{\times}_S S$, set $y = f(x)$ and $s = p(x)$ and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{K}_x & \longrightarrow & R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{K}) \\ \big\downarrow & & \big\uparrow \\ R\Psi_f \mathcal{K}_{x \leftarrow y} & \longrightarrow & R\Gamma(Y_{(y)} \times_{S_{(s)}} t, R\Psi_f \mathcal{K}). \end{array}$$

The condition (1) (resp. (2)) is equivalent to that the top (resp. bottom) horizontal arrow is an isomorphism for every $x \leftarrow t$.

The left vertical arrow is an isomorphism since $y = f(x)$. Let $T \subset S$ denote the reduced closed subscheme whose underlying set is the closure of the image of t and let $f_T \colon X_T \to Y_T$ denote the base change of f by the closed immersion $T \to S$. Then the right vertical arrow is identified with the base change morphism $R\Gamma(Y_{(y)} \times_{S_{(s)}} t, R\Psi_f \mathcal{K}) \to R\Gamma(Y_{T,(y)} \times_{T_{(s)}} t, R\Psi_{f_T} \mathcal{K}|_{X_T}) = R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{K})$. By the assumption that the formulation of $R\Psi_f \mathcal{K})$ commutes with the finite base change, the right vertical arrow is an isomorphism for every $x \leftarrow t$. Hence the assertion follows. □

1.3 Semi-continuity of the Swan conductor

We reformulate the main result of Deligne-Laumon in [14] in Proposition 1.13 below. Let $f \colon X \to S$ be a flat morphism of relative dimension 1 and let $Z \subset X$ be a closed subscheme. Assume that $X - Z$ is smooth over S and that Z is quasi-finite and flat over S. Let $\mathcal{K}$ be a constructible complex on X such that the restrictions of the cohomology sheaves on $X - Z$ are locally constant.

Let $s \to S$ be a geometric point with *algebraically closed* residue field. For a geometric point x of Z above s, the normalization of the strict localization $X_{s,(x)}$ is the disjoint union $\amalg_i X_i$ of finitely many spectra of strictly local discrete valuation rings with residue fields equal to that of s. Let K_i denote the fraction field of X_i for each component i and let $\bar{\eta}_i = \mathrm{Spec}\ \bar{K}_i \to X_i$ denote the geometric generic point defined by a separable closure. For a Λ-representation V of the absolute Galois group $G_{K_i} = \mathrm{Gal}(\bar{K}_i/K_i)$, the

Swan conductor $\mathrm{Sw}_{K_i}V \in \mathbf{N}$ is defined [14] and the total dimension is defined as the sum $\dim \mathrm{tot}_{K_i}V = \dim V + \mathrm{Sw}_{K_i}V$.

The stalk $\mathcal{H}^q(\mathcal{K})_{\bar\eta_i}$ for each integer q defines a Λ-representation of the absolute Galois group G_{K_i} and hence the total dimension $\dim \mathrm{tot}_{K_i}\mathcal{K}_{\bar\eta_i}$ is defined as the alternating sum $\sum_q(-1)^q \dim \mathrm{tot}_{K_i}\mathcal{H}^q(\mathcal{K})_{\bar\eta_i}$. We define the Artin conductor by

$$a_x(\mathcal{K}|_{X_s}) = \sum_i \dim \mathrm{tot}_{K_i}\mathcal{K}_{\bar\eta_i} - \dim \mathcal{K}_x. \tag{1.15}$$

We define a function $\varphi_{\mathcal{K}}$ on X supported on Z by

$$\varphi_{\mathcal{K}}(x) = a_x(\mathcal{K}|_{X_s}) \tag{1.16}$$

for $s = f(x)$. The derivative $\delta(\varphi_{\mathcal{K}})$ on $X \overset{\leftarrow}{\times}_S S$ is defined by (1.2).

Proposition 1.13 ([14, Théorème 2.1.1])**.** *Let S be a noetherian scheme and $f\colon X \to S$ be a flat morphism of relative dimension 1. Let $Z \subset X$ be a closed subscheme quasi-finite over S such that $U = X - Z$ is smooth over S. Let $\mathcal{K}$ be a constructible complex on X such that the restrictions of cohomology sheaves on $X - Z$ are locally constant.*

1. *The objects $R\Psi_f\mathcal{K}$ and $R\Phi_f\mathcal{K}$ are constructible and their formations commutes with any base change. The function $\varphi_{\mathcal{K}}$ (1.16) is constructible and satisfies*

$$\dim R\Phi_f\mathcal{K}_{x\leftarrow t} = \delta(\varphi_{\mathcal{K}})(x \leftarrow t). \tag{1.17}$$

2. *Assume $\mathcal{K} = j_!\mathcal{F}[1]$ for the open immersion $j\colon U = X - Z \to X$ and a locally constant sheaf $\mathcal{F}$ on U and that Z is flat over S. Then, we have $\delta(\varphi_{\mathcal{K}}) \geqq 0$. The function $\varphi_{\mathcal{K}}$ is* flat *over S if and only if $f\colon X \to S$ is universally locally acyclic relatively to $\mathcal{K} = j_!\mathcal{F}[1]$.*

Proof. We sketch and/or recall an outline of proof with some simplifications.

1. The constructibility and the commutativity with base change follows from Proposition 1.7.1 and the local acyclicity of smooth morphism. By devissage, it suffices to show the remaining assertions in the case where $\mathcal{K} = j_!\mathcal{F}[1]$ for the open immersion $j\colon U = X - Z \to X$ and a locally constant sheaf $\mathcal{F}$ on U.

By the commutativity with base change, the equality (1.17) is reduced to the case where S is the spectrum of a complete discrete valuation ring with algebraically closed residue field. Further by base change and the normalization, we may assume that X is normal and that its generic fiber is smooth. By devissage, we may assume that Z is flat over S. In this case, (1.17) was first proved in [14], under an extra assumption that X is smooth, by constructing a good compactification using a deformation argument. Later it was reproved together with a generalization in [13] using the semi-stable reduction theorem of curves without using the deformation argument.

Since the left hand side of (1.17) is constructible by Proposition 1.7.1, the function $\varphi_{\mathcal{K}}$ is also constructible by Lemma 1.2.1.

2. The complex $R\Phi_f\mathcal{K}$ is acyclic except at degree 0 by Lemma 1.10. Hence the assertions follow from the equality (1.17) and Corollary 1.9. □

Corollary 1.14. *Assume further that Z is finite over S and that $\mathcal{K} = j_!\mathcal{F}$ for a locally constant sheaf $\mathcal{F}$ on U. Then, the function $f_*\varphi_{\mathcal{K}}$ (1.3) on S is lower semi-continuous. The function $f_*\varphi_{\mathcal{K}}$ is locally constant if and only if $f\colon X \to S$ is universally locally acyclic relatively to $\mathcal{K} = j_!\mathcal{F}$.*

Proof. It follows from Proposition 1.13, Lemma 1.2.3 and Corollary 1.9. □

We give a slight generalization of Proposition 1.13. Let

$$\begin{array}{ccccc} Z & \xrightarrow{\subset} & X & \xrightarrow{f} & Y \\ & & & \searrow^{p} \quad \swarrow_{g} & \\ & & & S & \end{array} \tag{1.18}$$

be a commutative diagram of morphisms of finite type of schemes such that $g\colon Y \to S$ is flat of relative dimension 1 and that $Z \subset X$ is a closed subscheme quasi-finite over S. Assume that, for every geometric point $x \to X$, if we set $y = f(x)$ and $s = p(x)$ and define $T_{(x)} \subset Y_{(y)}$ to be the image of the finite scheme $Z \times_X X_{(x)}$ over $S_{(s)}$ by $f_{(x)}\colon X_{(x)} \to Y_{(y)}$, then the complement $Y_{(y)} - T_{(x)}$ is essentially smooth over $S_{(s)}$.

Let $\mathcal{L}$ be an object of $D^b_c(X \overset{\leftarrow}{\times}_Y Y)$ such that the restrictions of cohomology sheaves on $Y_{(y)} - T_{(x)}$ are locally constant for every geometric point $x \to X$. Then, similarly as (1.16), we define a function $\psi_{\mathcal{L}}$ on $X \overset{\leftarrow}{\times}_Y Y$ by

$$\psi_{\mathcal{L}}(x \leftarrow w) = a_w(\mathcal{L}|_{Y_{(y)} \times_{S_{(s)}} t}) \tag{1.19}$$

where $y = f(x), s = p(x)$ and $t = g(w)$. We also define a function $\delta(\psi_{\mathcal{L}})$ on $X \overset{\leftarrow}{\times}_S S$ by (1.6).

Proposition 1.15. *Let the notation be as above. Let $\mathcal{L}$ be an object of $D^b_c(X \overset{\leftarrow}{\times}_Y Y)$ and $x \leftarrow t$ be a point of $X \overset{\leftarrow}{\times}_S S$. Set $y = f(x)$ and $s = p(x)$ and assume that the restriction of cohomology sheaf $\mathcal{H}^q\mathcal{L}$ on $Y_{(y)} - T_{(x)}$ is locally constant for every q. Then, we have*

$$\dim R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow t} - \dim R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow s} = \delta(\psi_{\mathcal{L}})(x \leftarrow t). \tag{1.20}$$

Proof. By the canonical isomorphisms $R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow t} \to R\Gamma(Y_{(y)} \times_{S_{(s)}} S_{(t)}, \mathcal{L}|_{Y_{(y)} \times_{S_{(s)}} S_{(t)}})$ and $R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow s} \to R\Gamma(Y_{(y)}, \mathcal{L}|_{Y_{(y)}}) = \mathcal{L}_y$ (1.10), we obtain a distinguished triangle $\to R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow s} \to R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow t} \to R\Phi_{g_{(y)}}(\mathcal{L}|_{Y_{(y)}})_{y\leftarrow t} \to$. Hence it follows from Proposition 1.13.1. □

In fact, (1.17) is a special case of (1.21) below where $X = Y$.

Corollary 1.16. *We keep the notation in Proposition* 1.15. *Let $\mathcal{K}$ be an object of $D^b_c(X)$ such that $\mathcal{L} = R\Psi_f\mathcal{K}$ is an object of $D^b_c(X \overset{\leftarrow}{\times}_Y Y)$. Assume that $\mathcal{L}$ and a point $x \leftarrow t$ of $X \overset{\leftarrow}{\times}_S S$ satisfies the condition in Proposition* 1.15. *Then, we have*

$$\dim R\Phi_p\mathcal{K}_{x\leftarrow t} = \delta(\psi_{\mathcal{L}})(x \leftarrow t). \tag{1.21}$$

Proof. By the isomorphisms $R\Psi_p\mathcal{K} \to R\overleftarrow{g}_*\mathcal{L}$ and $R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow s} \to \mathcal{L}_y \to \mathcal{K}_x$, we obtain a distinguished triangle $\to R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow s} \to R\overleftarrow{g}_*\mathcal{L}_{x\leftarrow t} \to R\Phi_p\mathcal{K}_{x\leftarrow t} \to$. Hence it follows from (1.20). □

We consider the diagram (1.18) satisfying the condition there and assume further that $g\colon Y \to S$ is smooth. Let $\mathcal{K}$ be an object of $D^b_c(X)$ and assume that $p\colon X \to S$ is locally acyclic relatively to $\mathcal{K}$ and that the restriction of $f\colon X \to Y$ to the complement $X - Z$ is locally acyclic relatively to the restriction of $\mathcal{K}$.

We define a function $\varphi_{\mathcal{K},f}$ on Z as follows. By Proposition 1.7.1, the complex $R\Phi_f\mathcal{K}$ is constructible. For a geometric point x of Z, set $y = f(x)$ and let $s = p(x) \to S$ be a geometric point with algebraically closed residue field. Let u be the geometric generic point of the spectrum $Y_{s,(y)}$ of a strictly local discrete valuation ring with the same residue field as that of s. Then, the cohomology of the stalk $R\Phi_f\mathcal{K}_{x\leftarrow u}$ define Λ-representations of the absolute Galois group G_{K_u} of the fraction field K_u of $Y_{s,(y)}$ and hence the total dimension $\dim \mathrm{tot}_y R\Phi_f\mathcal{K}_{x\leftarrow u}$ is defined as the alternating sum. Similarly as (1.16), we define a function $\varphi_{\mathcal{K},f}$ on Z by

$$\varphi_{\mathcal{K},f}(x) = \dim \mathrm{tot}_y R\Phi_f\mathcal{K}_{x\leftarrow u}. \tag{1.22}$$

Proposition 1.17. *We consider the diagram* (1.18) *satisfying the condition there and assume that $Y \to S$ is smooth. Let $\mathcal{K}$ be an object of $D^b_c(X)$ and assume that $p\colon X \to S$ is locally acyclic relatively to $\mathcal{K}$ and that the restriction of $f\colon X \to Y$ to the complement $X - Z$ is locally acyclic relatively to the restriction of $\mathcal{K}$. Then, the function $\varphi_{\mathcal{K},f}$ on Z is constructible and* flat *over S. If Z is étale over S, it is locally constant.*

Proof. We apply Proposition 1.15 to $\mathcal{L} = R\Psi_f\mathcal{K}$. The assumption in Proposition 1.15 that $\mathcal{H}^q\mathcal{L} = R^q\Psi_f\mathcal{K}$ on $Y_{(y)} - T_{(x)} \subset Y_{(y)} = x \overleftarrow{\times}_Y Y$ is locally constant for every q is satisfied for every geometric point x of X by Corollary 1.8. Hence the function $\psi_{\mathcal{L}}$ (1.19) for $\mathcal{L} = R\Psi_f\mathcal{K}$ is defined as a function on $X \overleftarrow{\times}_Y Y$.

In order to apply Lemma 1.5, we show $\psi_{\mathcal{L}}(x \leftarrow w) = \sum_{z\in Z_{(x)}\times_{Y_{(y)}} w} \varphi_{\mathcal{K},f}(z)$ for a point $x \leftarrow w$ of $Z \overleftarrow{\times}_Y Y$ such that w is supported on the image $T_{(x)} \subset Y_{(y)}$ of $Z_{(x)}$. By the assumption that $Y \to S$ is smooth, the Milnor fiber $Y_{(w)} \times_{S_{(t)}} t$ is the spectrum of a discrete valuation ring. Let u be its geometric point dominating the generic point regarded as a geometric point of $Y_{(w)}$. Then by (1.19) and (1.15), we have $\psi_{\mathcal{L}}(x \leftarrow w) = \dim \mathrm{tot}_w(R\Psi_f\mathcal{K}_{x\leftarrow u}) - \dim(R\Psi_f\mathcal{K}_{x\leftarrow w})$. We apply Proposition 1.7.2 to $f\colon X \to Y$ and specializations $y \leftarrow w \leftarrow u$ to compute the right hand side. Then, the distinguished triangle (1.14) implies that the right hand side equals $\sum_{z\in Z_{(x)}\times_{Y_{(y)}} w} \dim \mathrm{tot}_w(R\Phi_f\mathcal{K}_{z\leftarrow u}) = \sum_{z\in Z_{(x)}\times_{Y_{(y)}} w} \varphi_{\mathcal{K},f}(z)$ as required.

Therefore, by applying Lemma 1.5, we obtain $\delta(\psi_{\mathcal{L}}) = \delta(\varphi_{\mathcal{K},f})$ as functions on $Z \overleftarrow{\times}_S S$. Since $R\Phi_p\mathcal{K} = 0$, the function $\psi_{\mathcal{L}}$ is flat over S by (1.21). Hence, the function $\varphi_{\mathcal{K},f}$ is also flat over S. Since it is flat over S, the function $\varphi_{\mathcal{K},f}$ is constructible by Lemma 1.2.1.

If Z is finite over S, the function $p_*\varphi_{\mathcal{K},f}$ is locally constant by Lemma 1.2.3. □

2 Singular support and the stability of vanishing cycles

2.1 Preliminaries on conic subsets and proper intersection

Let E be a vector bundle on a scheme X. Recall that a closed subset S of E is said to be *conic* if it is stable under multiplication. We study the intersection of a conic closed subset with a sub vector bundle.

Lemma 2.1. *Let $0 \to V \to E \to \bar{E} \to 0$ be an exact sequence of vector bundles over a locally noetherian scheme X. For a closed conic subset $S \subset E$, the following conditions are equivalent.*

(1) *The restriction of $E \to \bar{E}$ on S is quasi-finite.*

(2) *The restriction of $E \to \bar{E}$ on S is finite.*

(3) *The intersection $S \cap V$ is contained in the 0-section.*

Proof. (1)⇒(3): The intersection $S \cap V$ is the intersection of S with the inverse image of the 0-section of $\bar{E}$ by $E \to \bar{E}$. Since it is a conic subset of V, quasi-finiteness implies that it is a subset of the 0-section.

(3)⇒(2): We regard E as the complement of the associated projective space bundle $\mathbf{P}(E)$ in $\mathbf{P}(E \oplus \mathbf{A}^1)$ and similarly for $\bar{E}$. Let $\mathbf{P}(E \oplus \mathbf{A}^1)' \to \mathbf{P}(E \oplus \mathbf{A}^1)$ be the blow-up at $\mathbf{P}(V) \subset \mathbf{P}(E) \subset \mathbf{P}(E \oplus \mathbf{A}^1)$. Then the morphism $E \to \bar{E}$ is extended to $\mathbf{P}(E \oplus \mathbf{A}^1)' \to \mathbf{P}(\bar{E} \oplus \mathbf{A}^1)$. The condition (3) implies that the closure $\bar{S}$ of $S \subset E$ in $\mathbf{P}(E \oplus \mathbf{A}^1)$ does not meet $\mathbf{P}(V)$ and hence defines a closed subset of $\mathbf{P}(E \oplus \mathbf{A}^1)'$. Since the restriction of $\mathbf{P}(E \oplus \mathbf{A}^1)' \to \mathbf{P}(\bar{E} \oplus \mathbf{A}^1)$ to the complement of the inverse image of $\mathbf{P}(V)$ is affine, its restriction to $\bar{S}$ is finite and the assertion follows.

(2)⇒(1): Clear. □

Recall that if $Y \to X$ is an unramified morphism of schemes then, locally on Y, there exists an immersion to an étale scheme over X [7, IV-4, Corollaire (18.4.7)]. We generalize the definition of regular immersions to unramified morphisms.

Definition 2.2. *We say that an unramified morphism $Y \to X$ is* regular of codimension r *if, locally on Y, the exists a regular immersion of codimension r to an étale scheme over X. If $Y \to X$ is an unramified morphism regular of codimension r, the conormal sheaf $\mathcal{N}_{Y/X}$ is defined as a locally free $\mathcal{O}_Y$-module of rank r by étale descent. We also call the associated vector bundle $T^*_Y X$ the conormal bundle.*

Lemma 2.3. *Let S be a locally noetherian scheme and $Y \to X$ be an unramified morphism of schemes locally of finite type over S such that the conormal sheaf $\mathcal{N}_{Y/X}$ is locally generated by r sections. For a point $y \in Y$ and its images $x \in X$ and $s \in S$, the following conditions are equivalent:*

(1) *The morphism $Y \to S$ is flat and the unramified morphism $Y \to X$ is regular of codimension r on a neighborhood of y in Y.*

(2) *The morphism $X \to S$ is flat on a neighborhood of x and the unramified morphism $Y_s \to X_s$ is regular of codimension r on a neighborhood of y.*

Proof. Since the assertion is étale local on Y, we may assume that $Y \to X$ is an immersion. Then, it follows from [7, IV-1, Chap. 0 Proposition (15.1.16) b)⇔c)] (cf. [7, IV-4, Proposition (19.2.4) b)⇔c)]). For the convenience of the reader, we record a proof of the following Lemma, which is a key ingredient in the proof of [7, IV-1, Chap. 0 Proposition 15.1.16 b)⇔c)].

Lemma 2.4. *Let $A \to B$ be a local homomorphism of noetherian local rings and k be the residue field of A. Then, for $f \in \mathfrak{m}_B$, the following conditions are equivalent:*

(1) *$\bar{B} = B/fB$ is flat over A and f is a non-zero divisor of B.*

(2) *B is flat over A and $\bar{f}$ is a non-zero divisor of $B \otimes_A k$.*

Proof. (2)⇒(1) Let C denote the chain complex defined by the multiplication $f\colon B \to B$ where the second B is put on degree 0. We have $H_0(C) = \bar{B} = B/fB, H_1(C) = K = \mathrm{Ker}(f\colon B \to B)$ and $H_q(C) = 0$ for $q \neq 0, 1$. For the spectral sequence $E^2_{p,q} = Tor^A_p(H_q(C), k) \Rightarrow Tor^A_{p+q}(C, k)$, the condition (2) implies $Tor^A_r(C, k) = 0$ for $r \neq 0$.

Hence, we obtain $E^2_{1,0} = Tor^A_1(\bar{B}, k) = 0$. Since $\bar{B}$ is ideally separated as an A-module, it is flat over A by [3, n° 5.2 Théorème 1 (iii)⇒(i)]. Since $\bar{B}$ is flat over A, we obtain $E^2_{p,0} = 0$ for $p \neq 0$ and $E^2_{0,1} = K \otimes_A k = 0$. Hence by Nakayama's lemma, we have $K = 0$ and f is a non-zero divisor of B.

(1)⇒(2) The exact sequence $0 \to B \to B \to \bar{B} \to 0$ induces exact sequences $0 = Tor^A_1(\bar{B}, k) \to B \otimes_A k \to B \otimes_A k$ and $Tor^A_1(B, k) \to Tor^A_1(B, k) \to Tor^A_1(\bar{B}, k) = 0$. The first exact sequence means that $\bar{f} \in B \otimes_A k$ is a non-zero divisor. The second exact sequence shows that the finitely generated B-module $Tor^A_1(B, k)$ is 0 by Nakayama's lemma. Since B is ideally separated as an A-module, it is flat over A by [3, n° 5.2 Théorème 1 (iii)⇒(i)]. □

Definition 2.5. *Let S be a locally noetherian scheme, X be a scheme of finite type over S and $T \to X$ be an immersion.*

1. *We say that an unramified morphism $Y \to X$ regular of codimension r meets T* properly *over S if for every point $s \in S$, for every irreducible component P of the fiber T_s and for every irreducible component Q of $P \times_X Y$, we have $\dim Q = \dim P - r$.*

2. *Let Y be a smooth scheme over S of relative dimension r and $f\colon X \to Y$ be a morphism over S. We say that the fibers of $f\colon X \to Y$ meets T* properly *if its graph $X \to X \times_S Y$ meets $T \times_S Y$ properly over Y.*

In Definition 2.5.2, the fibers of $f\colon X \to Y$ meets T properly if and only if the fibers of the composition $T \to Y$ meets T properly.

Proposition 2.6. *Let S be a locally noetherian scheme, X be a* flat *scheme of finite type over S and $T \to X$ be an immersion. For an unramified morphism $Y \to X$ regular of codimension r, a closed point $y \in Y$ and the image $s \in S$ of y, we consider the following conditions:*

(1) *There exists an open neighborhood V of y in Y such that $V \to S$ is flat and that $V \to X$ meets T properly over S.*

(2) *There exists an open neighborhood V of y in Y_s such that the unramified morphism $V \to X_s$ is regular of codimension r and meets T_s properly.*

We have (1)⇒(2). *The conditions* (1) *and* (2) *are equivalent if there exists an integer d satisfying* $\dim P = d$ *for every point $s \in S$ and for every irreducible component P of the fiber T_s. If $T \to S$ is smooth of relative dimension d, they are further equivalent to the following conditions* (3) *and* (4)*:*

(3) *There exists an open neighborhood V of y in Y such that $V \to S$ is flat and that $T \times_X V \to S$ is flat of relative dimension $d - r$.*

(4) *There exists an open neighborhood V of y in Y_s such that the unramified morphisms $V \to X_s$ and $V \times_{X_s} T_s \to T_s$ are regular of codimension r.*

Proof. (1)⇒(2) The unramified morphism $V_s \to X_s$ is regular of codimension r by Lemma 2.3 (1)⇒(2). The rest follows from Definition 2.5.

(2)⇒(1) By Lemma 2.3 (2)⇒(1), $Y \to S$ is flat on a neighborhood of y. Let $g\colon T \times_X Y \to S$ denote the canonical morphism. Then, by [7, VI-3, Théorème (13.1.3)] the function $n(v) = \dim_v g^{-1}(g(v))$ defined on $T \times_X Y$ is upper semi-continuous. Since $n(v) \geqq d - r$, the subset of $T \times_X Y$ where the equality $n(v) = d - r$ holds is open. Since the condition

(2) implies $n(y) = d - r$, we have $n(v) = d - r$ on a neighborhood of y. Hence $Y \to X$ meets T properly over B on a neighborhood of y.

(2)⇔(4) For a regular local ring, a sequence of elements of the maximal ideal is a regular sequence if and only if it is a part of a system of parameters [7, IV-1, Chap. 0 Corollaire (16.5.6)]. Hence the assertion follows.

(3)⇔(4) It suffices to apply Lemma 2.3 to the unramified morphisms $Y \to X$ and $Y \times_X T \to T$. □

Corollary 2.7. *Let X be a scheme of finite type over a field k and $T \to X$ be an immersion. Let Y be a smooth scheme over k and $f\colon X \to Y$ be a morphism over k.*

1. *If $f\colon X \to Y$ is flat, then the following conditions are equivalent:*

(1) *The fibers of $f\colon X \to Y$ meets T properly.*

(2) *For every closed point y of Y, the regular immersion $X_y \to X$ meets $T \subset X$ properly.*

2. *Assume $T = X$ and $Y = C$ is a smooth curve over k. Then, the following conditions are equivalent:*

(1) *$f\colon T \to C$ is an open mapping.*

(2) *For every irreducible component P of T, the restriction $f|_P\colon P \to C$ is an open mapping.*

(3) *The fibers of $f\colon T \to C$ meets T properly.*

Proof. 1. By Proposition 2.6(1)⇔(2), the condition (1) is equivalent to that for every closed point y of Y, the regular immersion $X_y \to X \times_S y$ meets $T \times_S y$ properly. This is equivalent to the condition (2).

2. (1)⇔(2) We may assume that T is reduced. Then, since C is a smooth curve, the condition (1) is equivalent to that $f\colon T \to C$ is flat. Similarly, the condition (2) is equivalent to that, for every irreducible component P of T, the restriction $f|_P\colon P \to C$ is flat. Hence, we have (1)⇔(2).

(2)⇔(3) We may assume that $T = P$ is integral. Then by Proposition 2.6(1)⇔(2), the condition (3) is equivalent to that for every closed point y of C, the immersion $T_y \to T$ is regular of codimension 1. This is equivalent to that $T \to C$ is flat. □

Corollary 2.8. *Let S be a locally noetherian scheme, X be a flat scheme of finite type over S and $T \to X$ be an immersion. Let Y be a smooth scheme over S and $f\colon X \to Y$ be a flat morphism over S. Assume that the fiber of $f\colon X \to Y$ meets T properly.*

1. *Let $W \to Y$ be an unramified morphism of codimension r of flat schemes over S. Then the unramified morphism $V = W \times_Y X \to X$ of codimension r of flat schemes over S meets T properly over S.*

2. *Let $g\colon Y \to C$ be a flat morphism of smooth schemes over S. Then the fibers of the composition $g \circ f\colon X \to C$ meets T properly over S.*

Proof. 1. By Proposition 2.6(1)⇔(2), we may assume that $S = \operatorname{Spec} k$ for a field k. Since the assertion is étale local, we may assume that $W \to Y$ is a regular immersion. Let $w \in W$ be a closed point and $t_1, \ldots, t_r \in \mathcal{O}_{Y,w}$ be elements defining W on a neighborhood of w. Since $\mathcal{O}_{Y,w}$ is a regular local ring, we may extend it to a regular sequence $t_1, \ldots, t_d$. Let P be an irreducible component of T. Then, since the closed fiber $w \times_Y P \subset P$ defined by $t_1, \ldots, t_d$ is of codimension d, the closed subscheme $V \times_X P = W \times_Y P \subset P$ defined by $t_1, \ldots, t_r$ is of codimension r on a neighborhood of the fiber $w \times_Y P \subset P$. Hence the assertion follows.

2. By Proposition 2.6(1)⇔(2), we may also assume that $S = \operatorname{Spec} k$ for a field k. Then, it suffices to apply 2. to the closed imersion $Y_c \to Y$ of the fiber at closed point $c \in C$. □

Lemma 2.9. *Let $f\colon X \to S$ be a surjective morphism of irreducible schemes of finite type over a field k and let r be the dimension of the generic fiber X_η. Define a decreasing sequence of closed subsets of S by $S_i = \{s \in S \mid \dim X_s \geqq r+i\}$. Let D be a Cartier divisor of S meeting every irreducible component of every S_i properly. Then, if P is an irreducible component of $X \times_S D$, the closure Q of its image in S is an irreducible component of D.*

Proof. We have

$$\dim S_i + r + i \leqq \dim S_i \times_S X \leqq \dim X = \dim S + r$$

and the equality in the second inequality means $i = 0$. Hence we have $\dim S_i + i \leqq \dim S$. If the generic point of Q is contained in $S_i - S_{i+1}$, by the assumption that D meets S_i properly, we have

$$\dim P = \dim Q + r + i \leqq \dim S_i - 1 + r + i \leqq \dim S - 1 + r$$

and the equality in the first inequality means Q is an irreducible component of $S_i \cap D$. Since $\dim P = \dim X - 1 = \dim S + r - 1$, we have equalities everywhere. Thus, $i = 0$ and Q is an irreducible component of $S_0 \cap D = D$. □

2.2 Non-characteristic morphisms

We define in the next subsection a singular support of a constructible sheaf as a conic closed subset of the cotangent bundle by requiring a condition on the local acyclicity. To state the condition, we introduce some terminology.

For a smooth scheme X of dimension d over a field k, the cotangent bundle T^*X is the vector bundle $\mathbf{V}(\Omega^1_{X/k})$ associated to the locally free $\mathcal{O}_X$-module $\Omega^1_{X/k}$ of rank d. In this article, we use a contra-Grothendieck convention $\mathbf{V}(\mathcal{E}) = \operatorname{Spec} S^\bullet_{\mathcal{O}_X}\mathcal{E}^\vee$ to denote the vector bundle associated to a locally free $\mathcal{O}_X$-module $\mathcal{E}$. More generally for a smooth morphism $X \to S$, we define the relative cotangent bundle by $T^*X/S = \mathbf{V}(\Omega^1_{X/S})$.

For a regular immersion $Y \to X$, let $T^*_Y X$ denote the conormal bundle and $T^*_Y X \to Y \times_X T^*X$ the canonical morphism. We identify the 0-section of T^*X with the conormal bundle $T^*_X X$ of the identity $X \to X$. The cotangent bundle T^*X is defined as the conormal bundle of the diagonal immersion $X \to X \times X$.

First, we define the non-characteristicity for a morphism to a curve.

Definition 2.10. *Let X be a smooth scheme over a field k. Let $S = (S_i)_{i\in I}$ be a finite family of conic closed subsets of the cotangent bundle T^*X and let $T_i = S_i \cap T^*_X X \subset X$ be the intersections with the 0-section. Let C be a smooth curve over k and $f\colon X \to C$ be a flat morphism over k.*

1. *We say that $f\colon X \to C$ is* non-characteristic *with respect to S if the following conditions* (i) *and* (ii) *are satisfied:*

(i) *The inverse image of the union $\bigcup_{i\in I} S_i$ by the canonical map $df\colon X \times_C T^*C \to T^*X$ is a subset of the 0-section.*

(ii) *For every $i \in I$, the restriction $f|_{T_i}\colon T_i \to C$ of f is an* open *mapping.*

2. *We say that a closed point u of X is an* isolated characteristic point *of $f\colon X \to C$ with respect to S if there exists a neighborhood V of u such that the restriction $V - \{u\} \to C$ is non-characteristic with respect to the family of intersections $S_i \cap T^*(V - \{u\})$.*

The condition (i) in Definition 2.10.1 means that locally on X, if ω is a basis of T^*C, the intersection of $\bigcup_{i\in I} S_i$ with the section of T^*X defined by $f^*\omega$ is empty. If $S = (S_0)$ consists of the 0-section $S_0 = T^*_X X \subset T^*X$, a flat morphism $f\colon X \to C$ is non-characteristic with respect to S if and only if it is smooth. Thus, we may regard the non-characteristic condition as a notion analogous to the smoothness. If a family S contains the 0-section $S_0 = T^*_X X \subset T^*X$, a flat morphism $f\colon X \to C$ non-characteristic with respect to S is smooth. If $f\colon X \to C$ is a smooth morphism, the morphism $df\colon X \times_C T^*C \to T^*X$ is an injection and hence the condition (i) in Definition 2.10.1 is equivalent to that the intersection of $\bigcup_{i\in I} S_i$ with the image of $df\colon X \times_C T^*C \to T^*X$ is a subset of the 0-section $T^*_X X$.

Since S_i is conic and closed, the intersection $T_i = S_i \cap X$ equals the image of S_i by the projection $T^*X \to X$. If S_i is irreducible, then as its image, T_i is also irreducible. If we regard T_i as a closed subscheme of X with the reduced scheme structure, the condition (ii) in Definition 2.10.1 is equivalent to that $f|_{T_i}\colon T_i \to C$ is flat.

If a singular support $SS\mathcal{K}$ contains the 0-section $T^*_X X \subset T^*X$, an isolated characteristic point of $f\colon X \to C$ with respect to $\mathcal{K}$ is an isolated singular point of f.

A flat morphism $f\colon W \to C$ from an étale scheme W over X defines an unramified morphism $W \to X \times C$ of flat schemes over C. We introduce a terminology for such family of morphisms to be non-characteristic.

Definition 2.11. *Let the notation $X, S = (S_i)_{i\in I}$ and $T_i = S_i \cap T^*_X X \subset X$ be as in Definition 2.10.*

1. *Let B be a smooth scheme over k and*

$$i\colon W \longrightarrow X \times B \tag{2.1}$$

be an unramified morphism regular of codimension r of flat schemes over B. We say that $i\colon W \to X \times B$ over B is non-characteristic *with respect to S if the following conditions* (i) *and* (ii) *are satisfied:*

(i) *By the composition $T^*_W(X \times B) \to T^*(X \times B/B) \times_X W = T^*X \times_X W \to T^*X$ of the canonical morphism and the projection, the inverse image of $\bigcup_{i\in I} S_i$ to the conormal bundle $T^*_W(X \times B)$ is a subset of the 0-section.*

(ii) *The unramified morphism $W \to X \times B$ meets $T_i \times B$ properly over B for every $i \in I$.*

2. *Let $f\colon X \to Y$ be a flat morphism to a smooth scheme over k. We say that $f\colon X \to Y$ is* non-characteristic *with respect to S if the graph $X \to X \times Y$ of $f\colon X \to Y$ over Y is non-characteristic with respect to S.*

In the following, by abuse of terminology, we say $W \to X \times B$ meets T_i properly, instead of saying it meets $T_i \times B$ properly over B if there is no fear of confusion. Proposition 2.6 shows that non-charctericity condition is checked fiberwise. The following lemma implies that Definition 2.10.1 is equivalent to a special case of Definition 2.11.

Lemma 2.12. *Let $f\colon X \to C$ be a flat morphism to a smooth curve over k. Let $S = (S_i)_{i\in I}$ be a finite family of conic closed subsets of the cotangent bundle T^*X. Then, the following conditions are equivalent:*

(1) *The morphism* $f\colon X \to C$ *is non-characteristic with respect to* S *in the sense of Definition* 2.10.1.

(2) *The morphism* $f\colon X \to C$ *is non-characteristic with respect to* S *in the sense of Definition* 2.11.2.

Proof. The morphisms $X \to C$ and the immersion $X \to X \times C$ of schemes over C define the upper and the lower exact sequences in the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_X \otimes_{\mathcal{O}_C} \Omega^1_{C/k} & \longrightarrow & \Omega^1_{X/k} & \longrightarrow & \Omega^1_{X/C} & \longrightarrow & 0 \\
 & & \| & & \| & & \\
\mathcal{N}_{X/(X\times C)} & \longrightarrow & \mathcal{O}_X \otimes_{\mathcal{O}_{X\times C}} \Omega^1_{(X\times C)/C} & \longrightarrow & \Omega^1_{X/C} & \longrightarrow & 0
\end{array}$$

respectively. The condition (i) in Definition 2.10.1 is formulated in terms of the fibers of the morphism $X \times_C T^*C \to T^*X$ defined by the upper left arrow and the morphism $T^*_X(X \times C) \to T^*X$ in the condition (i) in Definition 2.11.1 is defined by the lower left arrow. Hence these conditions are equivalent.

For the conditions (ii), the equivalence follows from Corollary 2.7.2 (1)$\Leftrightarrow$(3). $\square$

For families $S = (S_i)_{i\in I}$ and $S' = (S'_j)_{j\in J}$ of closed conic subsets of T^*X, we say that S' is a refinement of S if there exists a decomposition $J = \amalg_{i\in I} J_i$ satisfying $S_i = \bigcup_{j\in J_i} S'_j$ for every $i \in I$. If $S' = (S'_j)_{j\in J}$ is a refinement of $S = (S_i)_{i\in I}$, then the non-characteristicity with respect to S' implies that with respect to S. For the pull-back by étale morphism, the following properties hold.

Lemma 2.13. *Let* $Y \to X$ *be an étale morphism and let* $T^*Y \to T^*X$ *be the canonical morphism. Assume* S_i *is* irreducible *for every* $i \in I$. *Let* $S' = (S'_i)_{i\in I}$ *be the family of the pull-backs* $S'_i = S_i \times_{T^*X} T^*Y$ *and* $S'' = (S''_j)_{j\in J}$ *be the family of their irreducible components.*

1. *If* S''_j *is an irreducible component of* S'_i, *then* $T''_j = S''_j \cap Y$ *is an irreducible component of* $T'_i = S'_i \cap Y$.

2. *Let* B *be a smooth scheme over* k *and* $W \to Y \times B$ *be an unramified morphism regular of codimension* r *of flat schemes over* B. *Then, the morphism* $W \to Y \times B$ *is non-characteristic with respect to the pull-back* S' *of* S *if and only if it is non-characteristic with respect to the refinement* S''.

Proof. 1. We canonically identify the morphism $T^*Y \to T^*X$ with the base change $Y \times_X T^*X \to T^*X$. Then, since $S'_i = S_i \times_X Y \to S_i$ is flat, the mapping $S''_j \to S_i$ is dominant by [7, IV-2, Proposition (2.3.4)]. Hence $T''_j \to T_i$ is also dominant. Since $T'_i \to T_i \times_X Y$ is an isomorphism, T''_j is an irreducible component of T'_i.

2. It follows from 1. $\square$

Let $i\colon Y \to X$ be the immersion of a smooth subscheme of codimension c and $S = (S_i)_{i\in I}$ be a finite family of conic closed subsets of the cotangent bundle T^*X. Here, by abuse of notation, we use the same letter i to denote the immersion and the index. Assume that the regular immersion $i\colon Y \to X$ is non-characteristic with respect to S. Then, we define a family $i^{-1}S = (i^{-1}S_i)_{i\in I}$ of closed conic subsets $i^{-1}S_i \subset T^*Y$ by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$.

For continuous mappings $f\colon Z \to X$ and $g\colon Z \to Y$ such that $g\colon Z \to Y$ is a closed mapping, the image of a closed subset S of X by the correspondence $X \leftarrow Z \to Y$ is defined to be the closed subset $g(f^{-1}(S))$ of Y. Note that by Lemma 2.1 (3)⇒(2) and by the assumption that $i\colon Y \to X$ is non-characteristic with respect to S, the restriction of the surjection $Y \times_X T^*X \to T^*Y$ on the intersection $S'_i = S_i \cap (Y \times_X T^*_X X)$ is finite and its image $i^{-1}S_i \subset T^*Y$ is closed.

Lemma 2.14. *Let $i\colon Y \to X$ be the immersion of a smooth subscheme of codimension c non-characteristic with respect to S. Let B be a smooth scheme and $W \to Y \times B$ be an unramified morphism regular of codimension r of flat schemes over B.*

Then, $W \to Y \times B$ is non-characteristic with respect to $i^{-1}S$ if and only if the unramified morphism $W \to X \times B$ regular of codimension $r + c$ is non-characteristic with respect to S.

Proof. By the morphism of exact sequences of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W \times_Y T^*_Y X & \longrightarrow & T^*_W(X \times B) & \longrightarrow & T^*_W(Y \times B) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W \times_Y T^*_Y X & \longrightarrow & W \times_X T^*X & \longrightarrow & W \times_Y T^*Y & \longrightarrow & 0, \end{array}$$

the condition (i) in Definition 2.11.1 for $W \to Y \times B$ and for $W \to X \times B$ are equivalent.

By the assumption that the immersion $i\colon Y \to X$ is non-characteristic, the regular immersion $Y \to X$ meets T_i properly for every $T_i = S_i \cap T^*_X X$. Hence the condition (ii) that $W \to Y \times B$ meets $T_i \times_X Y$ properly is also equivalent to that $W \to X \times B$ meets T_i properly. □

Assume that Y is a smooth divisor of X. If S_i is irreducible and is regarded as a reduced scheme, the condition (ii) in Definition 2.11.1 is equivalent to that $S_i \times_X Y \to S_i$ is a divisor of S_i since $S_i \to T_i$ is surjective. If we regard T_i as a reduced closed subscheme of X, the same condition on T_i means that $T_i \times_X Y$ is a divisor of T_i. If $S_i \subset T^*X$ is of dimension d, then $i^{-1}S_i \subset T^*Y$ is of dimension $d - 1$.

Let $f\colon Y \to X$ be a smooth morphism and $S = (S_i)_{i \in I}$ be a finite family of conic closed subsets of the cotangent bundle T^*X. We define a family $f^{-1}S = (f^{-1}S_i)_{i \in I}$ of closed conic subsets $f^{-1}S_i \subset T^*Y$ by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$.

Let $W \to Y \times B$ be an unramified morphism regular of codimension r. Let w be a closed point of W. Replacing W by a neighborhood of w, we decompose the unramified morphism $W \to Y \times B$ into the composition $W \to V \to Y \times B$ of a regular immersion of codimension r and an étale morphism. Let $w \in V$ also denote the image of w and t be an element of the maximal ideal $\mathfrak{m}_w \subset \mathcal{O}_{V,w}$. Replacing V by a neighborhood of w where t is defined and we define a morphism $V \to X \times B'$ where $B' = B \times \mathbf{A}^1$.

Lemma 2.15. *Let $f\colon Y \to X$ be a smooth morphism of relative dimension 1 and let $W \to Y \times B$ be an unramified morphism of flat schemes over B as above. Assume that $W \to Y \times B$ over B is non-characteristic with respect to the pull-back $f^{-1}S$. Assume that every S_i is of dimension $d = \dim X$ and that, for every T_i, its irreducible components have the same dimension. Then, for a closed point w of W such that the residue field $k(w)$ is separable over k, one of the following holds:*

(1) *On a neighborhood of w, the composition $W \to Y \times B \to X \times B$ is unramified regular of codimension $r - 1$ and is non-characteristic with respect to S.*

(2) *There exists a finitely many proper closed subsets of the fiber $T^*_w(V/B)$ of the relative cotangent bundle such that if $dt \in T^*_w(V/B)$ is not contained in their union, the following*

condition is satisfied: On a neighborhood of w, the morphism $V \to X \times B'$ is étale, $W \to B'$ is flat and the unramified morphism $W \to X \times B'$ over B' regular of codimension r is non-characteristic with respect to S.

Proof. In the case (1), if the composition $W \to Y \times B \to X \times B$ is unramified, it is regular of codimension $r-1$ and is non-characteristic with respect to S.

Let $b \in B$ be the image of w and t be a function defined on an étale neighborhood V of w in $X \times B$. Then, after shrinking V if necessary, the morphism $W \to V \to X \times B'$ satisfies the condition in (2) of Lemma 2.15 if its base change by $b \to B$ satisfies the condition in (2) of Lemma 2.15 by Proposition 2.6 (2)⇒(1) and by the assumption that the irreducible components of every T_i have the same dimension. Hence, it is reduced to the case where $B = \{b\}$ consists of a single point.

We assume $B = \{b\}$ consists of a single point such that $k(b)$ is a finite separable extension of k. Let $\mathfrak{m}_w \subset \mathcal{O}_{V,w}$ be the maximal ideal and we identify the fiber T^*_wV with $\mathfrak{m}_w/\mathfrak{m}_w^2$. The morphism $V \to X \times B'$ is étale at w if dt is not in the image of $w \times_x T^*_xX$ where $x \in X$ denotes the image of w. Then, the morphism $W \to X \times B'$ is unramified regular of codimension r.

We consider the flatness of $W \to B'$. Since the immersion $W \to V$ is regular of codimension r, we have $r \leqq \dim V$. First, we consider the case where $r = \dim V$. Since W is smooth over k of dimension $\dim V - r = 0$, it is étale over k and $W \to X \times B$ is unramified in this case.

We assume $r < \dim V$. Set $A = \mathcal{O}_{V,w}$ and $\bar{A} = \mathcal{O}_{W,w}$. Then, the local ring $\bar{A}$ is of Cohen-MaCaulay and we have $\dim \bar{A} = \dim V - r \geqq 1$. Hence any prime ideal $\mathfrak{p}$ of $\bar{A}$ associated to the $\bar{A}$-module $\bar{A}$ is minimal and is not equal to the maximal ideal $\bar{\mathfrak{m}}_w \subset \bar{A}$. Its image in $\bar{\mathfrak{m}}_w/\bar{\mathfrak{m}}_w^2$ is not the whole space by Nakayama's lemma. Thus, $W \to B' = B \times \mathbf{A}^1$ is flat if $dt \in T^*_wV = \mathfrak{m}_w/\mathfrak{m}_w^2$ is not contained in the inverse image by the surjection $\mathfrak{m}_w/\mathfrak{m}_w^2 \to \bar{\mathfrak{m}}_w/\bar{\mathfrak{m}}_w^2$ of the images of minimal prime ideals of $\bar{A}$.

We consider the condition (i) in Definition 2.11.1. We identify $w \times_x T^*_xX$ as a subspace of T^*_wV by the pull-back by the smooth morphism $V \to X$. By the assumption that $W \to Y \times B$ over B is non-characteristic with respect to the pull-back $f^{-1}S$, the pull-back by the morphism $w \times_W T^*_WV \to T^*_wV$ of the image of $w \times_x (T^*_xX \cap S) \subset w \times_x T^*_xX \subset T^*_wV$ is a subset of 0. Here S denote the union $\bigcup_{i\in I} S_i$ by abuse of notation. Hence, if dt is not contained in the translate by the image of $w \times_W T^*_WV \to T^*_wV$ of $w \times_x (T^*_xX \cap S) \subset T^*_wV$, the morphism $W \to X \times B'$ satisfies on a neighborhood of w the condition (i) in Definition 2.11.1.

We consider the condition (ii) in Definition 2.11.1. The argument is similar to that on the flatness of $W \to B'$. Let $T_i = S_i \cap T^*_XX$ and assume $w \in T_i \times_X V$. Since the immersion $W \to V$ is regular of codimension r and by the assumption that $W \to Y \times B$ meets $T_{i,Y} = T_i \times_X Y$ properly, we have $r \leqq \dim T_{i,Y} = \dim T_i + 1$. First, we consider the case where $r = \dim T_i + 1$. Since $\dim S_i = \dim X = d$, the intersection $T^*_xX \cap S_i$ with the fiber has dimension $\dim(T^*_xX \cap S_i) \geqq \dim S_i - \dim T_i = d + 1 - r$. Since $\dim w \times_W T^*_WY = r$, we have $\dim w \times_W T^*_WY + \dim(T^*_xX \cap S_i) \geqq d+1 = \dim T^*_wV > d = \dim(w \times T^*_xX)$. Since the inverse image in $w \times_W T^*_WY$ of $T^*_xX \cap S_i$ is a subset of 0, the morphism $w \times_W T^*_WY \to T^*_wV$ is an injection and its image is not a subspace of the image of $w \times T^*_xX$. Thus the canonical morphism $T^*_WY \to T^*(Y/X)$ and hence $W \times_X T^*X \to T^*W$ are surjections at w and the morphism $W \to X \times B$ is unramified at w in this case.

We assume $r < \dim T_{i,Y} = \dim T_i + 1$ and set $\bar{A}_i = \mathcal{O}_{W \times_X T_i, w}$. Then similarly as above, we have $\dim(\bar{A}_i) = \dim T_i + 1 - r \geqq 1$ and no minimal prime ideal is the maximal ideal of $\bar{A}_i$. Thus the unramified morphism $W \to X \times B'$ over B' meets T_i properly if $dt \in T^*_u V = \mathfrak{m}_w/\mathfrak{m}_w^2$ is not contained in the inverse image by the surjection $\mathfrak{m}_w/\mathfrak{m}_w^2 \to \bar{\mathfrak{m}}_{w,i}/\bar{\mathfrak{m}}_{w,i}^2$ of the images of minimal prime ideals of $\bar{A}_i$ where $\bar{\mathfrak{m}}_{w,i} \subset \bar{A}_i$ denotes the maximal ideal. Then, the morphism $W \to X \times B'$ satisfies on a neighborhood of w the condition (ii) in Definition 2.11.1. □

Let $f\colon X \to Y$ be a morphism of smooth schemes over k. Let $S = (S_i)_{i\in I}$ be a finite family of closed conic subsets of T^*X. We assume that the following condition is satisfied.

(Q) For every S_i and for every irreducible component P of the inverse image of S_i by the canonical morphism $X \times_Y T^*Y \to T^*X$, the intersection $Q = P \cap (X \times_Y T^*_Y Y)$ with the 0-section satisfies the following conditions:

(Q1) If $\dim Q \geqq \dim Y$, the fibers of $f\colon X \to Y$ meet Q properly and we have $P = Q$.

(Q2) If $\dim Q < \dim Y$, the restriction $Q \to Y$ of f is finite.

In the case (Q1), we define $f_!P \subset T^*Y$ to be the 0-section of T^*Y supported on the closure of the image of Q. In the case (Q2), we define an irreducible conic closed subset $f_!P \subset T^*Y$ to be the image of $P \subset Q \times_Y T^*Y \subset X \times_Y T^*Y$ by $X \times_Y T^*Y \to T^*Y$. We define a finite family $f_!S$ of irreducible conic closed subsets of T^*Y to be that consisting of $f_!P$ for irreducible components P of the inverse images of S_i by the canonical morphism $X \times_Y T^*Y \to T^*X$.

Lemma 2.16. *Let $f\colon X \to Y$ be a flat morphism of smooth schemes over k. Let $S = (S_i)_{i\in I}$ be a finite family of closed conic subsets of T^*X satisfying the condition* (Q) *above and define $f_!S$ as above.*

1. *Let $W \to Y \times B$ be an unramified morphism of codimension r of flat schemes over B. Assume that $W \to Y \times B$ is non-characteristic with respect to $f_!S$. Then its base change $V \to X \times B$ is non-characteristic with respect to S.*

2. *Let $g\colon Y \to C$ be a flat morphism to a smooth curve and define a closed subset $Z \subset X$ to be the union of Q as in the condition* (Q2) *finite over Y. If y is an isolated characteristic point of $g\colon Y \to C$, then $x \in Z \cap f^{-1}(y)$ is an isolated characteristic point of $g \circ f\colon X \to C$.*

Proof. 1. By the condition (Q1) and by the definition of $f_!S$, the condition (i) in Definition 2.11.1 for $f_!S$ implies that for S. Since $W \to B$ is flat, it meets $Y \times B$ properly. Let S_i be a member of S and $T_i = S_i \cap T^*_X X$ be the intersection with the 0-section. We show that $V \to X \times B$ meets $T_i \times B$ properly. Since T_i regarded as a subset of the 0-section $X \times_Y T^*_Y Y$ is contained in the inverse image of S_i by $X \times_Y T^*Y \to T^*X$, there exists an irreducible component P of the inverse image of S_i by $X \times_Y T^*Y \to T^*X$ such that the intersection $Q = P \cap (X \times_Y T^*_Y Y)$ equals T_i.

If $\dim T_i = \dim Q \geqq \dim Y$, then, by (Q1), the fibers of $X \to Y$ meets $Q = T_i$ properly. Since $W \to Y \times B$ over B meets $Y \times B$ properly, the immersion $V \to X \times B$ also meets $T_i \times B$ properly by Corollary 2.8.1. Assume $\dim T_i = \dim Q < \dim Y$. Then, the intersection $f_!P \cap T^*_Y Y$ with the 0-section equals the image $f(Q) = f(T_i)$. Since $W \to Y \times B$ is non-characteristic with respect to $f_!S$, it meets $f(T_i) \times B$ properly over B. Since $T_i \to f(T_i)$ is finite, the base change $V \to X \times B$ also meets $T_i \times B$ properly over B. Hence $V \to X \times B$ is non-characteristic with respect to S.

2. We may assume that the restriction of g on $Y - \{y\}$ is non-characteristic with respect to $f_!S$. Then, by 1., the restriction of $g \circ f$ on $X - f^{-1}(y)$ is non-characteristic with respect to S. As in the proof of 1., the condition (i) in Definition 2.10.1 for S is satisfied on the complement of Z. Hence it is is satisfied on the complement of the finite set $Z \cap f^{-1}(y)$. For Q satisfying $\dim Q \geqq \dim Y$, by Corollary 2.8.2 and by the assumption that the fibers of $f\colon X \to Y$ meets Q properly in (Q1), the fibers of $g \circ f\colon X \to C$ meets Q properly. Hence the condition (i) in Definition 2.10.1 for S is also satisfied on the complement of $Z \cap f^{-1}(y)$. □

2.3 Singular support and the local acyclicity

Let k denote a perfect field of characteristic $p > 0$ and Λ a finite field of characteristic $\neq p$. For a constructible complex $\mathcal{K}$ of Λ-modules on X, its support is the smallest closed subset Z of X such that $\mathcal{K}$ is acyclic on the complement $X - Z$.

Definition 2.17. *Let X be a smooth scheme of dimension d over k and $\mathcal{K}$ be a constructible complex of Λ-modules on X. Let $S = (S_i)_{i \in I}$ be a finite family of irreducible conic closed subsets of dimension d of the cotangent bundle T^*X.*

1. *For an integer $r \geqq 0$, we define a condition on S:*

(SSr) *Let B be a smooth scheme over k and $i\colon W \to X \times B$ be an unramified morphism regular of codimension $q \leqq r$ of flat schemes over B. Then the composition $f\colon W \to B$ with the second projection is* locally acyclic *relatively to the pull-back to W of $\mathcal{K}$ if the morphism $i\colon W \to X \times B$ over B is* non-characteristic *with respect to S.*

2. *If S satisfies the condition* (SS1) *and if the support of $\mathcal{K}$ is a subset of the intersection with the* 0*-section $\bigcup_{i \in I} S_i \cap T^*_X X \subset X$, we say that the union $\bigcup_{i \in I} S_i$ is a* singular support *of $\mathcal{K}$.*

Note that if S satisfies (SSr) and if $S' \supset S$ is another finite family of irreducible conic closed subschemes of codimension d of T^*X, it also satisfies (SSr). Hence the condition (SS1) does *not* uniquely determine the singular support. However, by abuse of notation, if S is a singular support of $\mathcal{K}$ and we write $S = SS\mathcal{K}$. In (SSr) the local acyclicity can be replaced by the universal local acyclicity, see Lemma 2.19.1 below. By the generic local acyclicity [5, Corollaire 2.16], the empty family and hence any finite family of irreducible conic closed subschemes of codimension d of T^*X satisfy (SS0). We study the meaning of the condition (SSr) in Lemma 2.23 later in this subsection.

If S contains the 0-section of the cotangent bundle T^*X and if an unramified morphism $i\colon W \to X \times B$ of codimension r is non-characteristic with respect to S, then the canonical morphism $T^*_W(X \times B) \to T^*X$ is injective and the composition $W \to B$ is smooth by the Jacobian criterion of smoothness. Hence, if the cohomology sheaf $\mathcal{H}^q(\mathcal{K})$ is locally constant for every integer q, the 0-section of the cotangent bundle T^*X is a singular support of $\mathcal{K}$ by the local acyclicity of smooth morphism.

Let $U = X - D$ be the complement of a divisor D with simple normal crossings, $j\colon U \to X$ be the open immersion and let $\mathcal{F}$ be a locally constant constructible sheaf on U. If $\mathcal{F}$ is tamely ramified along D, the singular support $SS(j_!\mathcal{F})$ is described as follows. Let $D_1, \ldots, D_m$ be the irreducible components of D and, for a subset $I \subset \{1, \ldots, m\}$, let $D_I = \bigcap_I D_i$. Then, the family $S = (T^*_{D_I} X)_{I \subset \{1,\ldots,m\}}$ of the conormal bundles is a singular support of $j_!\mathcal{F}$ by [8, 1.3.3 (i)].

The following immediate consequence of the flatness of the total dimension, Proposition 1.17, will play a crucial role in the definition of the characteristic cycle.

Lemma 2.18. *Let the notation be as in Definition* 2.17. *Let B be a smooth scheme over k, let $C \to B$ be a smooth morphism of relative dimension* 1 *and*

$$X \times B \longrightarrow C \tag{2.2}$$

be a flat morphism of smooth schemes over B. Let $Z \subset X \times B$ be a closed subscheme quasi-finite over B such that the restriction $X \times B - Z \to X \times C$ of the morphism induced by (2.2) *is non-characteristic with respect to a singular support S of $\mathcal{K}$. Then, the function $\varphi_{\mathrm{pr}_1^*\mathcal{K},f}$ on Z defined by*

$$\varphi_{\mathrm{pr}_1^*\mathcal{K},f}(z) = \dim \mathrm{tot}_{f(z)}\phi_z(\mathcal{K}|_{W_b}, f|_b) \tag{2.3}$$

is flat over B and constructible. In particular if Z is étale over B, the function $\varphi_{\mathrm{pr}_1^*\mathcal{K},f}$ is locally constant.

Proof. We apply Proposition 1.17 to the commutative diagram

$$\begin{array}{ccc} X \times B & \xrightarrow{\ f\ } & C \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & B. & \end{array} \tag{2.4}$$

The projection $p\colon X \times B \to B$ is locally acyclic relatively to $\mathrm{pr}_1^*\mathcal{K}$ by the generic universal local acyclicity [5, Corollaire 2.16]. By (SS1), the restriction of $f\colon X \times B \to C$ to $X \times B - Z$ is locally acyclic relatively to the pull-back of $\mathcal{K}$. Hence the assertion follows by Proposition 1.17. □

Lemma 2.19. *Let S be a finite family of irreducible conic closed subsets of dimension d and $\mathcal{K}$ be a constructible complex of Λ-modules on X.*

1. *In the condition* (SSr)*, one can replace* locally acyclic *by* universally locally acyclic.
2. *If $\mathcal{H}^q\mathcal{K}$ is locally constant for every q, the* 0*-section T^*_XX satisfies* (SSd) *for $\mathcal{K}$ and for $d = \dim X$.*
3. *Assume that S satisfies* (SSr) *for the restriction of $\mathcal{K}$ on the complement of a finite closed subset $Z \subset X$. Then, the union of S and the fiber T^*_ZX satisfies* (SSr) *for $\mathcal{K}$.*
4. *If a finite morphism $f\colon X \to Y$ is* unramified *and if S satisfies* (SSr) *for $\mathcal{K}$ and contains the* 0*-section T^*_XX, the image f_*S by the correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$ also satisfies* (SSr) *for $f_*\mathcal{K}$.*

Proof. 1. Let $i\colon W \to X \times B$ be an unramified morphism regular of codimension $q \leqq r$ non-characteristic with respect to $S = SS\mathcal{K}$ as in (SSr). We show that the base change of $W \to B$ by any morphism $B' \to B$ is locally acyclic relatively to the pull-back of $\mathcal{K}$. By a standard limit argument, we may assume that $B' \to B$ is of finite type. Since the local acyclicity is preserved by finite base change, it is reduced to the case where $B' = \mathbf{A}^n \times B$. Then, it suffices to apply (SSr) to the base change of the morphism $i\colon W \to X \times B$ by the flat morphism $\mathbf{A}^n \times B \to B$.

2. If an unramified morphism $i\colon W \to X \times B$ is non-characteristic with respect to $S \supset T^*_XX$, the composition $W \to B$ is smooth. Hence, the assertion is nothing but the local acyclicity of smooth morphism.

3. If the image of $W \to X \times B$ meets the inverse image of a closed point $x \in Z$ by the projection, it is not non-characteristic with respect to $S \supset T^*_x X$. Hence the assertion follows.

4. Let B be a smooth scheme over k and $W \to Y \times B$ be an unramified morphism regular of codimension $q \leqq r$ of flat schemes over B. Assume that $W \to Y \times B$ is non-characteristic with respect to the push-forward f_*S. We show that the composition $W \to B$ is locally acyclic relatively to the pull-back of $f_*\mathcal{K}$.

By the assumption that S contains $T^*_X X$, the morphism $W \to Y \times B$ meets the image of the finite morphism $X \to Y$ properly over B. Hence the unramified morphism $W \times_Y X \to X \times B$ is regular of codimension q and $W \times_Y X \to B$ is flat by Proposition 2.6(4)⇒(3). We show that the unramified morphism $W \times_Y X \to X \times B$ is non-characteristic with respect to S.

For the condition (i) in Definition 2.11.1, it is clear from the definition of f_*S. The image $S'_i = f_*S_i$ of S_i by the correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$ is irreducible since $T^*X \leftarrow X \times_Y T^*Y$ is a surjection of vector bundles. The intersection $T'_i = S'_i \cap T^*_Y Y$ with the 0-section equals the image $f(T_i)$ of the intersection $T_i = S_i \cap T^*_X X$ with the 0-section and $T_i \to T'_i$ is finite and surjective. Since $W \to Y \times B$ meets T'_i properly over B, the unramified morphism $W \times_Y X \to X \times B$ meets T_i properly over B.

Since S is assumed to satisfy (SSr) for $\mathcal{K}$, $W \times_Y X \to B$ is locally acyclic relatively to the pull-back of $\mathcal{K}$. Since $W \times_Y X \to W$ is finite, $g\colon W \to B$ is also locally acyclic relatively to the pull-back of $f_*\mathcal{K}$. □

Let $Y \subset X$ be a smooth irreducible divisor. Recall that the closed immersion $i\colon Y \to X$ is non-characteristic with respect to S if it satisfies the following conditions:

(i) The intersection $S \cap T^*_Y X$ with the conormal bundle is a subset of the 0-section.

(ii) The immersion $Y \to X$ meets every $T_i = S_i \cap T^*_X X$ properly.

The condition (ii) implies that the inverse image $Y \times_X T^*X \subset T^*X$ of $Y \subset X$ meets every component S_i of $S = (S_i)_{i \in I}$ properly. Further by (i) and Lemma 2.1 (3)⇒(2), the restriction of the canonical surjection $Y \times_X T^*X \to T^*Y$ to the intersection $S_i \cap (Y \times_X T^*X)$ is finite. Hence the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$ defines a closed conic subset $i^{-1}S_i \subset T^*Y$ and its irreducible components are of dimension $d - 1 = \dim Y$.

We define the pull-back $i^!S$ of S.

Definition 2.20. *Let $S = (S_i)_{i \in I}$ be a singular support of $\mathcal{K}$ on X and let Y be a smooth divisor of X. Assume that the immersion $i\colon Y \to X$ is non-characteristic with respect to S. We define a family $i^!S$ of irreducible closed subsets of T^*Y to be the refinement of $i^{-1}S = (i^{-1}S_i)_{i \in I}$ consisting of their irreducible components.*

Lemma 2.21. *Let $\mathcal{K}$ be a constructible complex of Λ-modules on X and let $S = (S_i)_{i \in I}$ be a singular support of $\mathcal{K}$. Let Y be a smooth divisor such that the closed immersion $i\colon Y \to X$ is non-characteristic with respect to S.*

If S satisfies the condition (SSr) *for $\mathcal{K}$, then $i^!S$ satisfies the condition* (SS$r-1$) *for $i^*\mathcal{K}$.*

Proof. Let B be a smooth scheme over k and $W \to Y \times B$ be an unramified morphism regular of codimension $q \leqq r - 1$ of flat schemes over B. Assume that $W \to Y \times B$ is non-characteristic with respect to $i^!S$. We show that the composition $W \to B$ is locally acyclic relatively to the pull-back $i^*\mathcal{K}$.

Since $i^! S$ is a refinement of $i^{-1}S$, the morphism $W \to Y \times B$ is non-characteristic with respect to $i^{-1}S$. Hence the unramified morphism $W \to X \times B$ regular of codimension $q + 1 \leqq r$ is non-characteristic with respect to S by Lemma 2.14 and the composition $W \to B$ is locally acyclic relatively to the pull-back of $\mathcal{K}$ by (SSr). $\square$

Let $f\colon Y \to X$ be a smooth morphism and let $S = (S_i)_{i \in I}$ be a finite family of irreducible conic closed subsets of dimension d of the cotangent bundle T^*X. Similarly as Definition 2.20, we define the pull-back f^*S to be the refinement of the family $f^{-1}S = (f^{-1}S_i)_{i \in I}$ consisting of irreducible components of the images $f^{-1}S_i$ by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$.

Proposition 2.22. *Let $f\colon Y \to X$ be a smooth morphism and assume that S satisfies* (SSr) *for $\mathcal{K}$. Then the pull-back f^*S of S satisfies* (SSr) *for $f^*\mathcal{K}$.*

Proof. Since the assertion is étale local on Y, it suffices to show the case where $Y \to X$ is smooth of relative dimension 1 by the induction on relative dimension.

Let B be a smooth scheme over k and let $W \to Y \times B$ be an unramified morphism regular of codimension $q \leqq r$ of flat schemes over B. Assume that $W \to Y \times B$ over B is non-characteristic with respect to the pull-back f^*S. We show that the morphism $W \to B$ is locally acyclic relatively to the pull-back of $\mathcal{K}$.

For an irreducible component S'_j of $f^{-1}S_i$, the intersection $T'_j = S'_j \cap T^*_Y Y$ is an irreducible component of $f^{-1}T_i = (f^{-1}S_i) \cap T^*_Y Y$. Hence, $W \to Y \times B$ over B is non-characteristic with respect to the inverse image $f^{-1}S$.

Let w be a closed point of W. Then, by Lemma 2.15, one of the following holds on a neighborhood of w:

(1) The composition $W \to Y \times B \to X \times B$ is unramified of codimension $q - 1 < r$ and is non-characteristic with respect to S.

(2) There exist a decomposition $W \to V \to Y \times B$ by a regular immersion of codimension r and an étale morphism, a smooth scheme $B' \to B$ of relative dimension 1 and an étale morphism $V \to X \times B'$ of flat schemes over B' such that the unramified morphism $W \to X \times B'$ over B' is regular of codimension $q \leqq r$ and is non-characteristic with respect to S.

Since the question is local on W, it suffices to consider the two cases separately after replacing W by an open neighborhood of w. Assume that (1) holds. Then the composition $W \to X \times B \to B$ is locally acyclic relatively to the pull-back of $\mathcal{K}$ since S is assumed to satisfy (SSr) for $\mathcal{K}$.

Assume that (2) holds. Then the composition $W \to X \times B' \to B'$ is locally acyclic relatively to the pull-back of $\mathcal{K}$ since S is assumed to satisfy (SSr) for $\mathcal{K}$. Since $B' \to B$ is smooth, the composition $W \to B$ is also locally acyclic relatively to the pull-back of $\mathcal{K}$ by [8, Corollaire 2.7]. $\square$

The condition (SSr) is a consequence of the compatibility of the construction of singular support with smooth pull-back and the local acyclicity of non-characteristic morphism in a more general setting.

Lemma 2.23. *Let $0 \leqq r \leqq d = \dim X$ be an integer. Assume that for every smooth morphism $f\colon Y \to X$, any flat morphism $g\colon Y \to Z$ of relative dimension $d - q$ for $0 \leqq q \leqq r$ is locally acyclic relatively to the pull-back $f^*\mathcal{K}$ if $g\colon Y \to Z$ is non-characteristic with respect to the pull-back of f^*S. Then, S satisfies the condition* (SSr).

Proof. Let B be a smooth scheme over k and let $W \to X \times B$ be an unramified morphism regular of codimension $q \leqq r$ of flat schemes over B. Assume that it is non-characteristic with respect to S. We show that the composition $W \to B$ is locally acyclic relatively to the pull-back $f^*\mathcal{K}$. The morphism $W \to B$ is flat of relative dimension $d - q$.

First we show the case where the composition $f: W \to X$ is smooth. The morphism $W \to X \times B$ is the composition $W \to W \times B \to X \times B$ of the graph of $W \to B$ and $f \times \mathrm{id}_B$. In the commutative diagram

$$\begin{array}{ccc} T^*_W(W \times B) & \longrightarrow & W \times_{W\times B} T^*(W \times B)/B = \quad T^*W \\ \uparrow & & \uparrow \\ T^*_W(X \times B) & \longrightarrow & W \times_{X\times B} T^*(X \times B)/B = W \times_X T^*X, \end{array}$$

the cokernel of the horizontal arrows are canonically isomorphic. Hence, the regular immersion $W \to W \times B$ is non-characteristic with respect to f^*S and $W \to B$ is locally acyclic relatively to the pull-back $f^*\mathcal{K}$.

We reduce the general case to the case where $f: W \to X$ is smooth. Since the question is local on W, we may assume that the unramified morphism $W \to X \times B$ is the composition $W \to V \to X \times B$ of a regular immersion of codimension q defined by functions $t_1, \ldots, t_q$ and an étale morphism. Let $B' = B \times \mathbf{A}^q$ and define an unramified morphism $V \to X \times B' = X \times B \times \mathbf{A}^q$ of codimension q by $t_1, \ldots, t_q$. Then, we obtain a cartesian diagram

$$\begin{array}{ccccc} V & \longrightarrow & X \times B' & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ W & \longrightarrow & X \times B & \longrightarrow & B \end{array}$$

where the vertical arrows are the immersions defined by $t_1 = \cdots = t_q = 0$. Then, by Proposition 2.6(2)⇒(1), after replacing V by a neighborhood of W if necessary, $V \to B'$ is flat of relative dimension $d - q$ and $V \to X \times B'$ is non-characteristic with respect to the pull-back of S. Since V is étale over $X \times B$, it is smooth over X and the morphism $V \to B'$ is locally acyclic relatively to the pull-back of $\mathcal{K}$. Hence, its base change $W \to B$ by an immersion $B \to B'$ is also locally acyclic relatively to the pull-back of $\mathcal{K}$. □

We formulate an analogue of the condition (SSr) for a complex on the vanishing topos.

Definition 2.24. *Let $f: X \to Y$ be a morphism of smooth schemes over k and $\mathcal{K}$ be a constructible complex of Λ-modules on $X \overleftarrow{\times}_Y Y$. For a finite family $S = (S_i)_{i\in I}$ of irreducible conic closed subsets of dimension $d = \dim Y$ of the cotangent complex T^*Y and an integer $r \geqq 0$, we define a condition:*

(SSr) *Let B be a smooth scheme over k and $i: W \to Y \times B$ be an unramified morphism regular of codimension $q \leqq r$ of flat schemes over B. Then the composition $W \to B$ with the second projection is* locally acyclic (Definition 1.11) *relatively to the pull-back to $(X \times_Y W) \overleftarrow{\times}_W W$ of $\mathcal{K}$ if the morphism $i: W \to Y \times B$ over B is* non-characteristic *with respect to S.*

Lemma 2.25. *Let $f: X \to Y$ be a flat morphism of smooth schemes. Let $S = (S_i)_{i\in I}$ be a finite family of closed conic subsets of dimension d of T^*X satisfying the condition* (Q) *before Lemma* 2.16*. Assume that in the case* (Q2)*, we have* $\dim P = \dim Y$*.*

1. *The members of the finite family $f_! S$ defined there are closed conic subsets of dimension* $\dim Y$ *of* T^*Y.

2. *Assume that the formation of $R\Psi_f\mathcal{K}$ commutes with finite base change. If S satisfies* (SSr) *in the sense of Definition* 2.17 *for $\mathcal{K}$, then $f_!S$ satisfies* (SSr) *in the sense of Definition* 2.24 *for $R\Psi_f\mathcal{K}$.*

Proof. 1. It follows from the assumption on $\dim P$ and that $Q \to Y$ is finite in the case (Q2).

2. Let $W \to Y \times B$ be an unramified morphism of codimension r of flat schemes over B and let $V \to X \times B$ be the base change. Assume that $W \to Y \times B$ is non-characteristic with respect to $f_!S$. We show that $W \to B$ is locally acyclic relatively to the pull-back of $R\Psi_f\mathcal{K}$ to $V \overset{\leftarrow}{\times}_W W$.

By Lemma 2.16.1, $V \to X \times B$ is non-characteristic with respect to S. Since S satisfies (SSr) for $\mathcal{K}$, the morphism $V \to B$ is locally acyclic relatively to the pull-back of $\mathcal{K}$. Hence, by the assumption that the formation of $R\Psi_f\mathcal{K}$ commutes with finite base change and by Lemma 1.12, the morphism $W \to B$ is locally acyclic relatively to the pull-back of $R\Psi_f\mathcal{K}$. □

2.4 Singular support and ramification

We briefly recall ramification theory [1], [17] and that there exists a singular support after removing a closed subscheme of codimension $\geqq 2$ if necessary.

First, we recall the definition of the total dimension of a Galois representation of a local field. Let K be a henselian discrete valuation field with residue field of characteristic $p > 0$ and $G_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ be the absolute Galois group. Then, the (non-logarithmic) filtration $(G_K^r)_{r \geqq 1}$ by ramification groups is defined in [1]. It is a decreasing filtration by closed normal subgroups indexed by rational numbers $\geqq 1$.

For a real number $r \geqq 1$, we define subgroups $G_K^{r+} \subset G_K^{r-}$ by $G_K^{r+} = \overline{\bigcup_{s>r} G_K^s}$ and $G_K^{r-} = \bigcap_{s<r} G_K^s$. It is proved in [1] that G_K^1 is the inertia group $I = \mathrm{Ker}(G_K \to G_F)$ where G_F denotes the absolute Galois group of the residue field F and G_K^{1+} is the wild inertia group P that is the pro-p Sylow subgroup of I. It is also proved that $G_K^{r-} = G_K^r$ for rational numbers $r > 1$ and $G_K^{r-} = G_K^{r+}$ for irrational numbers $r > 1$.

Let Λ be a finite field of characteristic $\neq p$ and let V be a continuous representation of G_K on a Λ-vector space of finite dimension. Then, since $P = G_K^{1+}$ is a pro-p group and since $G_K^{r-} = G_K^r$ for rational r and $G_K^{r-} = G_K^{r+}$ for irrational r, there exists a unique decomposition $V = \bigoplus_{r \geqq 1} V^{(r)}$ called the slope decomposition characterized by the condition that the G_K^{r+} fixed part $V^{G_K^{r+}}$ is equal to the sum $\bigoplus_{s \leqq r} V^{(s)}$. Then, the *total dimension* of V is defined by

$$\dim \mathrm{tot}_K V = \sum_{r \geqq 1} r \cdot \dim V^{(r)}. \tag{2.5}$$

It is clear from the definition that $\dim \mathrm{tot}_K V$ is a rational number satisfying $\dim \mathrm{tot}_K V \geqq \dim V$. The equality is equivalent to that $P = G_K^{1+}$ acts on V trivially, namely, the action of G_K on V is tamely ramified. In the classical case where the residue field F is perfect, we recover the classical definition of the total dimension $\dim \mathrm{tot}_K V = \dim V + \mathrm{Sw}_K V$

We study a geometric case where X is a smooth scheme over a perfect field k of characteristic $p > 0$. Let D be a reduced divisor and $U = X - D$ be the complement. Let

$\mathcal{K}$ be a constructible complex of Λ-modules such that the restriction of the cohomology sheaf $\mathcal{H}^q\mathcal{K}$ to U is locally constant for every q. Let ξ be the generic point of an irreducible component of D. Then, the local ring $\mathcal{O}_{X,\xi}$ is a discrete valuation ring and the fraction field K of its henselization is called the local field at ξ. The stalk of $\mathcal{H}^q\mathcal{K}$ at the geometric point of U defined by a separable closure $K_{\rm sep}$ defines a Λ-representation V^q of the absolute Galois group G_K.

For a rational number $r > 1$, the graded quotient ${\rm Gr}^r G_K = G_K^r/G_K^{r+}$ is a profinite abelian group annihilated by p [17, Corollary 2.28] and its dual group is related to differential forms as follows. We define ideals $\mathfrak{m}_{K_{\rm sep}}^{(r)}$ and $\mathfrak{m}_{K_{\rm sep}}^{(r+)}$ of the valuation ring $\mathcal{O}_{K_{\rm sep}}$ by $\mathfrak{m}_{K_{\rm sep}}^{(r)} = \{x \in K_{\rm sep} \mid {\rm ord}_K x \geqq r\}$ and $\mathfrak{m}_{K_{\rm sep}}^{(r+)} = \{x \in K_{\rm sep} \mid {\rm ord}_K x > r\}$ where ${\rm ord}_K$ denotes the valuation normalized by ${\rm ord}_K(\pi) = 1$ for a uniformizer π of K. The residue field $\bar F$ of $\mathcal{O}_{K_{\rm sep}}$ is an algebraic closure of F and the quotient $\mathfrak{m}_{K_{\rm sep}}^{(r)}/\mathfrak{m}_{K_{\rm sep}}^{(r+)}$ is an $\bar F$-vector space of dimension 1. A canonical injection

$$ch\colon Hom_{\mathbf{F}_p}({\rm Gr}^r G_K, \mathbf{F}_p) \to Hom_{\bar F}(\mathfrak{m}_{\bar K}^{(r)}/\mathfrak{m}_{\bar K}^{(r+)}, \Omega^1_{X/k,\xi} \otimes \bar F) \tag{2.6}$$

is also defined [17, Corollary 2.28].

We say that the ramification of $\mathcal{F}$ along D is isoclinic of slope $r \geqq 1$ if $V = V^{(r)}$ in the slope decomposition. The ramification of $\mathcal{F}$ along D is isoclinic of slope 1 if and only if the corresponding Galois representation V is tamely ramified. Assume that the ramification of $\mathcal{F}$ along D is isoclinic of slope $r > 1$. Assume also that the finite field Λ contains a primitive p-th root of 1 and identify $\mathbf{F}_p$ with a subgroup of $\Lambda^\times$. Then, $V = V^{(r)}$ is further decomposed by characters $V = \bigoplus_{\chi\colon {\rm Gr}^r G_K \to \mathbf{F}_p} \chi^{\oplus m(\chi)}$. For a character χ appearing in the decomposition, the twisted differential form $ch(\chi)$ defined on a finite covering of a dense open scheme of D is called a characteristic form of $\mathcal{F}$.

Assume that $U = X - D$ is the complement of a divisor with simple normal crossings D and let $D_1, \ldots, D_m$ be the irreducible components of D. We say the ramification of $\mathcal{F}$ along D is isoclinic of slope $R = \sum_i r_i D_i$ if the ramification of $\mathcal{F}$ along D_i is isoclinic of slope r_i for every irreducible component D_i of D.

In [17, Definition 3.1], we define the condition for ramification of $\mathcal{F}$ along D to be non-degenerate. We say that the ramification of $\mathcal{F}$ is non-degenerate along D if it admits étale locally a direct sum decomposition $\mathcal{F} = \bigoplus_j \mathcal{F}_j$ such that each $\mathcal{F}_j$ is isoclinic of slope $R_j \geqq D$ for a $\mathbf{Q}$-linear combination of irreducible components of D and that the ramification of $\mathcal{F}_j$ is non-degenerate along D at multiplicity R_j. For the definition of the last condition, we refer to [17, Definition 3.1]. It implies that the characteristic forms are extended to differential forms on the boundary without zero. Note that there exists a closed subset of codimension at least 2 such that on its complement, the ramification of $\mathcal{F}$ along D is non-degenerate.

We introduce a slightly stronger condition that implies local acyclicity. We say that the ramification of $\mathcal{F}$ is *strongly* non-degenerate along D if it satisfies the condition above with $R_j \geqq D$ replaced by $R_j = D$ or $R_j > D$. Here, the inequality $R_j > D$ means that the coefficient in R_j of every irreducible component of D is > 1. Note that if the ramification of $\mathcal{F}$ along D is non-degenerate, on the complement of the singular locus of D, the ramification of $\mathcal{F}$ along D is strongly non-degenerate.

We also briefly recall the definition of the characteristic cycle [17, Definition 3.5] in the strongly non-degenerate and isoclinic case. Assume first that $R = D$. Then, the locally constant sheaf $\mathcal{F}$ on U is tamely ramified along D and the characteristic cycle is defined by

$$\text{Char } j_!\mathcal{F} = (-1)^d \text{rank } \mathcal{F} \cdot \sum_I [T^*_{D_I} X] \tag{2.7}$$

where $T^*_{D_I}X$ denotes the conormal bundle of the intersection $D_I = \bigcap_I D_i$ for a set of indices $I \subset \{1, \ldots, m\}$.

Assume $R = \sum_i r_i D_i > D = \sum_i D_i$. For each irreducible component, we have a decomposition by characters $V = \bigoplus_{\chi\colon \mathrm{Gr}^r G_{K_i} \to \mathbf{F}_p} \chi^{\oplus m(\chi)}$. Further, the characteristic form of each character χ appearing in the decomposition defines a sub line bundle L_χ of the pull-back $D_\chi \times_X T^*X$ of the contingent bundle to a finite covering $\pi_\chi\colon D_\chi \to D_i$ by the non-degenerate assumption. Then, the characteristic cycle in the case $R > D$ is defined by

$$\text{Char } j_!\mathcal{F} = (-1)^d \Big(\text{rank } \mathcal{F} \cdot [T^*_X X] + \sum_i \sum_\chi \frac{r_i \cdot m(\chi)}{[D_\chi \colon D_i]} \pi_*[L_\chi] \Big). \tag{2.8}$$

In the general strongly non-degenerate case, the characteristic cycle is defined by the additivity and étale descent. If X is a curve, we recover the classical definition

$$\text{Char } j_!\mathcal{F} = -\Big(\text{rank } \mathcal{F} \cdot [T^*_X X] + \sum_{x \in D} \dim \text{tot}_x \mathcal{F} \cdot [T^*_x X] \Big). \tag{2.9}$$

Proposition 2.26 ([17, Proposition 3.15])**.** *Assume that the ramification of $\mathcal{F}$ is strongly non-degenerate along D. Then, the support of the characteristic cycle defined by* (2.7) *and* (2.8) *satisfies* (SSr) *for every integer $r \geqq 0$.*

Proof. The construction of the characteristic cycle in the general strongly non-degenerate case commutes with smooth pull-back. Hence, by Lemma 2.23, it suffices to show that if a flat morphism $f\colon X \to Y$ is non-characteristic with respect to the support S of the characteristic cycle of $\mathcal{K}$ defined above, then it is locally acyclic relatively to $\mathcal{K}$. Since the 0-section $T^*_X X$ is contained in the singular support, the morphism $f\colon X \to Y$ is smooth. By the local acyclicity of smooth morphism, it suffices to show the local acyclicity at D. Since the assertion is étale local, we may assume that $\mathcal{F}$ is isoclinic of slope $R = D$ or $R > D$ and that the ramification of $\mathcal{F}$ is non-degenerate along D at multiplicity R.

If $R = D$, the locally constant sheaf $\mathcal{F}$ on U is tamely ramified along D. Since the singular support is the union $S = \bigcup_I T^*_{D_I} X$ of the conormal bundles of the intersections $D_I = \bigcap_I D_i$ of irreducible components $D_1, \ldots, D_m$ of D for all $I \subset \{1, \ldots, m\}$, the non-characteristicity means that the inverse image of D in X is a divisor with simple normal crossings relatively to $X \to Y$. Hence, the assertion follows from [8, 1.3.3 (i)].

Assume $R > D$. By Proposition 2.6 (1)⇒(3) applied to the graph $X \to X \times B$ of S and the immersion $D_i \times B \to X \times B$ for irreducible components D_i of D, the restriction of $f\colon X \to B$ to D is flat. Hence the assertion follows from [17, Proposition 3.15]. □

Consequently, a singular support exists after removing a closed subscheme of codimension at least 2 if necessary.

Corollary 2.27. *Assume that there exists a finite set Z of closed points of $D = X - U$ such that the ramification of $\mathcal{F}$ is strongly non-degenerate along $D - Z$. Then, a singular support of $j_!\mathcal{F}$ exists. In particular, if $\dim X = 2$, a singular support of $j_!\mathcal{F}$ always exists.*

Proof. It follows from Proposition 2.26 and Lemma 2.19.3. □

The construction of the characteristic cycle is compatible with the pull-back by non-characteristic morphisms, [17, Proposition 3.8]. We state it for a non-characteristic immersion. Let $i\colon Y \to X$ be an immersion of smooth subscheme meeting D transversally non-characteristic with respect to the singular support $SSj_!\mathcal{F}$ and define $i^!\mathrm{Char}\ j_!\mathcal{F}$ to be $(-1)^{\dim X - \dim Y}$-times the image of $\mathrm{Char}\ j_!\mathcal{F}$ by the correspondence $T^*X \leftarrow Y \times_X T^*X \to T^*Y$. Then, we have

$$\mathrm{Char}\ i^*(j_!\mathcal{F}) = i^!\mathrm{Char}\ j_!\mathcal{F}. \tag{2.10}$$

The definition of the characteristic cycle is slightly generalized to complexes as follows. Let X and a smooth divisor D be as above and let $\mathcal{K}$ be a constructible complex of Λ-modules on X. Assume that for every q, the restriction of $\mathcal{H}^q\mathcal{K}$ to D is locally constant, that the restriction of $\mathcal{H}^q\mathcal{K}$ on the complement $U = X - D$ is locally constant and that its ramification along D is non-degenerate. Then, the characteristic cycle of $\mathcal{K}$ is defined by

$$\mathrm{Char}\ \mathcal{K} = \sum_q (-1)^q \mathrm{Char}\ j_!j^*\mathcal{H}^q\mathcal{K} + (-1)^{d-1}\sum_q (-1)^q \mathrm{rank}\ \mathcal{H}^q\mathcal{K}|_D \cdot [T^*_D X]. \tag{2.11}$$

Let C be a smooth curve and t be a local parameter at a closed point x of C. For a 1-cycle A on the cotangent bundle T^*C meeting the section dt properly on a neighborhood of the fiber of x, let $(A, dt)_x$ denote the intersection number.

Lemma 2.28. *Let the notation be as above. Let $W \subset D \times_X \mathbf{P}(TX)$ be the open subset consisting of the tangent vectors at closed point x of smooth curve C meeting D transversely at x such that the immersion $C \to X$ is non-characteristic at x with respect to the union of the singular supports of $j_!\mathcal{H}^q\mathcal{K}_U$ for the open immersion $j\colon U \to X$.*

1. [17, Corollary 3.9.2] *Let $i\colon C \to X$ be a closed immersion of a smooth curve over k and x be a closed point of $C \cap D$. If the tangent vector of C at x is contained in W, for the Artin conductor* (1.15), *we have*

$$-a_x(i^*\mathcal{K}) = (i^!\mathrm{Char}\ \mathcal{K}, dt)_x. \tag{2.12}$$

2. *Let $f\colon Y \to X$ be a morphism of smooth schemes over k and $i\colon C \to Y$ be a closed immersion of smooth curve meeting the pull-back f^*D transversely at a closed point y. Assume that the composition $g\colon C \to X$ is non-characteristic with respect to $\mathcal{K}$ at y. Then on a neighborhood of y, the pull-back $D' = f^*D$ is smooth and the ramification of the pull-back to $Y - D'$ of the cohomology sheaves $\mathcal{H}^q\mathcal{K}$ are non-degenerate and consequently we have an equality*

$$-a_y(g^*\mathcal{K}) = (i^!\mathrm{Char}\ f^*\mathcal{K}, dt)_y. \tag{2.13}$$

Proof. 1. By [17, Corollary 3.9.2] and (2.9), we have

$$i^!\mathrm{Char}\ \mathcal{K} = \mathrm{Char}\ i^*\mathcal{K} = -(\mathrm{rank}\ i^*\mathcal{K}_\eta \cdot [T^*_C C] + a_x(i^*\mathcal{K}) \cdot [T^*_x C])$$

on a neighborhood of the fiber $T^*_x C$ where η denotes the generic point of C. Hence the assertion follows.

2. Since D' meets C transversely at y, the divisor D' is smooth on a neighborhood of y. Since $g\colon C \to X$ is non-characteristic with respect to $\mathcal{K}$ at y, the morphism $f\colon Y \to X$ is also non-characteristic with respect to $\mathcal{K}$ at y on a neighborhood of y in the sense of [17, Definition 3.7.2]. The equality (2.13) follows from (2.12). □

We generalize the construction of the singular support and the characteristic cycle to vanishing topos.

Definition 2.29. *Let X be a smooth scheme over a field k and $j\colon U \to X$ be the open immersion of the complement of a smooth divisor D. Let Z be a scheme quasi-finite over D. Let $\mathcal{K}$ be a constructible complex of Λ-modules on $Z \overset{\leftarrow}{\times}_X X$.*

1. *We say that $\mathcal{K}$ is non-degenerate along D if the following conditions are satisfied for every $q \in \mathbf{Z}$: The restrictions of the cohomology sheaf $\mathcal{H}^q\mathcal{K}$ to $Z \overset{\leftarrow}{\times}_X U$ and to $Z \overset{\leftarrow}{\times}_X D$ are locally constant. Étale locally on D, there exists a locally constant sheaf $\mathcal{F}_q$ on U such that the ramification along D is non-degenerate and that the restriction of $\mathcal{H}^q\mathcal{K}$ to $Z \overset{\leftarrow}{\times}_X U$ is isomorphic to the pull-back of $\mathcal{F}_q$.*

2. *Assume that $\mathcal{K}$ is non-degenerate along D. We define* Char $\mathcal{K}$ *to be the locally constant function on Z with the value $\sum_q (-1)^q \mathrm{Char}\ \mathcal{H}^q\mathcal{K}$.*

Lemma 2.30. *Let X be a smooth scheme over a perfect field k and $j\colon U \to X$ be the open immersion of the complement of a divisor D with simple normal crossings. Let Z be a scheme quasi-finite over D. Let $\mathcal{K}$ be a constructible complex of Λ-modules on $Z \overset{\leftarrow}{\times}_X X$.*

1. *There exists an open subscheme $X^\circ \subset X$ such that $D^\circ = D \times_X X^\circ \subset D$ is dense and that, for $Z^\circ = Z \times_X X^\circ$ and $U^\circ = U \times_X X^\circ$, the restrictions of $\mathcal{H}^q\mathcal{K}$ to $Z^\circ \overset{\leftarrow}{\times}_{X^\circ} U^\circ$ and to $Z^\circ \overset{\leftarrow}{\times}_{X^\circ} D^\circ$ are locally constant for every $q \in \mathbf{Z}$.*

2. *After further shrinking X° if necessary, the pull-back of $\mathcal{H}^q\mathcal{K}$ to $Z^\circ \overset{\leftarrow}{\times}_{X^\circ} X^\circ$ is non-degenerate along D for every $q \in \mathbf{Z}$.*

Proof. 1. It follows from the description of a constructible sheaf in [11, 1.3].

2. By the definition of the site defining $Z \overset{\leftarrow}{\times}_X U$, there exists an étale scheme X' over X such that the intersection of the image is dense in D and locally constant sheaf $\mathcal{F}_q$ on $U' = U \times_X X'$ such that $\mathcal{H}^q\mathcal{K}$ is the pull-back of $\mathcal{F}_q$ for every q. Since $\mathcal{F}_q = 0$ except for finitely many q, the assertion follows. □

Lemma 2.31. *Let X be a smooth scheme of finite type over a perfect field k and $U = X - D$ be the complement of a smooth divisor of X. Let Z be a scheme quasi-finite over X such that the image of Z is contained in D and let $\mathcal{K}$ be a constructible complex of Λ-modules on $Z \overset{\leftarrow}{\times}_X X$. Assume that the restrictions of $\mathcal{H}^q\mathcal{K}$ to $Z \overset{\leftarrow}{\times}_X U$ and to $Z \overset{\leftarrow}{\times}_X D$ are locally constant and that the pull-back of $\mathcal{H}^q\mathcal{K}$ to $Z \overset{\leftarrow}{\times}_X U$ is non-degenerate along D for every $q \in \mathbf{Z}$.*

Let $f\colon Y \to X$ be a morphism of smooth schemes over k and $i\colon C \to Y$ be a closed immersion of smooth curve meeting the pull-back f^*D transversely at a geometric point y. Define a cartesian diagram

$$\begin{array}{ccccc} W & \longrightarrow & V & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{\ i\ } & Y & \longrightarrow & X \end{array}$$

and let w be a geometric point of W above y. Assume that the composition $C \to X$ is non-characteristic with respect to $\mathcal{K}$ on a neighborhood of $w \leftarrow y$ in $W \overset{\leftarrow}{\times}_C C$. Then the pull-back $D' = f^*D$ is smooth on a neighborhood of y and the ramification of the pull-back to $U \times_X Y$ of the cohomology sheaves $\mathcal{H}^q\mathcal{K}$ are non-degenerate along D' on a neighborhood of $w \leftarrow y$ in $V \overset{\leftarrow}{\times}_Y Y$. Further, for a local parameter t of C at y, we have an equality

$$-a_w(\mathcal{K}|_{W\overset{\leftarrow}{\times}_C C}) = (i^!\mathrm{Char}\ (\mathcal{K}|_{V\overset{\leftarrow}{\times}_Y Y}), dt)_w. \tag{2.14}$$

Proof. It follows from Lemma 2.28.2. □

2.5 Stability of vanishing cycles

Let u be an isolated characteristic point (Definition 2.10.2) of $f\colon X \to C$ with respect to $\mathcal{K}$. By abuse of notation, let $df\colon X \to T^*X$ denote the section on a neighborhood of u defined by the pull-back of a local basis of T^*C on a neighborhood of $f(u)$. Then, for an irreducible component S_i of the singular support, u is an isolated point of the pull-back $df^{-1}(S_i)$ if u is contained in it and hence the intersection number $(S_i, df)_u$ is defined as its multiplicity. More generally, for a linear combination $\sum_i a_iS_i$, the intersection number $(\sum_i a_iS_i, df)_u$ is defined.

Assuming the existence of a singular support satisfying (SS1), we deduce the stability of the total dimension of the space of nearby cycles at isolated characteristic points from the consequence Lemma 2.18 of the flatness of the Swan conductor Proposition 1.17. For morphisms $f, g\colon X \to Y$ of schemes and a closed subscheme $Z \subset X$ defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we say that $g \equiv f \bmod \mathcal{I}_Z$ if their restrictions to Z are the same.

Proposition 2.32. *Let X be a smooth scheme of dimension d over a perfect field k and $\mathcal{K}$ be a constructible complex of Λ-modules. Assume that there exists a singular support $S = SS\mathcal{K} \subset T^*X$ satisfying the condition* (SS1). *Let $f\colon X \to C$ be a flat morphism to a smooth curve C such that $u \in X$ be an isolated characteristic point of f with respect to $\mathcal{K}$.*

Then, there exists an integer $N \geqq 2$ such that for every morphism $g\colon V \to C$ on an étale neighborhood V of u satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, the closed point u is an isolated characteristic point of g with respect to $\mathcal{K}$, the morphism g is flat at u and we have

$$\dim \mathrm{tot}\phi_u(\mathcal{K}, f) = \dim \mathrm{tot}\phi_u(\mathcal{K}, g) \tag{2.15}$$

and $(S_i, df)_{T^*X,u} = (S_i, dg)_{T^*V,u}$ for every irreducible component S_i of the singular support $SS\mathcal{K} = \bigcup_i S_i$.

Proof. By taking an étale morphism to $\mathbf{A}^1_k$ on a neighborhood of $f(u) \in C$ and by replacing f by the composition, we may assume $C = \mathbf{A}^1_k = \mathrm{Spec}\ k[s]$ and $f(u) = 0$.

By the assumption that u is an isolated characteristic point with respect to f, it is an isolated point of the inverse image $ds^*(S_i) \subset X$ by the section $ds\colon X \to T^*X$ for every irreducible component S_i of the singular support $SS\mathcal{K} = \bigcup_i S_i$.

Let $N \geqq 2$ be an integer such that $\mathfrak{m}_u^{N-2}$ annihilates $ds^*(S_i)$. For a morphism $g\colon V \to C$ on an étale neighborhood V of u satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, we have $dg \equiv df \bmod \mathfrak{m}_u^{N-1}$. Hence, u is an isolated characteristic point with respect to g by Nakayama's lemma and we have an equality $(S_i, df)_{T^*X,u} = (S_i, dg)_{T^*V,u}$ for every irreducible component S_i of the singular support $SS\mathcal{K} = \bigcup_i S_i$.

Let $T_i \subset X$ be the intersection of an irreducible component $S_i \subset T^*X$ of the singular support $SS\mathcal{K} = \bigcup_i S_i$ with the 0-section $X = T^*_X X$, regarded as a reduced closed subscheme. Since T_i is irreducible, it is integral. By the assumption that u is an isolated characteristic point, the restriction of f to $T_i \cap (X - \{u\})$ is flat. Then, the pullback $f^*(s)$ of the coordinate s of $C = \mathbf{A}^1_k$ in the local integral ring $\mathcal{O}_{T_i,u}$ is not 0 for any $T_i \neq \{u\}$. Hence, there exists an integer $N \geqq 1$ such that $f^*(s)$ in $\mathcal{O}_{T_i,u}/\mathfrak{m}_u^N$ is not 0 for any $T_i \neq \{u\}$. Then, the $g \equiv f \bmod \mathfrak{m}_u^N$ implies $g^*(s)$ in $\mathcal{O}_{T_i,u}/\mathfrak{m}_u^N$ is not 0 for any $T_i \neq \{u\}$. and that the restriction of $g\colon V \to C$ to the pull-back of T_i is flat at u for any $T_i \neq \{u\}$. Similarly, $g\colon V \to C$ itself is flat at u. Thus u is also an isolated characteristic point of $g\colon V \to C$.

We show the equality (2.15). By replacing X by V, we may assume $X = V$. Define a commutative diagram

$$(2.16) \qquad \begin{array}{ccc} X \times \mathbf{A}^1 & \xrightarrow{\quad h \quad} & C \times \mathbf{A}^1 \\ & {\scriptstyle \mathrm{pr}_2} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & \mathbf{A}^1 = \mathrm{Spec}\ k[t] & \end{array}$$

by $h = (1-t)f + tg$. It is a homotopy connecting f to g. By the assumption on g, we have $h \equiv f \bmod \mathfrak{m}_u^N$. Hence, for every $c \in k$, u is an isolated characteristic point of the fiber $h_c\colon X \to C$ of h at $t = c$. Since X and $C \times \mathbf{A}^1$ are flat over $\mathbf{A}^1$, there exists a neighborhood $W \subset X \times \mathbf{A}^1$ of $\{u\} \times \mathbf{A}^1$ such that $h\colon X \times \mathbf{A}^1 \to C \times \mathbf{A}^1$ over $\mathbf{A}^1$ is non-characteristic on $W - (\{u\} \times \mathbf{A}^1)$. Hence by (SS1), the morphism $h\colon X \times \mathbf{A}^1 \to C \times \mathbf{A}^1$ is locally acyclic on $W - (\{u\} \times \mathbf{A}^1)$ relatively to the pull-back $\mathrm{pr}_1^*\mathcal{K}$.

We apply Lemma 2.18 to the restriction of the diagram (2.16) and $\mathrm{pr}_1^*\mathcal{K}$ to $W \subset X \times \mathbf{A}^1$ and the closed subscheme $Z = \{u\} \times \mathbf{A}^1 \subset W$ étale over $\mathbf{A}^1$. By Lemma 2.18, the function $\varphi_{\mathrm{pr}_1^*\mathcal{K},h}$ on Z is locally constant. Since $Z = \{u\} \times \mathbf{A}^1$ is connected, it is a constant function and we obtain $\varphi_{\mathrm{pr}_1^*\mathcal{K},h}(0) = \varphi_{\mathrm{pr}_1^*\mathcal{K},h}(1)$ namely the equality (2.15). $\square$

3 Local Radon transform

3.1 Morphism defined by a pencil

In this subsection, we fix some notations and terminology to define and study local Radon transform later in this section. Let X be a quasi-projective integral smooth scheme of dimension d over a field k and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X, \mathcal{L})$ be a sub k-vector space of finite dimension such that the canonical morphism $E \otimes_k \mathcal{O}_X \to \mathcal{L}$ is a surjection and induces an immersion

$$X \to \mathbf{P} = \mathbf{P}(E^\vee) = \mathrm{Proj}_k S^\bullet E.$$

We use a contra-Grothendieck notation also for a projective space $\mathbf{P}(E)(k) = (E - \{0\})/k^\times$.

Let $\mathbf{P}^\vee = \mathbf{P}(E)$ be the dual of $\mathbf{P}$. The universal hyperplane $\mathbf{H} = \{(x, H) \mid x \in H\} \subset \mathbf{P} \times \mathbf{P}^\vee$ is defined by the identity $\mathrm{id} \in \mathrm{End}(E)$ regarded as a section $F \in \Gamma(\mathbf{P} \times \mathbf{P}^\vee, \mathcal{O}(1,1)) = E \otimes E^\vee$. By the canonical injection $\Omega^1_{\mathbf{P}/k}(1) \to E \otimes \mathcal{O}_{\mathbf{P}}$, the universal hyperplane $\mathbf{H}$ is identified with the covariant projective space bundle $\mathbf{P}(T^*\mathbf{P})$ associated to the cotangent bundle $T^*\mathbf{P}$. Further, the identity of $\mathbf{H}$ is the same as the morphism $\mathbf{H} = \mathbf{P}(T^*_{\mathbf{H}}(\mathbf{P} \times \mathbf{P}^\vee)) \to \mathbf{H} = \mathbf{P}(T^*\mathbf{P})$ induced by the locally splitting injection $\mathcal{N}_{\mathbf{H}/\mathbf{P}\times\mathbf{P}^\vee} \to \mathrm{pr}_1^*\Omega^1_{\mathbf{P}/k}$.

The fibered product $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ is the intersection of $X \times \mathbf{P}^\vee$ with $\mathbf{H}$ in $\mathbf{P} \times \mathbf{P}^\vee$ and is the universal family of hyperplane sections. We consider the universal family of hyperplane sections

$$X \xleftarrow{\ q\ } X \times_{\mathbf{P}} \mathbf{H} \xrightarrow{\ p\ } \mathbf{P}^\vee = \mathbf{P}(E). \tag{3.1}$$

Lemma 3.1. *Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee)$.*

1. *The morphism $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is flat and the immersion $X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee$ is regular of codimension* 1.

2. *Let $T \subset X$ be an integral closed subscheme and define a subspace $E' = \mathrm{Ker}(E \to \Gamma(T, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_T))$ and $\mathbf{P}'^\vee = \mathbf{P}(E') \subset \mathbf{P}^\vee = \mathbf{P}(E)$. Then, $T \times \mathbf{P}'^\vee \subset X \times \mathbf{P}^\vee$ is contained in $T \times_{\mathbf{P}} \mathbf{H} \subset X \times_{\mathbf{P}} \mathbf{H}$ and the complement $(T \times_{\mathbf{P}} \mathbf{H}) - (T \times \mathbf{P}'^\vee)$ is the largest open subscheme where the regular immersion $X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee$ of flat scheme over $\mathbf{P}^\vee$ meets $T \times \mathbf{P}^\vee$ properly. The codimension of $E' \subset E$ is strictly larger than* $\dim T$.

Proof. The fiber product $T \times_{\mathbf{P}} \mathbf{H}$ is the intersection of $T \times \mathbf{P}^\vee$ with the divisor $\mathbf{H}$ in $\mathbf{P} \times \mathbf{P}^\vee$. If a hyperplane $H \in \mathbf{P}^\vee$ is contained in $\mathbf{P}'^\vee$, it contains T. If otherwise, H meets T properly. Hence we have an inclusion $T \times \mathbf{P}'^\vee \subset T \times_{\mathbf{P}} \mathbf{H}$ and the complement $(T \times_{\mathbf{P}} \mathbf{H}) - (T \times \mathbf{P}'^\vee)$ is the largest open subset where the regular immersion $\mathbf{H} \to \mathbf{P} \times \mathbf{P}^\vee$ over $\mathbf{P}^\vee$ meets $T \times \mathbf{P}^\vee$ properly, by Proposition 2.6.

Applying this to $X = T$, we obtain the assertion 1 since X is smooth and $\mathbf{P}'^\vee$ is empty in this case further by Proposition 2.6. This also implies the assertion that the complement $(T \times_{\mathbf{P}} \mathbf{H}) - (T \times \mathbf{P}'^\vee)$ is the largest open subscheme where the regular immersion $X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee$ of flat scheme over $\mathbf{P}^\vee$ meets $T \times \mathbf{P}^\vee$ properly in 2.

Since the subspace $E/E' \subset \Gamma(T, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_T)$ defines an immersion $T \to \mathbf{P}(E/E')$, we have $\dim E/E' - 1 \geqq \dim T$. □

Let $S \subset T^*X$ be a closed conic subset. Define $\widetilde{S} \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ to be the pull-back of S by the surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ and

$$\mathbf{P}(\widetilde{S}) \subset \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H} \tag{3.2}$$

the projectivization. If S is irreducible of dimension d, then $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ is also irreducible and is of codimension d, unless $S = T^*_X X$ and X is an open subscheme of $\mathbf{P}$. In the following, we will exclude this case. We will give a sufficient condition for the restriction to $\mathbf{P}(\widetilde{S})$ of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ to be generically finite at Corollary 3.17.1.

Lemma 3.2. *Let $S \subset T^*X$ be a closed conic subset. For a point z of $X \times_{\mathbf{P}} \mathbf{H}$, corresponding to the pair (u, H) of a hyperplane $H \subset \mathbf{P}^\vee$ and $u \in X \cap H$, the following conditions are equivalent:*

(1) *z is a point of $\mathbf{P}(\widetilde{S})$.*

(2) *The fiber at u of the inverse image of $S \subset T^*X$ to the conormal bundle by the canonical morphism $T^*_{X \times_{\mathbf{P}} \mathbf{H}}(X \times \mathbf{P}^\vee) \to T^*X \times_X (X \times \mathbf{P}^\vee) \to T^*X$ is a subset of the* 0*-section.*

Proof. The immersion $X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee = X \times \mathbf{P}(E)$ to the projective space bundle is defined by the restriction of the canonical morphism $T^*_{\mathbf{H}}(\mathbf{P} \times \mathbf{P}^\vee) \to \mathbf{H} \times_{\mathbf{P} \times \mathbf{P}^\vee} T^*(\mathbf{P} \times \mathbf{P}^\vee/\mathbf{P}^\vee) = \mathbf{H} \times_{\mathbf{P}} T^*\mathbf{P}$ inducing the canonical morphism $T^*_{X \times_{\mathbf{P}} \mathbf{H}}(X \times \mathbf{P}^\vee) \to T^*X \times_X (X \times \mathbf{P}^\vee)$. Hence the assertion follows. □

We construct the universal family of morphisms defined by pencils. Let $\mathbf{G} = \mathrm{Gr}(1, \mathbf{P}^\vee)$ be the Grassmannian variety parametrizing lines in $\mathbf{P}^\vee$. The universal line $\mathbf{D} \subset \mathbf{P}^\vee \times \mathbf{G}$ is canonically identified with the flag variety parametrizing pairs (H, L) of points H of $\mathbf{P}^\vee$ and lines L passing through H. It is the same as the flag variety $\mathrm{Fl}(1, 2, E)$ parametrizing pairs of a line and a plane including the line in E.

The projective space $\mathbf{P}^\vee$ and the Grassmannian variety $\mathbf{G}$ are also equal to the Grassmannian varieties $\mathrm{Gr}(1, E)$ and $\mathrm{Gr}(2, E)$ parametrizing lines and planes in E respectively. The projections $\mathbf{P}^\vee \leftarrow \mathbf{D} \rightarrow \mathbf{G}$ sending a pair (H, L) to the line L and to the hyperplane H are the canonical morphisms $\mathrm{Gr}(1, E) \leftarrow \mathrm{Fl}(1, 2, E) \rightarrow \mathrm{Gr}(2, E)$. By the projection $\mathbf{D} \rightarrow \mathbf{P}^\vee$, it is also canonically identified with the projective space bundle associated to the tangent bundle $\mathbf{D} = \mathbf{P}(T\mathbf{P}^\vee)$.

Let $\mathbf{A} \subset \mathbf{P} \times \mathbf{G}$ be the universal family of the intersections of hyperplanes parametrized by lines. The scheme $\mathbf{A}$ is also canonically identified with the Grassmann bundle $\mathrm{Gr}(2, T^*\mathbf{P})$ over $\mathbf{P}$. In the diagram (3.3) below, the left vertical arrow and the lower line are the canonical morphisms $\mathrm{Gr}(1, X \times_{\mathbf{P}} T^*\mathbf{P}) \rightarrow \mathrm{Gr}(1, E) \leftarrow \mathrm{Fl}(1, 2, E) \rightarrow \mathrm{Gr}(2, E)$. Hence the fiber product $\mathbf{D} \times_{\mathbf{P}^\vee} (X \times_{\mathbf{P}} \mathbf{H})$ is canonically identified with the blow-up $(X \times \mathbf{G})' \rightarrow X \times \mathbf{G}$ at the intersection $\mathbf{A} \cap (X \times \mathbf{G}) = X \times_{\mathbf{P}} \mathbf{A}$. Thus we obtain a commutative diagram

$$(3.3)\qquad \begin{array}{ccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & (X \times \mathbf{G})' & \longrightarrow & X \times \mathbf{G} \\ \big\downarrow & & \big\downarrow{\scriptstyle \tilde{p}} & & \big\downarrow{\scriptstyle p} \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{D} & \longrightarrow & \mathbf{G}. \end{array}$$

where the left square is cartesian. The canonical morphism $(X \times \mathbf{G})' \rightarrow X \times \mathbf{G}$ is an isomorphism on the complement $(X \times \mathbf{G})^\circ = (X \times \mathbf{G}) - (X \times_{\mathbf{P}} \mathbf{A})$.

For a line $L \subset \mathbf{P}^\vee$, define $p_L \colon X_L \rightarrow L$ by the cartesian diagram

$$(3.4)\qquad \begin{array}{ccc} X_L & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\ {\scriptstyle p_L}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ L & \longrightarrow & \mathbf{P}^\vee. \end{array}$$

It is also the fiber of $\tilde{p} \colon (X \times \mathbf{G})' \rightarrow \mathbf{D}$ at the point of $\mathbf{G}$ corresponding to L. We put $X_L^\circ = X_L \cap (X \times \mathbf{G})^\circ$ and let $p_L^\circ \colon X_L^\circ \rightarrow L$ be the restriction of $p_L \colon X_L \rightarrow L$. If L is the line passing through hyperplanes $H_0 \neq H_\infty$, the scheme X_L° is the complement of the axis $A_L = H_0 \cap H_\infty$. If A_L meets X properly, the scheme X_L is the blow-up of X at the intersection $X \cap A_L$.

Lemma 3.3. *Let $S \subset T^*X$ be a closed conic subset of dimension d and $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ be as in* (3.2). *Let $L \subset \mathbf{P}^\vee$ be a line and $p_L^\circ \colon X_L^\circ \rightarrow L$ be the morphism defined by the pencil. Let $x \in X_L^\circ$ be a point and ω be a basis of Ω_L^1 on a neighborhood $p_L(x)$ and define a section $p_L^{\circ *}\omega$ of T^*X on a neighborhood of x.*

1. *The following conditions on x are equivalent:*

(1) *x is contained in the intersection $X_L^\circ \cap \mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$.*

(2) *The section $p_L^{\circ *}\omega$ of T^*X meets S in the fiber of x.*

2. *Suppose that x is an isolated point of the intersection $X_L^\circ \cap \mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$. Then, we have an equality*

$$(3.5)\qquad (S, p_L^{\circ *}\omega)_{T^*X, x} = (\mathbf{P}(\widetilde{S}), X_L^\circ)_{X \times_{\mathbf{P}} \mathbf{H}, x}$$

of the intersection numbers.

Proof. 1. Let $(\mathbf{P}\times\mathbf{G})^\circ \subset \mathbf{P}\times\mathbf{G}$ be the complement of $\mathbf{A}$. Then, in the exact sequence

$$(3.6)\qquad 0\to (\mathbf{P}\times\mathbf{G})^\circ\times_{\mathbf{D}} T^*\mathbf{D}/\mathbf{G} \longrightarrow T^*(\mathbf{P}\times\mathbf{G})^\circ/\mathbf{G} \longrightarrow T^*(\mathbf{P}\times\mathbf{G})^\circ/\mathbf{D}\to 0,$$

the left injection induces a morphism $\tilde p_L^{\circ *}\colon X_L^\circ\times_L T^*L\to X_L^\circ\times_{\mathbf{P}} T^*\mathbf{P}$. This is a lifting of the morphism $p_L^{\circ *}\colon X_L^\circ\times_L T^*L\to T^*X_L^\circ$ with respect to the surjection $X\times_{\mathbf{P}} T^*\mathbf{P}\to T^*X$. Let $\overline{\tilde p_L^{\circ *}\omega}$ be the section of $X\times_{\mathbf{P}}\mathbf{H}=\mathbf{P}(X\times_{\mathbf{P}} T^*\mathbf{P})\to X$ defined on a neighborhood of x by $\tilde p_L^{\circ *}\omega$. The condition (1) is equivalent to that the image of x by $\overline{\tilde p_L^{\circ *}\omega}$ is contained in $\mathbf{P}(\widetilde S)$. Thus, it suffices to show that the inclusion $X_L\to X\times_{\mathbf{P}}\mathbf{H}$ is the same as the section $\overline{\tilde p_L^{\circ *}\omega}$ of $X\times_{\mathbf{P}}\mathbf{H}=\mathbf{P}(X\times_{\mathbf{P}} T^*\mathbf{P})\to X$ on a neighborhood of x.

The inclusion $X_L\to X\times_{\mathbf{P}}\mathbf{H}\subset X\times\mathbf{P}^\vee$ is induced by the immersion $\mathbf{H}\to\mathbf{P}\times\mathbf{P}^\vee$ defined by the left injection in the exact sequence

$$(3.7)\qquad 0\to T^*_{\mathbf{H}}(\mathbf{P}\times\mathbf{P}^\vee) \longrightarrow \mathbf{H}\times_{\mathbf{P}\times\mathbf{P}^\vee} T^*(\mathbf{P}\times\mathbf{P}^\vee)/\mathbf{P}^\vee \longrightarrow T^*\mathbf{H}/\mathbf{P}^\vee\to 0.$$

of vector bundles on $(\mathbf{P}\times\mathbf{G})^\circ$. Since the exact sequence (3.6) coincides with the pull-back of (3.7), the assertion follows from Lemma 3.2.

2. In the notation of the proof of 1., we have

$$(S, p_L^{\circ *}\omega)_{T^*X,x} = (\widetilde S, \tilde p_L^{\circ *}\omega)_{X\times_{\mathbf{P}}T^*\mathbf{P},x} = (\mathbf{P}(\widetilde S), \overline{\tilde p_L^{\circ *}\omega})_{X\times_{\mathbf{P}}\mathbf{H},x}.$$

Since the right hand side equals that of (3.5) by the proof of 1., the assertion follows. □

3.2 Morphism defined by a pencil on a hyperplane section

Assume that X is quasi-projective and let $E\subset\Gamma(X,\mathcal{L})$ be a subspace of finite dimension defining a closed immersion $X\to\mathbf{P}=\mathbf{P}(E^\vee)$. We identify the Grassmannian variety $\mathbf{G}=\mathrm{Gr}(2,E)$ parametrizing subspaces of dimension 2 of E with the Grassmannian variety $\mathbf{G}=\mathrm{Gr}(1,\mathbf{P}^\vee)$ parametrizing lines in $\mathbf{P}^\vee$. The universal family $\mathbf{A}\subset\mathbf{P}\times\mathbf{G}$ of linear subspace of codimension 2 of $\mathbf{P}=\mathbf{P}(E^\vee)$ consists of pairs (x,L) of a point x of the axis $A_L\subset\mathbf{P}$ of a line $L\subset\mathbf{P}^\vee$. The intersection $X\times_{\mathbf{P}}\mathbf{A}=(X\times\mathbf{G})\cap\mathbf{A}$ is canonically identified with the bundle $\mathrm{Gr}(2,X\times_{\mathbf{P}}T^*\mathbf{P})$ of Grassmannian varieties parametrizing rank 2 subbundles.

Lemma 3.4. *Let $E\subset\Gamma(X,\mathcal{L})$ be a subspace of finite dimension defining an immersion $X\to\mathbf{P}=\mathbf{P}(E^\vee)$.*

1. *Define an open subscheme*

$$(3.8)\qquad (X\times_{\mathbf{P}}\mathbf{A})^\flat\subset X\times_{\mathbf{P}}\mathbf{A}$$

to be the largest one flat of relative dimension $d-2$ over $\mathbf{G}$. If $\dim X\geqq 2$, the mapping $(X\times_{\mathbf{P}}\mathbf{A})^\flat\to\mathbf{G}$ is dominant and the closure of the image of the complement $X\times_{\mathbf{P}}\mathbf{A} - (X\times_{\mathbf{P}}\mathbf{A})^\flat$ is of codimension $\geqq 2$.

2. *Let $T\subset X$ be an irreducible closed subset and define*

$$(3.9)\qquad (T\times_{\mathbf{P}}\mathbf{A})^\flat\subset T\times_{\mathbf{P}}\mathbf{A}$$

to be the largest open subscheme where the regular immersion $\mathbf{A}\to\mathbf{P}\times\mathbf{G}$ of codimension 2 of flat schemes over $\mathbf{G}$ meets $T\times\mathbf{G}$ properly. Then, if $\dim T\geqq 2$, the mapping $(T\times_{\mathbf{P}}\mathbf{A})^\flat\to\mathbf{G}$ is dominant and the closure of the image of the complement $T\times_{\mathbf{P}}\mathbf{A} - (T\times_{\mathbf{P}}\mathbf{A})^\flat$ is of codimension $\geqq 2$. If $\dim T\leqq 1$, the closure of the image of $T\times_{\mathbf{P}}\mathbf{A}$ in $\mathbf{G}$ is of codimension $2-\dim T$.

Proof. 2. By replacing X by $\mathbf{P}$ and T by the closures in $\mathbf{P}$, we may assume that $T \to \mathbf{P}$ is a closed immersion. If the proper morphism $T \times_{\mathbf{P}} \mathbf{A} \to \mathbf{G}$ is not surjective, then for the line $L \subset \mathbf{P}$ corresponding to the point of $\mathbf{G}$ not in the image, the axis $A_L \subset \mathbf{P}$ does not meet T. Since the morphism $\mathbf{P} - A_L \to L$ is affine, its restriction $T \to L$ is finite and we have $\dim T \leqq 1$.

We regard $T \subset \mathbf{P}$ as a reduced closed subscheme. Then, $T \times_{\mathbf{P}} \mathbf{A}$ is also reduced and is flat over $\mathbf{G}$ on the complement of a closed subset of codimension $\geqq 2$. By the first assertion, $(T \times_{\mathbf{P}} \mathbf{A})^\flat$ contains the open subscheme where $T \times_{\mathbf{P}} \mathbf{A} \to \mathbf{G}$ is flat and the assertion follows.

If $\dim T = 0$, the image of $T \times_{\mathbf{P}} \mathbf{A}$ in $\mathbf{G}$ is of codimension 2. If $\dim T = 1$, since the image in $\mathbf{G}$ of the fibers of $T \times_{\mathbf{P}} \mathbf{A} \to T$ is not constant in $\mathbf{G}$, the closure of the image $T \times_{\mathbf{P}} \mathbf{A}$ in $\mathbf{G}$ is of codimension 1.

1. It suffices to apply 2 to $T = X$. □

We canonically identify the fiber product $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D}$ with the flag bundle $\mathrm{Fl}(1, 2, X \times_{\mathbf{P}} T^*\mathbf{P})$ parametrizing pairs of sub line bundles and rank 2 subbundles of $X \times_{\mathbf{P}} T^*\mathbf{P}$ with inclusions. Let $\mathbf{L} \subset (X \times_{\mathbf{P}} \mathbf{H}) \times_{\mathbf{P}} T^*\mathbf{P} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) \times_X (X \times_{\mathbf{P}} T^*\mathbf{P})$ denote the universal sub line bundle of the pull-back of $X \times_{\mathbf{P}} T^*\mathbf{P}$ on $X \times_{\mathbf{P}} \mathbf{H}$. Then the fiber product $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} = \mathrm{Fl}(1, 2, X \times_{\mathbf{P}} T^*\mathbf{P})$ is also canonically identified with the projective space bundle $\mathbf{P}(((X \times_{\mathbf{P}} \mathbf{H}) \times_{\mathbf{P}} T^*\mathbf{P})/\mathbf{L})$ associated to the quotient bundle $((X \times_{\mathbf{P}} \mathbf{H}) \times_{\mathbf{P}} T^*\mathbf{P})/\mathbf{L}$ over $X \times_{\mathbf{P}} \mathbf{H}$. The forgetting morphisms define a commutative diagram

$$(3.10) \qquad \begin{array}{ccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & \mathbf{P}(((X \times_{\mathbf{P}} \mathbf{H}) \times_{\mathbf{P}} T^*\mathbf{P})/\mathbf{L}) & & \\ & & = X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{D} & \longrightarrow & \mathbf{G} \end{array}$$

where the right square is cartesian.

For a conic closed subset $S \subset T^*X$, we define a closed subset

$$(3.11) \qquad \mathbf{Q}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{A}$$

to be the image of $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ by the upper line of (3.10) regarded as a correspondence. Since $\mathbf{D}$ is a $\mathbf{P}^1$-bundle of $\mathbf{G}$ and since $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ is a subset of codimension d, the subset $\mathbf{Q}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{A}$ is of codimension $\geqq d - 1$. We will give a sufficient condition to have an equality at Corollary 3.17.2.

Lemma 3.5. *Let $S \subset T^*X$ be a conic closed subset.*

1. *For a point $z \in X \times_{\mathbf{P}} \mathbf{A}$ corresponding to the pair (u, V) of $u \in X$ and a subspace V of dimension 2 of the fiber $u \times_{\mathbf{P}} T^*\mathbf{P}$ at u, the following conditions are equivalent:*

(1) *z is a point of $\mathbf{Q}(\widetilde{S})$.*

(2) *The intersection of V with $\widetilde{S} \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ is not a subset of 0.*

2. *Assume that S is irreducible of dimension d and let $T = S \cap T^*_X X$ be the intersection with the 0-section. If $\dim T \leqq 1$, we have $\mathbf{Q}(\widetilde{S}) = T \times_{\mathbf{P}} \mathbf{A}$.*

Proof. 1. By the definition of $\mathbf{Q}(\widetilde{S})$, the condition (1) is equivalent to that there exists a line $L \subset V \subset u \times_{\mathbf{P}} T^*\mathbf{P}$ such that the point $w \in X \times_{\mathbf{P}} \mathbf{H}$ corresponding to (u, L) is contained in $\mathbf{P}(\widetilde{S})$. Hence, it follows from Lemma 3.2.

2. By the assumption $\dim S = d$ and $\dim T \leqq 1$, the fiber $S \times_T u$ of a point $u \in T$ is of dimension $\geqq d-1$. Hence, by Lemma 2.1 (3)⇒(1), the intersection of a subspace $V \subset u \times_{\mathbf{P}} T^*\mathbf{P}$ of codimension 2 with the fiber of $\widetilde{S}$ at u is not contained in 0. Thus, by 1, we have $T \times_{\mathbf{P}} \mathbf{A} \subset \mathbf{Q}(\widetilde{S})$. The other inclusion is obvious. □

We give another description of $\mathbf{Q}(\widetilde{S})$. For a vector bundle V over a scheme X and the associated projective space bundle, a canonical morphism $\mathbf{P}(V) \times_X \mathbf{P}(V) - \mathbf{P}(V) \to \mathrm{Fl}(1,2,V)$ is defined by $(L, L') \mapsto (L, L+L')$. Applying this to $V = X \times_{\mathbf{P}} T^*\mathbf{P}$, we define a morphism

$$(X \times_{\mathbf{P}} \mathbf{H} - \mathbf{P}(\widetilde{S})) \times_X \mathbf{P}(\widetilde{S}) \to X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} = \mathbf{P}(((X \times_{\mathbf{P}} \mathbf{H}) \times_{\mathbf{P}} T^*\mathbf{P})/\mathbf{L}) \tag{3.12}$$

over $X \times_{\mathbf{P}} \mathbf{H}$.

Lemma 3.6. *Let $S \subset T^*X$ be a closed conic subset and $\mathbf{Q}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{A}$ be the image of the projectivization $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ by the correspondence in the upper line*

$$X \times_{\mathbf{P}} \mathbf{H} \longleftarrow X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \longrightarrow X \times_{\mathbf{P}} \mathbf{A} \tag{3.13}$$

of (3.10).

1. *On the inverse image of $X \times_{\mathbf{P}} \mathbf{H} - \mathbf{P}(\widetilde{S})$, the pull-back of $\mathbf{Q}(\widetilde{S})$ by $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \to X \times_{\mathbf{P}} \mathbf{A}$ is equal to the image of* (3.12).

2. *Assume that S is irreducible of dimension d and that S is not the 0-section $T^*_X X$ or a fiber $T^*_u X$ of a closed point. Then, the morphism* (3.12) *is generically finite.*

Proof. 1. We define a commutative diagram

$$\begin{array}{ccccc} X \times_{\mathbf{P}} (\mathbf{H} \times_{\mathbf{P}} \mathbf{H}) & \longleftarrow & X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} (\mathbf{D} \times_{\mathbf{G}} \mathbf{D}) & \xrightarrow{\mathrm{pr}_2} & X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\ X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \end{array} \tag{3.14}$$

by

$$\mathrm{Gr}(1, X \times_{\mathbf{P}} T^*\mathbf{P}) \longleftarrow \mathrm{Fl}(1,2, X \times_{\mathbf{P}} T^*\mathbf{P}) \longrightarrow \mathrm{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P}).$$

The right vertical arrow is the same as the lower right horizontal arrow, the right square is cartesian and the upper left horizontal arrow is the self product of the lower left horizontal arrow. The pull-back of $\mathbf{Q}(\widetilde{S})$ is defined as the image of $\mathbf{P}(\widetilde{S})$ by the correspondence from the lower left to upper right via lower right. By the diagram (3.14), it is the same as that by the correspondence via upper left. The upper left horizontal arrow induce an isomorphism

$$X \times_{\mathbf{P}} (\mathbf{H} \times_{\mathbf{P}} \mathbf{H} - \mathbf{H}) \longleftarrow X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} (\mathbf{D} \times_{\mathbf{G}} \mathbf{D} - \mathbf{D})$$

on the complements of the diagonals since it maps the triple (V, L, L') of rank 2 subbundles spanned by different sub line bundles to (L, L'). Hence, the assertion follows.

2. By the assumption, there exists a point x of X and lines L, L' in the fiber $T^*_x\mathbf{P}$ such that $(x, L) \notin \mathbf{P}(\widetilde{S})$ and $(x, L') \in \mathbf{P}(\widetilde{S})$. Then, the fiber of (3.12) at $(x, (L+L')) \in \mathbf{Q}(\widetilde{S})$ is the intersection $\mathbf{P}(L+L') \cap \mathbf{P}(\widetilde{S})$ and is finite. □

Let $H \subset \mathbf{P} = \mathbf{P}(E^\vee)$ be a hyperplane. It corresponds to a line $L_H \subset E$ and the dual projective space $H^\vee$ is the projective space $\mathbf{P}(E')$ of the quotient space $E' = E/L_H$.

Let $L \subset H^\vee = \mathbf{P}(E')$ be a line. It corresponds to a subspace of E' of dimension 2 and further to a subspace $W = W_L$ of E of dimension 3 containing L_H. Further, the line L is canonically identified with the subspace of $\mathbf{G} = \mathrm{Gr}(2, E)$ parametrizing planes in E containing L_H and contained in W_L.

Let $Y = X \cap H$ be the hyperplane section and $p_L\colon Y_L \to L$ be the morphism defined by the pencil L. Under the identification above, we have a cartesian diagram

$$\begin{array}{ccc} Y_L & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \\ {\scriptstyle p_L}\downarrow & & \downarrow \\ L & \longrightarrow & \mathbf{G}. \end{array} \tag{3.15}$$

We construct the universal family of the diagram (3.15). Let $\mathbf{B} = \mathrm{Fl}(1, 3, E)$ and $\mathbf{C} = \mathrm{Fl}(1, 2, 3, E)$ denote the flag varieties parametrizing pairs of a line and a subspace of dimension 3 with inclusion and triples of a line, a plane and a subspace of dimension 3 with inclusions respectively. The pair (H, L) of a hyperplane $H \subset \mathbf{P}$ and a line $L \subset H^\vee$ in the dual projective space corresponds to a point of $\mathbf{B}$ defined by the flag $L_H \subset W_L \subset E$.

Similarly as the diagram (3.3), we consider a commutative diagram

$$\begin{array}{ccccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & Y_{\mathbf{B}} & \longleftarrow & Y'_{\mathbf{B}} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{B} & \longleftarrow & \mathbf{C} & \longrightarrow & \mathbf{G} \end{array} \tag{3.16}$$

where the left and the right squares are cartesian and the bottom horizontal arrows

$$\mathbf{P}(E) \longleftarrow \mathrm{Fl}(1, 3, E) \longleftarrow \mathrm{Fl}(1, 2, 3, E) \longrightarrow \mathrm{Gr}(2, E) \tag{3.17}$$

and the top middle arrow

$$\mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) \times_{\mathbf{P}(E)} \mathrm{Fl}(1, 3, E) \longleftarrow \mathrm{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P}) \times_{\mathrm{Gr}(2,E)} \mathrm{Fl}(1, 2, 3, E)$$

are the forgetful morphisms. Define a morphism $\mathrm{Fl}(1, 3, X \times_{\mathbf{P}} T^*\mathbf{P}) \to Y_{\mathbf{B}}$ to be the canonical morphism $\mathrm{Fl}(1, 3, X \times_{\mathbf{P}} T^*\mathbf{P}) \to \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) \times_{\mathbf{P}(E)} \mathrm{Fl}(1, 3, E)$. Since $X \times_{\mathbf{P}} T^*\mathbf{P}$ is a twist of a sub vector bundle of codimension 1 of $E \times X$, it is a regular immersion of codimension $2 = 3 - 1$. The top middle arrow is the blow-up of $Y_{\mathbf{B}}$ at $\mathrm{Fl}(1, 3, X \times_{\mathbf{P}} T^*\mathbf{P})$ and is an isomorphism on the complement $Y^\circ_{\mathbf{B}} = Y_{\mathbf{B}} - \mathrm{Fl}(1, 3, X \times_{\mathbf{P}} T^*\mathbf{P})$.

At the point of $\mathbf{B}$ corresponding to (H, L), the fiber of $Y^\circ_{\mathbf{B}} \subset Y_{\mathbf{B}}$ is $Y^\circ_L \subset Y = X \cap H$ and the fiber of $Y'_{\mathbf{B}} \to \mathbf{C}$ is $p_L\colon Y_L \to L$.

The following elementary lemma will be used in the proof of Theorem 4.8 to verify that the assumption in Lemma 2.31 is satisfied on a dense open for the universal family.

Lemma 3.7. *Under the correspondence*

$$T^*(\mathbf{C}/\mathbf{B}) \longleftarrow \mathbf{C} \times_{\mathbf{G}} T^*\mathbf{G} \longrightarrow T^*\mathbf{G} \tag{3.18}$$

defined by the lower line of (3.16)*, the image of the inverse image of the complement of the* 0*-section of* $T^*(\mathbf{C}/\mathbf{B})$ *equals the complement of the* 0*-section of* $T^*\mathbf{G}$*.*

Proof. Let $\mathbf{L} \subset \mathbf{W} \subset E \times \mathbf{B}$ denote the universal sub line bundle and the universal sub vector bundle of rank 3 on $\mathbf{B} = \mathrm{Fl}(1,3,E)$ and let $\mathbf{V} \subset E \times \mathbf{G}$ denote the universal sub bundle of rank 2 on $\mathbf{G} = \mathrm{Gr}(2,E)$. Let $\mathbf{L_C} \subset \mathbf{V_C} \subset \mathbf{W_C} \subset E \times \mathbf{C}$ denote their pull-backs on $\mathbf{C}$. Then, the relative cotangent bundle $T^*(\mathbf{C}/\mathbf{B})$ is canonically identified with the Hom-bundle $\mathit{Hom}(\mathbf{W_C}/\mathbf{V_C}, \mathbf{V_C}/\mathbf{L_C})$ and the cotangent bundle $T^*\mathbf{G}$ is canonically identified with $\mathit{Hom}(E \times \mathbf{G}/\mathbf{V}, \mathbf{V})$ respectively. Under these identifications, the canonical morphism $\mathbf{C} \times_{\mathbf{G}} T^*\mathbf{G} \to T^*(\mathbf{C}/\mathbf{B})$ is the morphism $\mathit{Hom}(E \times \mathbf{C}/\mathbf{V_C}, \mathbf{V_C}) \to \mathit{Hom}(\mathbf{W_C}/\mathbf{V_C}, \mathbf{V_C}/\mathbf{L_C})$ induced by the injection $\mathbf{W_C}/\mathbf{V_C} \to E \times \mathbf{C}/\mathbf{V_C}$ and the surjection $\mathbf{V_C} \to \mathbf{V_C}/\mathbf{L_C}$.

For a point of $\mathbf{G}$ corresponding to a subspace $V \subset E$ of dimension 2 and for a non-zero form corresponding to a linear mapping $f\colon E/V \to V$, there exists subspaces $W/V \subset E/V$ and $L \subset V$ of dimension 1 such that the composition $W/V \subset E/V \to V \to V/L$ is not the 0-mapping. Hence the assertion follows. □

3.3 Local Radon transform

We define local Radon transform using the universal family of hyperplane sections.

Definition 3.8. *Assume that X is quasi-projective and smooth of dimension d over a perfect field k of characteristic $p > 0$. Let Λ be a finite field of characteristic $\ell \neq p$ and $\mathcal{K}$ be a constructible complex of Λ-modules on X. For an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a finite dimensional subspace $E \subset \Gamma(X,\mathcal{L})$ defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee)$, we define the* local Radon transform

$$\mathcal{R}_E\mathcal{K} = R\Psi_p q^*\mathcal{K} \tag{3.19}$$

as an object of the derived category on the vanishing topos $(X \times_{\mathbf{P}} \mathbf{H}) \overleftarrow{\times}_{\mathbf{P}^\vee} \mathbf{P}^\vee$, using the universal family of hyperplane sections

$$X \xleftarrow{\ q\ } X \times_{\mathbf{P}} \mathbf{H} \xrightarrow{\ p\ } \mathbf{P}^\vee = \mathbf{P}(E). \tag{3.1}$$

We study the locus of local acyclicity of the right arrow $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ of (3.1) with respect to the pull-back $q^*\mathcal{K}$ by applying (SS1) to the regular immersion $X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee$. Let $S \subset T^*X$ be a singular support of $\mathcal{K}$ satisfying (SS1). Let $\widetilde{S} \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ denote the inverse image of $S = SS\mathcal{K} \subset T^*X$ by the surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ and let $\mathbf{P}(\widetilde{S}) \subset \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H}$ be the projectivization as in (3.2).

Let $T_i \subset X$ be the intersection of an irreducible component $S_i \subset T^*X$ of the singular support $SS\mathcal{K} = \bigcup_i S_i$ with the 0-section $X = T^*_X X$, regarded as a reduced closed subscheme. Define a subspace $E_i \subset E$ to be the kernel of the restriction mapping $E \subset \Gamma(X,\mathcal{L}) \to \Gamma(T_i, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{T_i})$ and $\mathbf{P}_i^\vee = \mathbf{P}(E_i) \subset \mathbf{P}^\vee = \mathbf{P}(E)$ as in Lemma 3.1.2.

Define $\mathbf{R}_{\geqq 1}(S) \subset \mathbf{R}(S) \subset X \times_{\mathbf{P}} \mathbf{H}$ to be the union

$$\mathbf{R}_{\geqq 1}(S) = \bigcup_{i:\dim T_i \geqq 1} (T_i \times \mathbf{P}_i^\vee) \subset \mathbf{R}(S) = \bigcup_i (T_i \times \mathbf{P}_i^\vee) \subset X \times_{\mathbf{P}} \mathbf{H} \tag{3.20}$$

If $T_i = \{x\}$ for a closed point x, we have $S_i = T^*_x X$ and $\mathbf{P}_i^\vee$ is the dual hyperplane $x^\vee$ and hence $T_i \times \mathbf{P}_i^\vee = \mathbf{P}(\widetilde{S}_i)$ is contained in $\mathbf{P}(\widetilde{S})$.

Lemma 3.9. *Assume that $S \subset T^*X$ satisfies* (SS1) *for $\mathcal{K}$ on X.*

1. *The complement of the union $\mathbf{P}(\widetilde{S}) \cup \mathbf{R}(S) \subset X \times_{\mathbf{P}} \mathbf{H}$ is the largest open subscheme where the regular immersion $p\colon X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee$ of codimension 1 of flat schemes over $\mathbf{P}^\vee$ is non-characteristic with respect to $S = SS\mathcal{K}$. Consequently, the morphism $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is universally locally acyclic relatively to $q^*\mathcal{K}$ on the complement of the same union $\mathbf{P}(\widetilde{S}) \cup \mathbf{R}(S)$.*

2. *Define $(X \times_{\mathbf{P}} \mathbf{H})^\square \subset X \times_{\mathbf{P}} \mathbf{H} - \mathbf{R}_{\geqq 1}(S)$ to be the largest open subset such that the intersection*

$$\mathbf{P}(\widetilde{S})^\square = (X \times_{\mathbf{P}} \mathbf{H})^\square \cap \mathbf{P}(\widetilde{S}) \tag{3.21}$$

is quasi-finite over $\mathbf{P}^\vee$. On $(X \times_{\mathbf{P}} \mathbf{H})^\square$, the construction of the local Radon transform $\mathcal{R}_E\mathcal{K} = R\Psi_p q^*\mathcal{K}$ commutes with base change.

Proof. 1. The immersion $X \times_{\mathbf{P}} \mathbf{H} \to X \times \mathbf{P}^\vee$ is regular of codimension 1 and $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is flat by Lemma 3.1.1. For a point z of $X \times_{\mathbf{P}} \mathbf{H}$, the condition (i) in Definition 2.11.1 is satisfied if and only if z is not contained in $\mathbf{P}(\widetilde{S})$ by Lemma 3.2. The condition (ii) in Definition 2.11.1 is satisfied if and only if z is not contained in $\mathbf{R}(S)$ by Lemma 3.1.2.

The universal local acyclicity follows from (SS1) and Lemma 2.19.1.

2. Since $\mathbf{P}(\widetilde{S})^\square$ is quasi-finite over $\mathbf{P}^\vee$, the assertion follows from 1 and Proposition 1.7.1. □

Lemma 3.10. *Assume that $S \subset T^*X$ satisfies* (SS1) *for $\mathcal{K}$ on X. Define subsets*

$$\mathbf{Z}(S) \subset (X \times \mathbf{G})^\triangledown \subset (X \times \mathbf{G})^\circ \tag{3.22}$$

to be the inverse images by $(X \times \mathbf{G})^\circ \subset (X \times \mathbf{G})' \to (X \times_{\mathbf{P}} \mathbf{H})$ of $\mathbf{P}(\widetilde{S})^\square \subset (X \times_{\mathbf{P}} \mathbf{H})^\square \subset X \times_{\mathbf{P}} \mathbf{H}$ defined in Lemma 3.9.2.

1. *For a point (u, L) of $(X \times \mathbf{G})^\circ$, the following conditions are equivalent:*

(1) *(u, L) is a point of $\mathbf{Z}(S) \subset (X \times \mathbf{G})^\triangledown$.*

(2) *$u \in X_L^\circ$ is an isolated characteristic point of $p_L^\circ\colon X_L^\circ \to L$ with respect to $\mathcal{K}$.*

2. *Define a function $\varphi_{\mathcal{K}}$ on $\mathbf{Z}(S) \subset (X \times \mathbf{G})^\triangledown$ by*

$$\varphi_{\mathcal{K}}(z) = \dim \mathrm{tot}_u \phi_u(\mathcal{K}, p_L^\circ) \tag{3.23}$$

for a point $z \in \mathbf{Z}(S)$ corresponding to the pair (u, L) of an isolated characteristic point u of the morphism $p_L^\circ\colon X_L^\circ \to L$ defined by $L \subset \mathbf{P}^\vee$. Then, $\varphi_{\mathcal{K}}$ is constructible and flat over $\mathbf{G}$.

Proof. 1. For $(u, L) \in (X \times \mathbf{G})^\circ$, it is contained in $T_i \times \mathbf{P}_i^\vee$ if and only if $T_i \cap X_L^\circ$ is a subset of the fiber of $p_L^\circ\colon X_L^\circ \to L$. Hence the assertion follows from Lemma 3.9.1 and Corollary 2.7.2.

2. It suffices to apply Lemma 2.18 to the diagram

$$\begin{array}{ccc} (X \times \mathbf{G})^\triangledown & \longrightarrow & \mathbf{D} \\ {\scriptstyle\cap}\big\downarrow & & \big\downarrow \\ X \times \mathbf{G} & \longrightarrow & \mathbf{G}. \end{array} \tag{3.24}$$

□

3.4 Second local Radon transform

We define a variant of the local Radon transform.

Definition 3.11. *Let the notation be as in Definition* 3.8 *except that*

$$X \xleftarrow{\ q\ } X\times_{\mathbf P}\mathbf A \xrightarrow{\ p\ } \mathbf G = \mathrm{Gr}(2,E) \tag{3.25}$$

denotes the universal linear section of codimension 2*. We define the* second local Radon transform

$$\mathcal R^{(2)}_E\mathcal K = R\Psi_p q^*\mathcal K. \tag{3.26}$$

as an object of the derived category on the vanishing topos $(X\times_{\mathbf P}\mathbf A)\overset{\leftarrow}{\times}_{\mathbf G}\mathbf G$.

We study the locus of local acyclicity of the right arrow $p\colon X\times_{\mathbf P}\mathbf A\to\mathbf G$ of (3.25) with respect to the pull-back $q^*\mathcal K$ by applying (SS2) to the regular immersion $X\times_{\mathbf P}\mathbf A\to X\times\mathbf G$.

Let $S\subset T^*X$ be a singular support of $\mathcal K$. We define a closed subset $\mathbf Q(\widetilde S)\subset X\times_{\mathbf P}\mathbf A$ (3.11) to be the image of $\mathbf P(\widetilde S)\subset X\times_{\mathbf P}\mathbf H$ (3.2) by the upper line

$$X\times_{\mathbf P}\mathbf H \longleftarrow X\times_{\mathbf P}\mathbf A\times_{\mathbf G}\mathbf D \longrightarrow X\times_{\mathbf P}\mathbf A \tag{3.13}$$

of (3.10) regarded as a correspondence. Let $(X\times_{\mathbf P}\mathbf A)^\flat\subset X\times_{\mathbf P}\mathbf A$ be the largest open subscheme flat of relative dimension $d-2$ over $\mathbf G$ as in Lemma 3.4.1. For an irreducible component S_i of $SS\mathcal K=\bigcup_i S_i$, let $T_i=S_i\cap T^*_XX$ denote the intersection with the 0-section and define $(T_i\times_{\mathbf P}\mathbf A)^\flat\subset T_i\times_{\mathbf P}\mathbf A$ to be the largest open subscheme where the regular immersion $\mathbf A\to\mathbf P\times\mathbf G$ of codimension 2 meets $T_i\times\mathbf G$ properly as in Lemma 3.4.2. We define closed subsets

$$\begin{aligned}\mathbf T_{\geqq 2}(S) &= \big(X\times_{\mathbf P}\mathbf A - (X\times_{\mathbf P}\mathbf A)^\flat\big)\cup\bigcup_{\dim T_i\geqq 2}\big(T_i\times_{\mathbf P}\mathbf A-(T_i\times_{\mathbf P}\mathbf A)^\flat\big)\subset\\ \mathbf T(S) &= \big(X\times_{\mathbf P}\mathbf A - (X\times_{\mathbf P}\mathbf A)^\flat\big)\cup\bigcup_i\big(T_i\times_{\mathbf P}\mathbf A-(T_i\times_{\mathbf P}\mathbf A)^\flat\big)\subset X\times_{\mathbf P}\mathbf A.\end{aligned} \tag{3.27}$$

Lemma 3.12. *Assume that* $S\subset T^*X$ *satisfies* (SS2) *for* $\mathcal K$ *on* X.

1. *The complement of the union* $\mathbf Q(\widetilde S)\cup\mathbf T(S)\subset X\times_{\mathbf P}\mathbf A$ *is the largest open subscheme where the regular immersion* $(X\times_{\mathbf P}\mathbf A)^\flat\to X\times\mathbf G$ *of codimension* 2 *of flat schemes over* $\mathbf G$ *is non-characteristic with respect to* $S=SS\mathcal K$*. Consequently, the morphism* $(X\times_{\mathbf P}\mathbf A)^\flat\to\mathbf G$ *is universally locally acyclic relatively to* $q^*\mathcal K$ *on the complement of the same union* $\mathbf Q(\widetilde S)\cup\mathbf T(S)$*.*

2. *Let* $(X\times_{\mathbf P}\mathbf A)^\square$ *be the largest open subscheme of* $X\times_{\mathbf P}\mathbf A-\mathbf T_{\geqq 2}(S)$ *such that the intersection*

$$\mathbf Q(\widetilde S)^\square = (X\times_{\mathbf P}\mathbf A)^\square\cap\mathbf Q(\widetilde S) \tag{3.28}$$

is quasi-finite over $\mathbf G$*. On* $(X\times_{\mathbf P}\mathbf A)^\square$*, the construction of the second local Radon transform* $\mathcal R^{(2)}_E\mathcal K=R\Psi_p q^*\mathcal K$ *commutes with base change.*

Proof. 1. The proof is similar to that of Lemma 3.9.1. The immersion $(X \times_{\mathbf{P}} \mathbf{A})^\flat \to X \times \mathbf{G}$ is regular of codimension 2 by Lemma 3.4.1. For a point z of $(X \times_{\mathbf{P}} \mathbf{A})^\flat$, the condition (i) in Definition 2.11.1 is satisfied if and only if z is not contained in $\mathbf{Q}(\widetilde{S})$ by Lemma 3.5. The condition (ii) in Definition 2.11.1 is satisfied if and only if z is not contained in $\mathbf{T}(S)$ by Lemma 3.4.2.

The universal local acyclicity follows from (SS2) and Lemma 2.19.1.

2. Similarly as Lemma 3.9.2, since $\mathbf{Q}(\widetilde{S})^\square$ is quasi-finite over $\mathbf{G}$, the assertion follows from 1 and Proposition 1.7.1. □

Lemma 3.13. *Let (H, L) be the pair of a hyperplane $H \subset \mathbf{P}$ and a line $L \subset H^\vee$ in the dual projective space. Let $Y = X \cap H$ be the hyperplane section and u be a point of $Y_L^\circ \subset Y$. Then, the following conditions are equivalent:*

(1) *The point $(u, H) \in X \times_{\mathbf{P}} \mathbf{H}$ is not contained in the union $\mathbf{P}(\widetilde{S}) \cup \mathbf{R}(S)$ and the image of $u \in Y_L^\circ$ in $X \times_{\mathbf{P}} \mathbf{A}$ is not contained in the union $\mathbf{Q}(\widetilde{S}) \cup \mathbf{T}(S)$.*

(2) *The immersion $i\colon Y = X \cap H \to X$ is non-characteristic at u with respect to S and the morphism $p_L^\circ\colon Y_L^\circ \to L$ is non-characteristic with respect to $i^! S$ at u.*

Proof. The first condition in (2) is equivalent to that in (1) by Lemma 3.9.1. The second condition in (2) is also equivalent to that in (1) by Lemma 3.6 and Lemma 3.12.1. □

We consider the commutative diagram

$$\begin{array}{ccccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & Y_{\mathbf{B}} & \longleftarrow & Y'_{\mathbf{B}} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{B} & \longleftarrow & \mathbf{C} & \longrightarrow & \mathbf{G} \end{array} \tag{3.16}$$

of the universal family of morphisms defined by pencils for hyperplane sections and the open subscheme $Y_{\mathbf{B}}^\circ \subset Y_{\mathbf{B}}$ where $Y'_{\mathbf{B}} \to Y_{\mathbf{B}}$ is an isomorphism.

Lemma 3.14. *Assume that $S \subset T^*X$ satisfies* (SS2) *for $\mathcal{K}$ on X. Define an open subscheme*

$$Y_{\mathbf{B}}^\triangledown \subset Y_{\mathbf{B}}^\circ \tag{3.29}$$

to be the intersection with the inverse images of $X \times_{\mathbf{P}} \mathbf{H} - (\mathbf{P}(\widetilde{S}) \cup \mathbf{R}(S))$ (see Lemma 3.9*) by $Y_{\mathbf{B}} \to X \times_{\mathbf{P}} \mathbf{H}$ and of $(X \times_{\mathbf{P}} \mathbf{A})^\square$ (see Lemma* 3.12.2*) by $Y'_{\mathbf{B}} \to X \times_{\mathbf{P}} \mathbf{A}$.*

1. *The morphism $Y_{\mathbf{B}}^\triangledown \to \mathbf{B}$ is locally acyclic relatively to the pull-back of $\mathcal{K}$.*

2. *Define a closed subset*

$$\mathbf{W}(S) \subset Y_{\mathbf{B}}^\triangledown \tag{3.30}$$

to be the inverse image of $\mathbf{Q}(\widetilde{S})^\square \subset (X \times_{\mathbf{P}} \mathbf{A})^\square$ by $Y_{\mathbf{B}}^\triangledown \subset Y_{\mathbf{B}} \to X \times_{\mathbf{P}} \mathbf{A}$. Then on the complement $Y_{\mathbf{B}}^\triangledown - \mathbf{W}(S)$, the morphism $Y_{\mathbf{B}}^\triangledown \to \mathbf{C}$ is locally acyclic relatively to the pull-back of $\mathcal{K}$.

3. *Define a function $\varphi_{i^*\mathcal{K}}$ on $\mathbf{W}(S) \subset Y_{\mathbf{B}}^\triangledown$ by*

$$\varphi_{i^*\mathcal{K}}(z) = \dim \mathrm{tot}_u \phi_u(i^*\mathcal{K}, p_L^\circ) \tag{3.31}$$

for $z \in Y_{\mathbf{B}}^\triangledown$ corresponding to (u, H, L) where u is an isolated characteristic point of $p_L^\circ\colon Y_L^\circ \to L$ and $Y = X \cap H$. Then, $\varphi_{i^*\mathcal{K}}$ is constructible and flat over $\mathbf{B}$.

Proof. 1. and 2. It follows from Lemma 3.9.1 and Lemma 3.12.1 respectively.

3. We apply Proposition 1.17 to the commutative diagram

$$\begin{array}{ccc} Y^{\triangledown}_{\mathbf{B}} & \xrightarrow{f} & \mathbf{C} \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & \mathbf{B}. & \end{array} \tag{3.32}$$

Then, since $\mathbf{Q}(\widetilde{S})^{\square} \to \mathbf{G}$ is quasi-finite, $\mathbf{W}(\widetilde{S}) \to \mathbf{B}$ is also quasi-finite and the assumptions in Proposition 1.17 is satisfied by the assertions 1 and 2. $\square$

3.5 Existence of good pencils

In this subsection, we assume that k is algebraically closed. Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^{\vee})$. We consider the following condition:

(E) For every pairs of distinct closed points $u \neq v$ of X, the composition

$$E \subset \Gamma(X, \mathcal{L}) \to \mathcal{L}_u/\mathfrak{m}_u^2\mathcal{L}_u \oplus \mathcal{L}_v/\mathfrak{m}_v^2\mathcal{L}_v \tag{3.33}$$

is a surjection.

For an integer $n \geqq 1$, let $E^{(n)} \subset \Gamma(X, \mathcal{L}^{\otimes n})$ denote the image of $S^nE \to \Gamma(X, \mathcal{L}^{\otimes n})$ induced by the inclusion $E \to \Gamma(X, \mathcal{L})$.

Lemma 3.15. *Let X be a quasi-projective scheme over an algebraically closed field k and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Assume that $E \subset \Gamma(X, \mathcal{L})$ defines an immersion $X \to \mathbf{P} = \mathbf{P}(E^{\vee})$.*

1. *For $n \geqq 3$, the subspace $E^{(n)} \subset \Gamma(X, \mathcal{L}^{\otimes n})$ satisfies the condition* (E) *above.*

2. *Let u be a closed point of X. Then, for an integer $n \geqq 1$, the composition $E^{(n)} \subset \Gamma(X, \mathcal{L}^{\otimes n}) \to \mathcal{L}_u^{\otimes n}/\mathfrak{m}_u^{n+1}\mathcal{L}_u^{\otimes n}$ is a surjection.*

Proof. 1. We may assume $X = \mathbf{P}^d, \mathcal{L} = \mathcal{O}(1), E = \Gamma(X, \mathcal{L})$ and $u = (0, \cdots, 0, 1), v = (1, 0, \cdots, 0)$. Then, the assertion is clear.

2. Similarly as in the proof of 1., we see that the case $n = 1$ holds. Hence, the assertion follows from the subjectivity of $S^n(\mathcal{L}/\mathfrak{m}_u^2\mathcal{L}) \to \mathcal{L}_u^{\otimes n}/\mathfrak{m}_u^{n+1}\mathcal{L}_u^{\otimes n}$. $\square$

Proposition 3.16. *Let X be a quasi-projective smooth scheme of dimension d over an algebraically closed field k and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^{\vee})$ and satisfying the condition* (E) *at the beginning of this subsection. Let u be a closed point, $\omega \in T_u^*X$ and let $S \subset T^*X$ be a closed subset of codimension d.*

1. *There exist sections $l_0, l_\infty \in E$ and a neighborhood U of u such that l_∞ is a basis of $\mathcal{L}|_U$ and that the function $f = l_0/l_\infty$ on U satisfies the following conditions:*

We have $df(u) = \omega$ and the section $df\colon U \to T^*X$ does not meet S on the complement $U - \{u\}$.

2. *Assume that S does not contain the fiber T_u^*X at u as a subset. Then, there exist sections $l_0, l_1, l_\infty \in E$ and a neighborhood U of u such that l_∞ is a basis of $\mathcal{L}|_U$ and that*

the functions $f = l_0/l_\infty$ and $g = l_1/l_\infty$ on U and the closed subscheme $Z \subset U$ defined by $f = g = 0$ satisfies the following conditions:

The point u is contained in Z, we have $df(u) = \omega$ and there exists a neighborhood Y of $(u, 0) \in Z \times \mathbf{A}^1$ such that the section $df - tdg\colon Z \times \mathbf{A}^1 \to T^*X$ does not meet S on the complement $Y - \{(u, 0)\}$.

Proof. Take an element $l_\infty \in E$ such that $l_\infty(u) \neq 0$. After shrinking X to a neighborhood of u if necessary, we identify $\mathcal{L}$ with $\mathcal{O}_X$ by the basis l_∞ and a section l with a function $f = l/l_\infty$.

Let $\mathcal{I}_\Delta \subset \mathcal{O}_{X\times X}$ be the ideal sheaf defining the diagonal $X \to X \times X$. Define a closed subscheme $P \subset X \times X$ by $\mathcal{I}_\Delta^2$ and let $p_1, p_2\colon P \to X$ denote the restrictions of the projections. Define a vector bundle V of rank $d + 1$ on X by $V = \mathbf{V}(p_{2*}p_1^*\mathcal{L})$. Since $\Omega^1_{X/k} = \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$, we have an exact sequence of vector bundles $0 \to T^*X \to V \to L \to 0$.

Set $X^\circ = X - \{u\}$ and let W be the k-vector space $\mathcal{L}_u/\mathfrak{m}_u^2\mathcal{L}_u$ regarded as a scheme over k. The condition (E) implies that the linear morphism

$$E \times X^\circ \to W \times (V \times_X X^\circ) \tag{3.34}$$

of vector bundles on X° defined by $l \to (l \bmod \mathfrak{m}_u^2,\ \mathrm{pr}_1^* l|_{X^\circ})$ is a surjection. Let $E_\omega \subset E$ denote the inverse image of $\omega \in T^*_uX = \mathfrak{m}_u\mathcal{L}_u/\mathfrak{m}_u^2\mathcal{L}_u \subset W$ by the surjection $E \to W$.

1. Define an affine morphism

$$a\colon E \times X^\circ \times \mathbf{A}^1 \to W \times (V \times_X X^\circ) \times \mathbf{A}^1$$

of vector bundles on $X^\circ \times \mathbf{A}^1$ by sending (l, x, t) to $(l/l_\infty \bmod \mathfrak{m}_u^2,\ \mathrm{pr}_1^*(l/l_\infty)(x) - t,\ t)$. This is a surjection and hence is flat, by the surjectivity of (3.34). Since $S \subset T^*X \subset V$ is of codimension $d+1$, the inverse image $a^{-1}(S) \subset E_\omega \times X^\circ \times \mathbf{A}^1$ of $\{\omega\} \times (S \times_X X^\circ) \times \mathbf{A}^1 \subset W \times (V \times_X X^\circ) \times \mathbf{A}^1$ is a closed subset of codimension $d + 1$. Since $\dim X = d$, the projection $p\colon E_\omega \times X^\circ \times \mathbf{A}^1 \to E_\omega$ is of relative dimension $d + 1$. Hence, there exists a dense open subscheme $E^\circ_\omega \subset E_\omega$ where the restriction of the projection p to $a^{-1}(S)$ is quasi-finite.

Take a section $l_0 \in E^\circ_\omega$. Then, since $a^{-1}(S) \cap p^{-1}(l_0)$ is finite, shrinking X further if necessary, we may assume that the intersection $a^{-1}(S) \cap p^{-1}(l_0)$ is empty. By $l_0 \in E_\omega$, we have $df(u) = \omega$. For $x \in X^\circ$, taking $t = f(x) = l_0/l_\infty(x)$, we see that $df(x) = \mathrm{pr}_1^*(l_0/l_\infty)(x) - t \in T^*_xX \subset V \times_X x$ is not contained in S since $a^{-1}(S) \cap p^{-1}(l_0)$ is empty.

2. Let $\omega' \in T^*_uX \subset W$ be an element not contained in S and let $E_{\omega'} \subset E$ denote the inverse image of ω' by the surjection $E \to W$. Define an affine morphism

$$b\colon E \times E \times X^\circ \times \mathbf{A}^1 \to W \times W \times (V \times_X X^\circ) \times \mathbf{A}^1$$

of vector bundles on $X^\circ \times \mathbf{A}^1$ by sending

$$(l, l', x, t) \mapsto (l/l_\infty \bmod \mathfrak{m}_u^2,\ l'/l'_\infty \bmod \mathfrak{m}_u^2,\ \mathrm{pr}_1^*(l/l_\infty)(x) - t \cdot \mathrm{pr}_1^*(l'/l_\infty)(x),\ t).$$

This is a surjection and hence is flat, by the surjectivity of (3.34). Since $S \subset T^*X \subset V$ is of codimension $d+1$, the inverse image $b^{-1}(S) \subset E_\omega \times E_{\omega'} \times X^\circ \times \mathbf{A}^1$ of $\{(\omega, \omega')\} \times (S \times_X X^\circ) \times \mathbf{A}^1 \subset W \times W \times (V \times_X X^\circ) \times \mathbf{A}^1$ is a closed subset of codimension $d + 1$. Since $\dim X = d$, the projection $q\colon E_\omega \times E_{\omega'} \times X^\circ \times \mathbf{A}^1 \to E_\omega \times E_{\omega'}$ is of relative dimension $d + 1$. Hence, there exists a dense open subscheme $(E_\omega \times E_{\omega'})^\circ \subset E_\omega \times E_{\omega'}$ where the restriction of the projection q to $b^{-1}(S)$ is quasi-finite.

Take a pair of sections $(l_0, l_1) \in (E_\omega \times E_{\omega'})^\circ$. Then, since $b^{-1}(S) \cap q^{-1}(l_0, l_1)$ is finite, shrinking X further if necessary, we may assume that the intersection $b^{-1}(S) \cap q^{-1}(l_0, l_1)$ is empty. Since $l_0/l_\infty \bmod \mathfrak{m}_u^2$ and $l_1/l_\infty \bmod \mathfrak{m}_u^2$ are contained in the subspace $T_u^*X \subset W$, we have $f(u) = g(u) = 0$ and $u \in Z$. By $l_0 \in E_\omega$, we have $df(u) = \omega$. By $l_1 \in E_{\omega'}$ and $\omega' \notin S \cap T_u^*X$, we have $dg(u) = \omega' \notin S$. Hence, $df(u) - tdg(u)$ is not contained in S except at finitely many closed point $t \in \mathbf{A}^1$. For $x \in X^\circ \cap Z$ and $t \in \mathbf{A}^1$, we have $f(x) = g(x) = 0$ and $df(x) - tdg(x) = \mathrm{pr}_1^*(l_0/l_\infty)(x) - t \cdot \mathrm{pr}_1^*(l_1/l_\infty)(x) \in T_x^*X \subset V \times_X x$ is not contained in S since $b^{-1}(S) \cap q^{-1}(l_0, l_1)$ is empty. □

Corollary 3.17. *Assume that X is quasi-projective over k and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee)$ and satisfying the condition* (E) *at the beginning of this subsection. Let $S \subset T^*X$ be an irreducible closed subset of dimension $d = \dim X$.*

1. *The restriction of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ to $\mathbf{P}(\widetilde{S})$ is generically finite.*
2. *Assume that $S \subset T^*X$ is not the fiber at a point. The restriction of the composition $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \to \mathbf{D} \to \mathbf{G}$ to the inverse image of $\mathbf{P}(\widetilde{S})$ is generically finite. Consequently, $\mathbf{Q}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{A}$ is of codimension $d - 1$ and the restriction of $X \times_{\mathbf{P}} \mathbf{A} \to \mathbf{G}$ to $\mathbf{Q}(\widetilde{S})$ is generically finite.*

Proof. 1. Let u be a closed point of X and $\omega \in T_u^*X$ be a differential form contained in S. We apply Proposition 3.16.1 to u, ω and S. Let L be the line spanned by l_0, l_∞ and $p_L\colon X_L \to L$ be the morphism defined by L. Then, u is not contained in the axis $X \cap A_L$ and is an isolated point of the intersection $X_L \cap \mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ by Lemma 3.3.1. Thus, the assertion follows from the cartesian diagram (3.4).

2. Let u be a closed point of X and $\omega \in T_u^*X$ be a differential form contained in S. We apply Proposition 3.16.2 to u, ω and S. Let L be the line spanned by l_0, l_1 and $p_L\colon X_L \to L$ be the morphism defined by L and $H \in L$ be the hyperplane defined by l_0. Then u is contained in the axis $X \cap A_L$ and $(u, H) \in (X \cap A_L) \times L \subset X_L$ is an isolated point of the intersection $((X \cap A_L) \times L) \cap \mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} \mathbf{H}$ by Lemma 3.3.1. Thus, the first assertion also follows from the cartesian diagram (3.4).

Since $\mathbf{P}(\widetilde{S}) \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ is of codimension d, its inverse image in $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D}$ is also of codimension d. Since $X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \to X \times_{\mathbf{P}} \mathbf{A}$ is of relative dimension 1, the assertion on $\mathbf{Q}(\widetilde{S})$ follows from the first assertion. □

Lemma 3.18. *Let u be a closed point of X, $N \geqq 1$ be an integer and $f \in \mathcal{O}_{X,u}/\mathfrak{m}_u^N$. Then, for $n \geqq N$, there exist $l_0, l_\infty \in E^{(n)} = \Gamma(X, \mathcal{L}^{\otimes n})$ such that $l_\infty(u) \neq 0$ and*

$$l_0/l_\infty \equiv f \bmod \mathfrak{m}_u^N. \tag{3.35}$$

Proof. Since E defines an immersion $X \to \mathbf{P}$, there exists a section $l \in E$ such that $l(u) \neq 0$. We identify $\mathcal{O}_{X,u}/\mathfrak{m}_u^n$ with $\mathcal{L}^{\otimes n}/\mathfrak{m}_u^n\mathcal{L}^{\otimes n}$ by the local basis $l_\infty = l^{\otimes n} \in E^{(n)}$ at u. Then, the assertion follows by Lemma 3.15.2. □

We show the existence of a good pencil to be used in the induction step in the proof of the index formula, Theorem 4.13.

Lemma 3.19. *There exist a line $L \subset \mathbf{P}^\vee$ and a point $H \in L$ satisfying the following conditions: The hyperplane $H \subset \mathbf{P}$ and the axis $A_L \subset \mathbf{P}$ of L meet X transversely. The immersions $i\colon Y = X \cap H \to X$ and $i'\colon Z = X \cap A_L \to Y$ are non-characteristic with* *respect to S and to $i^!S$ respectively. The morphism $p_L\colon X_L \to L$ has at most isolated non-characteristic points.*

Proof. Let $T_i = S_i \cap T^*_X X$, $r_i = \dim S_i - \dim T_i$ and let $T_{ij} = \{x \in T_i \mid \dim S_i \times_{T_i} x \geqq r_i + j\}$ for $j \geqq 0$. By Lemma 3.4.2, the open subscheme $V \subset \mathbf{G}$ consisting of lines L such that the axis $A_L \subset \mathbf{P}$ meets T_{ij} properly for every $i \in I$ and $j \geqq 0$ is dense.

For an irreducible component S_i of the singular support $SS\mathcal{K}$, let $\Delta_i \subset \mathbf{P}^\vee$ be the image of $\mathbf{P}(\tilde{S}_i)$ by $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$. There exists a closed subscheme $\Delta'_i \subset \Delta_i$ such that $\mathbf{P}(\tilde{S}_i) \to \Delta_i$ is finite outside Δ'_i and that Δ'_i is of codimension $\geqq 2$ in $\mathbf{P}^\vee$ since $\mathbf{P}(\tilde{S}_i) \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ is of codimension d and $X \times_{\mathbf{P}} T^*\mathbf{P} \to \mathbf{P}^\vee$ is of relative dimension $d-1$. We consider the diagram

$$\mathbf{P}^\vee \longleftarrow \mathbf{D} \longrightarrow \mathbf{G}. \tag{3.36}$$

as a correspondence. Then, since $\mathbf{D} \to \mathbf{G}$ is a $\mathbf{P}^1$-bundle and since $\Delta'_i \subset \mathbf{P}^\vee$ is of codimension $\geqq 2$, its image Σ_i in $\mathbf{G}$ by the correspondence (3.36) is a closed subset of at least codimension 1.

Let (H, L) be a point of $\mathbf{D} \subset \mathbf{P}^\vee \times \mathbf{G}$ such that $H \in \mathbf{P}^\vee$ is not contained in the image of the union $\mathbf{P}(\widetilde{S}) \cup \mathbf{R}(S) \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ and that $L \in \mathbf{G}$ is contained in V above but not contained in the image of the union $\mathbf{Q}(\widetilde{S}) \cup \mathbf{T}(S) \subset X \times_{\mathbf{P}} \mathbf{A}$ or $\bigcup_i \Sigma_i$. Since H is not in the image of the union of $\mathbf{P}(\widetilde{S})$ and $\mathbf{R}(S)$, the intersection $Y = X \cap H$ is a smooth divisor and the immersion $i\colon Y \to X$ is non-characteristic with respect to S by Lemma 3.9.1.

Further by Lemma 2.9 and $L \in V$, for every $S_i \subset T^*X$ and for every irreducible component P of $i^{-1}S_i \subset T^*Y$, the intersection $P \cap T^*_Y Y$ is an irreducible component of $Y \cap T_i$ for $T_i = S_i \cap T^*_X X$. Since L is not in the image of the union of $\mathbf{Q}(\widetilde{S})$ and $\mathbf{T}(S)$, the axis A_L meets X transversely, $X \cap A_L$ is a smooth divisor of $X \cap H$ and the immersion $i'\colon X \cap A_L \to X \cap H$ is non-characteristic with respect to $i^!S$ by Lemma 3.12.1. Since L does not meet Δ'_i, the morphism $p_L\colon X_L \to L$ has at most isolated non-characteristic points. □

4 Characteristic cycle

4.1 Characteristic cycle and the Milnor formula

We state and prove the existence of characteristic cycle satisfying the Milnor formula.

Theorem 4.1 (cf. [6, Principe p. 7])**.** *Let X be a smooth scheme of dimension d over a perfect field k of characteristic $p > 0$ and $\mathcal{K}$ be a constructible complex of Λ-modules on X. Assume that there exists a singular support $S = SS\mathcal{K} \subset T^*X$ satisfying the condition* (SS1)*. Then, there exists a unique cycle* Char $\mathcal{K}$ *of dimension d with $\mathbf{Z}[\frac{1}{p}]$-coefficient of T^*X supported on $SS\mathcal{K}$ satisfying the following condition:*

For every flat morphism $f\colon V \to C$ on an étale neighborhood V of a closed point $u \in X$ to a smooth curve C such that u is an isolated characteristic point *of f with respect to the pull-back of $\mathcal{K}$, we have*

$$-\dim \operatorname{tot}\phi_u(\mathcal{K}, f) = (\operatorname{Char} \mathcal{K}, df)_u. \tag{4.1}$$

The Milnor formula [4] and (4.1) imply that the coefficient of the 0-section $T^*_X X$ in the characteristic cycle Char $\mathcal{K}$ equals $(-1)^d$-times the rank of $\mathcal{K}$ on a dense open subscheme of X. Since a singular support exists for a surface by Corollary 2.27, the characteristic cycle is defined for a constructible sheaf on a surface and satisfies the Milnor formula (4.1) (cf. [18, (3.29)]).

In Section 2.4, we have defined the characteristic cycle Char $j_!\mathcal{F}$ for a locally constant constructible sheaf $\mathcal{F}$ on the complement $U = X - D$ under the assumption that the ramification of $\mathcal{F}$ along D is strictly non-degenerate. We will show in Theorem 4.11 that it is the same as that characterized in Theorem 4.1. In order to distinguish the two definitions, before the proof of Theorem 4.11, we let the characteristic cycle defined in Section 2.4 denoted by $\mathrm{Char}^\circ\mathcal{K}$. In the case where $\mathcal{F}$ is tamely ramified along D, Theorem 4.11 that means that the characteristic cycle defined in (2.7) satisfies (4.1) is shown by [19].

The proof of Theorem 4.1 will occupy the rest of this subsection. First, we define a characteristic cycle $\mathrm{Char}_E\mathcal{K}$ by choosing an embedding $X \to \mathbf{P} = \mathbf{P}(E)$ as in the last section, that may a priori depend on the choice.

Assume that X is quasi-projective over k and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Take a subspace $E \subset \Gamma(X, \mathcal{L})$ of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee)$. Replacing E by $E^{(n)}$ for $n \geqq 3$ if necessary as in Lemma 3.15.1, we assume that the condition (E) at the beginning of subsection 3.5 is satisfied.

To define the characteristic cycle $\mathrm{Char}_E\mathcal{K}$, we define the coefficients of irreducible components S_i of a singular support $SS\mathcal{K} = \bigcup_i S_i$. Let S_i be an irreducible component of $S = SS\mathcal{K}$ and $\mathbf{P}(\widetilde{S}_i) \subset \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H}$ be the projectivization of the inverse image of S_i by the surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ as in (3.2). Since E is assumed to satisfy the condition (E), the restriction of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ to $\mathbf{P}(\widetilde{S}_i)$ is generically finite by Corollary 3.17.1.

Since $\mathbf{P}(\widetilde{S}_i) \subset X \times_{\mathbf{P}} \mathbf{H}$ is of codimension d and $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is of relative dimension $d-1$, the closure $\Delta_i \subset \mathbf{P}^\vee$ of the image of $\mathbf{P}(\widetilde{S}_i)$ with the reduced scheme structure is an irreducible divisor. The local ring of $\mathbf{P}^\vee$ at the generic point η_i of Δ_i is a discrete valuation ring and the residue field of the generic point ξ_i of $\mathbf{P}(\widetilde{S}_i)$ is a finite extension of that of η_i. By Lemma 1.3, there exists a dense open subscheme $\mathbf{P}(\tilde{S}_i)^\circ$ of $\mathbf{P}(\tilde{S}_i) \subset X \times_{\mathbf{P}} \mathbf{H}$ quasi-finite over $\mathbf{P}^\vee$ such that the function $\varphi_{\mathbf{P}(\tilde{S}_i)^\circ}$ is a constant function. Its value is the inseparable degree $[\xi_i : \eta_i]_{\mathrm{insep}}$.

We consider the diagram

$$(4.2) \qquad \begin{array}{ccc} (X \times \mathbf{G})^\triangledown & \xrightarrow{\ f\ } & \mathbf{D}\,. \\ & {\scriptstyle p^\triangledown}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathbf{G} & \end{array}$$

Let $\varphi_{\mathcal{K}}$ be the function on $\mathbf{Z}(S)$ (Lemma 3.10)

$$(4.3) \qquad \varphi_{\mathcal{K}}(z) = \dim \mathrm{tot}\phi_u(\mathcal{K}, p_L^\circ)$$

defined in Lemma 3.10.2. By Lemma 3.10.2, the function $\varphi_{\mathcal{K}}$ is flat over $\mathbf{G}$. Recall that we have a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{Z}(S) & \longrightarrow & (X \times \mathbf{G})^{\triangledown} & \xrightarrow{f} & \mathbf{D} & \longrightarrow & \mathbf{G} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\mathbf{P}(\widetilde{S}) & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} & \longrightarrow & \mathbf{P}^{\vee} & &
\end{array}$$

where the left square is cartesian.

Since E is assumed to satisfy the condition (E) at the beginning of subsection 3.5, by Corollary 3.17.1, there exists a dense open subscheme Z_i° in the inverse image of $\mathbf{P}(\tilde{S}_i) \subset X \times_{\mathbf{P}} \mathbf{H}$ by $(X \times \mathbf{G})^{\triangledown} \to X \times_{\mathbf{P}} \mathbf{H}$ such that the function $\varphi_{\mathcal{K}}$ is constant. Define an integer a_i by

$$a_i = -\varphi_{\mathcal{K}}(z) \tag{4.4}$$

for $z \in Z_i^{\circ}$. Shrinking Z_i° if necessary, we assume that it is contained in the inverse image of $\mathbf{P}(\tilde{S}_i)^{\circ} \subset X \times_{\mathbf{P}} \mathbf{H}$ and that the intersection $Z_i^{\circ} \cap Z_{i'}$ is empty for $i' \neq i$. Then, on the dense open subset $\coprod_i Z_i^{\circ} \subset \mathbf{Z}(S)$, we have

$$-\varphi_{\mathcal{K}} = \sum_i \frac{a_i}{[\xi_i : \eta_i]_{\text{insep}}} \varphi_{\mathbf{P}(\tilde{S}_i)^{\circ}} \tag{4.5}$$

where by abuse of notation, we also write $\varphi_{\mathbf{P}(\tilde{S}_i)^{\circ}}$ for its pull-back to $\mathbf{Z}(S)$.

We define

$$\mathrm{Char}_E \mathcal{K} = \sum_i \frac{a_i}{[\xi_i : \eta_i]_{\text{insep}}} [S_i]. \tag{4.6}$$

Since the inseparable degree $[\xi_i : \eta_i]_{\text{insep}}$ is a power of p, the coefficients in $\mathrm{Char}_E \mathcal{K}$ are in $\mathbf{Z}[\frac{1}{p}]$. We will later prove that the coefficients are independent of E and $\mathcal{L}$ as a consequence of the Milnor formula (4.1). If $\mathcal{K}$ is a perverse sheaf, we will also prove $a_i \geqq 0$ in Corollary 4.5. The following interpretation in terms of the ramification theory is not used in the sequel.

Lemma 4.2.

$$a_i = -\dim \mathrm{tot}_{\eta_i} \phi_{\xi_i}(q^* \mathcal{K}, p)$$

Proof. It follows from Lemma 2.28.1 and $\mathbf{D} = \mathbf{P}(T^* \mathbf{P}^{\vee})$. □

We show that the characteristic cycle $\mathrm{Char}_E \mathcal{K}$ satisfies the Milnor formula (4.1) for a morphism defined by a pencil. Let $L \subset \mathbf{P}^{\vee}$ be a line and $A_L \subset \mathbf{P}$ be the intersection of hyperplanes contained in L. On the complement $X_L^{\circ} = X - (A_L \cap X)$, a morphism $p_L^{\circ} \colon X_L^{\circ} \to L$ is defined by sending a point to the unique hyperplane in L containing it.

Proposition 4.3. *Assume that $S = SS\mathcal{K} = \bigcup_i S_i \subset T^*X$ satisfies* (SS1) *for $\mathcal{K}$ on X. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and $E \subset \Gamma(X, \mathcal{L})$ be a k-vector space of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^{\vee})$ satisfying the condition* (E) *at the beginning of subsection* 3.5.

Let $L \subset \mathbf{P}^{\vee}$ be a line and $p_L^{\circ} \colon X_L^{\circ} \to L$ be the morphism defined by the pencil L. Let u be a closed point of X_L° such that u is an isolated characteristic point of $p_L^{\circ} \colon X_L^{\circ} \to L$ with respect to $\mathcal{K}$. Then, we have

$$-\dim \mathrm{tot} \phi_u(\mathcal{K}, p_L^{\circ}) = (\mathrm{Char}_E \mathcal{K}, dp_L^{\circ})_{T^*X, u}. \tag{4.7}$$

Proof. Let $L \subset \mathbf{P}^\vee$ be a line and $u \in X_L^\circ$ be a closed point. Then the image $z \in (X \times \mathbf{G})^\circ$ of u by the inclusion $X_L^\circ \to (X \times \mathbf{G})^\circ$ lies in $(X \times \mathbf{G})^\triangledown$ if and only if u satisfies the assumption in Proposition 4.3, by Lemma 3.9.

We define another flat function φ_A on $\mathbf{Z}(S)$ and compare it with $\varphi_{\mathcal{K}}$ defined at (4.3). Let A denote $\mathrm{Char}_E \mathcal{K}$ and define a cycle $\mathbf{P}(\widetilde{A})$ of $\mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H}$ supported on $\mathbf{P}(\widetilde{S})$ as in Lemma 3.3. Since $\mathbf{G}$ is regular, a coherent $\mathcal{O}_{(X\times\mathbf{G})^\circ}$-module is of finite tor-dimension as an $\mathcal{O}_{\mathbf{G}}$-module. Hence the pull-back of $\mathbf{P}(\widetilde{A})$ by $(X \times \mathbf{G})^\circ \to X \times_{\mathbf{P}} \mathbf{H}$ defines a function φ_A on $\mathbf{Z}(S)$ flat over $\mathbf{G}$ by Lemma 1.3. For a point $z \in \mathbf{Z}(S)$ corresponding to (u, L) such that $u \in X_L^\circ$ is an isolated characteristic point of $p_L^\circ \colon X_L^\circ \to L$, we have

$$\varphi_A(z) = (\mathrm{Char}_E \mathcal{K}, dp_L^\circ)_{T^*X_L^\circ, u} \tag{4.8}$$

by Lemma 3.3.

Since E is assumed to satisfy the condition (E) at the beginning of subsection 3.5, on a dense open subset $\coprod_i Z_i^\circ \subset \mathbf{Z}(S)$, we have an equality

$$-\varphi_{\mathcal{K}} = \varphi_A \tag{4.9}$$

by (4.5). Since $\varphi_{\mathcal{K}}$ and φ_A are functions on $\mathbf{Z}(S)$ flat over $\mathbf{G}$ the equality (4.9) holds everywhere on $\mathbf{Z}(S)$ by Lemma 1.2.2. Thus, by (4.3) and (4.8), the equality (4.7) is proved. □

Combining Proposition 4.3 with Proposition 2.32 and Lemma 3.18, we prove (4.1) with E replaced by $E^{(n)}$ for sufficiently large n.

Corollary 4.4. *Assume that there exists a singular support satisfying* (SS1) *for $\mathcal{K}$ on X. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module and $E \subset \Gamma(X, \mathcal{L})$ be a k-vector space of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee)$.*

Let $f \colon V \to C$ be a smooth morphism to a smooth curve over k defined on an étale neighborhood of a closed point u. Assume that u is an isolated characteristic point of f with respect to the singular support $S = SS\mathcal{K}$.

Then, there exists an integer $m \geqq 1$ such that for every $n \geqq m$, we have

$$-\dim \mathrm{tot}\phi_u(\mathcal{K}, f) = (\mathrm{Char}_{E^{(n)}} \mathcal{K}, df)_{T^*V, u}. \tag{4.10}$$

Proof. Let $N \geqq 2$ be an integer as in Proposition 2.32 for f and u. By Lemma 3.15.2 and Lemma 3.18, for $n \geqq m = \max(N, 3)$, the subspace $E^{(n)} \subset \Gamma(X, \mathcal{L}^{\otimes n})$ satisfies the condition (E) and there exists $l_0, l_\infty \in E^{(n)}$ such that the morphism $p_L \colon X_L^\circ \to L$ for the line L spanned by l_0, l_∞ satisfies $p_L^\circ \equiv f \bmod \mathfrak{m}_u^N$. Then, we have (4.7) with E replaced by $E^{(n)}$ by Proposition 4.3 and the both sides of (4.7) are equal to the corresponding sides of (4.10) by Proposition 2.32. Thus the assertion follows. □

Proof of Theorem 4.1. First, we prove the case where X is quasi-projective. We show that the characteristic cycle $\mathrm{Char}_E \mathcal{K}$ is independent of the choice of an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and $E \subset \Gamma(X, \mathcal{L})$ defining an immersion $X \to \mathbf{P}(E^\vee)$ and satisfying the condition (E) at the beginning of subsection 3.5.

Let S_i be an irreducible component of a singular support $SS\mathcal{K}$. Since E is assumed to satisfy the condition (E), we may take a point $u \in Z_i^\circ$ in the notation in the proof of Proposition 4.3 and a pencil $p_L^\circ \colon X_L^\circ \to L$ such that $u \in X_L^\circ \cap Z_i^\circ \subset X \times_{\mathbf{P}} T^*\mathbf{P}$. Then since $(S_{i'}, dp_L)_{T^*X^\circ_L,u} = 0$ for $i' \neq i$, the equality (4.7) implies that the coefficient of S_i in $\mathrm{Char}_E\mathcal{K}$ is the unique rational number that makes the equality (4.1) holds for $f = p^\circ_L$. Consequently, we have a uniqueness of the characteristic cycle.

Let $\mathcal{L}'$ be another ample invertible $\mathcal{O}_X$-module and $E' \subset \Gamma(X, \mathcal{L}')$ defining an immersion $X \to \mathbf{P}(E'^\vee)$ and let $N \geqq 2$ be an integer as in Proposition 2.32 for p°_L and u. Then, as in the proof of Corollary 4.4, there exists an integer $m \geqq 0$ such that for every $n \geqq m$, there exists a line $L' \subset \mathbf{P}(E'^{(n)})$ such that $p_L \equiv p_{L'} \bmod \mathfrak{m}^N_u$. We also have $\dim \mathrm{tot}_u\phi(\mathcal{K}, p_L) = \dim \mathrm{tot}_u\phi(\mathcal{K}, p_{L'})$, $(S_i, dp_L)_{T^*X^\circ_L,u} = (S_i, dp_{L'})_{T^*X^\circ_{L'},u}$ and $(S_{i'}, dp_L)_{T^*X^\circ_L,u} = (S_{i'}, dp_{L'})_{T^*X^\circ_{L'},u} = 0$ for $i' \neq i$. Hence, the coefficient of S_i in $\mathrm{Char}_E\mathcal{K}$ is equal to that in $\mathrm{Char}_{E'(n)}\mathcal{K}$ for every $n \geqq m$. Thus, $\mathrm{Char}_E\mathcal{K}$ is independent of E.

Define the characteristic cycle $\mathrm{Char}\,\mathcal{K}$ to be the common cycle. Then, it satisfies (4.1) by Corollary 4.4. Since the uniqueness holds, we obtain the general case by patching. □

Since the characterization (4.1) is an étale local condition, the formation of the characteristic cycle $\mathrm{Char}_E\mathcal{K}$ commutes with étale base change.

Corollary 4.5 ([6, Question p. 7]). *If $\mathcal{K}$ is a perverse sheaf on U, then the coefficients of* $\mathrm{Char}\,\mathcal{K}$ *satisfy* $\geqq 0$.

Proof. Let u be an isolated characteristic point of a morphism $f\colon X \to C$ to a curve. Then, since $\phi(\mathcal{K}, f)[-1]$ is a perverse sheaf by [9, Corollaire 4.6] and u is an isolated point of its support, it is acyclic except at degree 0. Hence $a_i = -\dim \mathrm{tot}\phi_u(\mathcal{K}, f) \geqq 0$. □

We show a generalization of the Milnor formula. Let $f\colon X \to Y$ be a flat morphism of smooth schemes over k. Let $S = \bigcup_i S_i$ be the union of a finite family of irreducible closed conic subsets of codimension d of T^*X. We assume that S satisfies the following condition (Q′) that is a modification of the condition (Q) before Lemma 2.16 modified as in Lemma 2.25:

(Q′) For every S_i and for every irreducible component P of the inverse image of S_i by the canonical morphism $X \times_Y T^*Y \to T^*X$, the intersection $Q = P \cap (X \times_Y T^*_Y Y)$ with the 0-section satisfies the following condition:

(Q1) If $\dim Q \geqq \dim Y$, the fibers of $f\colon X \to Y$ meet Q properly and we have $P = Q$.

(Q2′) If $\dim Q < \dim Y$, we have $\dim P = \dim Y$ and the restriction $Q \to Y$ of f is finite.

Define a closed subset $Z \subset X$ to be the union of Q appearing in (Q2′). It is finite over Y. A finite family $f_!S$ of closed conic subsets of codimension $\dim Y$ of T^*Y is defined as in Lemma 2.16. If S is a singular support of a constructible complex $\mathcal{K}$ of Λ-modules on X, the cycle $f_!\mathrm{Char}\,\mathcal{K}$ is also defined as their linear combination.

Lemma 4.6. *Let $f\colon X \to Y$ be a flat morphism of smooth schemes over k and $\mathcal{K}$ be a constructible complex of Λ-modules on X. Let S be a singular support of $\mathcal{K}$ satisfying the condition* (Q′) *above and define also $Z \subset X$ as above. Let x be a geometric point of X and $y = f(x)$ be the image.*

Assume that x is the unique point in the fiber Z_y. Then, for every smooth *morphism $g\colon Y \to C$ to a smooth curve such that y is an isolated characteristic point with respect to $f_!S$ and $v = g(y)$, the cycles $f_!\mathrm{Char}\,\mathcal{K}$ satisfies the Milnor formula,*

$$-\dim \mathrm{tot}_v\phi_y(R\Psi_f\mathcal{K}|_{x \overleftarrow{\times}_Y Y}, g) = (f_!\mathrm{Char}\mathcal{K}, dg)_y \tag{4.11}$$

Proof. Let $p\colon X \to C$ denote the composition $g \circ f\colon X \to C$. Then, by Lemma 2.16.2, the condition (Q′) implies that x is an isolated characteristic point of p with respect to S. Hence by the Milnor formula (4.1), we have

$$-\dim \mathrm{tot}_v \phi_x(\mathcal{K}, p) = (\mathrm{Char}\mathcal{K}, dp)_x \tag{4.12}$$

For the left hand side of (4.11), we have a canonical isomorphism $R\overleftarrow{g}_{(y)*}(R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y}) \to R\Psi_p\mathcal{K}|_{x\overleftarrow{\times}_C C}$ by (1.9). This implies the equality

$$\begin{aligned}\dim \mathrm{tot}_v \phi_y(R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y}, g) &= a_v(R\overleftarrow{g}_{(y)*}(R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y})) \\ &= a_v(R\Psi_p\mathcal{K}|_{x\overleftarrow{\times}_C C}) = \dim \mathrm{tot}_v \phi_x(\mathcal{K}, p)\end{aligned} \tag{4.13}$$

where a_v denotes the Artin conductor (1.15).

On the right hand side of (4.11), we have an equality $(f_!\mathrm{Char}\mathcal{K}, dg)_y = (\mathrm{Char}\mathcal{K}, dp)_x$ by the assumption that x is the unique point in the fiber Z_y. Hence the Milnor formula (4.12) implies the equality (4.11). □

Lemma 4.6 can be reformulated by introducing the characteristic cycle $\mathrm{Char}R\Psi_f\mathcal{K}$ as follows. Assume that $R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y}$ on $Y_{(y)}$ is constructible. Then, we may regard it as the pull-back of a constructible sheaf on an étale neighborhood of $y = f(x)$ and define the characteristic cycle $\mathrm{Char}R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y}$ as the pull-back. Since the construction of the characteristic cycle is compatible with the pull-back by étale morphism, it is well-defined. For example, if Y is a curve, we have

$$\mathrm{Char}\ R\Psi_f\mathcal{K}|_{u\overleftarrow{\times}_Y Y} = -\big(\mathrm{rank}\ \psi_u(\mathcal{K}, f)[T^*_{Y_{(y)}}Y_{(y)}] + \dim \mathrm{tot}_y \phi_u(\mathcal{K}, f)[T^*_y Y_{(y)}]\big). \tag{4.14}$$

Proposition 4.7. *Let the notation and the assumptions be as in Lemma* 4.6*. Assume further that S satisfies* (SSr) *for $r = \max(\dim Y, 1)$. Then, $R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y}$ is constructible and we have*

$$\mathrm{Char}R\Psi_f\mathcal{K}|_{x\overleftarrow{\times}_Y Y} \equiv (f_!\mathrm{Char}\mathcal{K})|_{x\overleftarrow{\times}_Y Y} \qquad \mathrm{mod}\ \langle T^*_{Y_{(y)}}Y_{(y)}\rangle \tag{4.15}$$

where the congruence means that the both cycles have the same coefficients except that of the 0*-section.*

Proof. By the condition (Q′), the graph $X \to X \times Y$ of f is a regular immersion of codimension $\dim Y \leqq r$ and non-characteristic with respect to S on the complement of Z. Since S is assumed to satisfy (SSr) and Z is finite over Y, the complex $R\Psi_f\mathcal{K}$ is constructible and the formation of $R\Psi_f\mathcal{K}$ commutes with base change, by Proposition 1.7.1. Since S is assumed to satisfy (SS1) for $\mathcal{K}$, the family $f_!S$ also satisfies the condition (SS1) for $R\Psi_f\mathcal{K}$, by Lemma 2.25. Hence it suffices to compare the coefficients except that for the 0-section. Thus, it follows from Lemma 4.6 and Proposition 3.16.1. □

4.2 Restriction to a divisor and the Euler-Poincaré characteristic

We prove that the construction of the characteristic cycles is compatible with the pull-back by non-characteristic immersion of a smooth divisor and derive an index formula for the Euler number.

Let $Y \subset X$ be a smooth irreducible divisor. Recall from Definition 2.11.1 that the closed immersion $i\colon Y \to X$ is non-characteristic with respect to S if it satisfies the following conditions (i) and (ii):

(i) The intersection $S \cap T^*_Y X$ with the conormal bundle is a subset of the 0-section.

(ii) The immersion $Y \to X$ meets every $T_i = S_i \cap T^*_X X$ properly.

We define $i^!\mathrm{Char}\ \mathcal{K}$ to be (-1)-*times* the image of $\mathrm{Char}\ \mathcal{K}$ by the correspondence

$$T^*X \longleftarrow Y \times_X T^*X \longrightarrow T^*Y$$

as in Definition 2.20.

Theorem 4.8. *Let $\mathcal{K}$ be a constructible complex of Λ-modules on X and let $S = SS\mathcal{K}$ be a singular support of $\mathcal{K}$ satisfying* (SS2). *Let Y be a smooth divisor such that the immersion $i\colon Y \to X$ is non-characteristic. Then, we have*

$$\mathrm{Char}\ i^*\mathcal{K} = i^!\mathrm{Char}\ \mathcal{K}. \tag{4.16}$$

In the proof, we will use the fact that Theorem 4.11 is proved in dimension 2 in [18, Proposition 3.20]. We will reduce the general case to this case by using the second Radon transform.

Proof. By Lemma 2.21, $i^!S$ is a singular support for $i^*\mathcal{K}$. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $i^!\mathrm{Char}\ \mathcal{K}$ satisfies the Milnor formula. Hence the question is local and we may assume that X is affine. Further we may assume that there exists an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$, a subspace $E \subset \Gamma(X, \mathcal{L})$ of finite dimension defining an immersion $X \to \mathbf{P} = \mathbf{P}(E^\vee)$ satisfying the condition (E) studied in Lemma 3.15 and a hyperplane $H \subset \mathbf{P}$ such that $Y = X \cap H$. First, we prove a Milnor formula for an isolated characteristic point of a morphism defined by a pencil.

Lemma 4.9. *Let $u \in Y$ be a closed point and $p_L\colon Y^\circ_L \to L$ be a morphism defined by a pencil $L \subset H^\vee$ such that u is an isolated characteristic point of $g = p_L\colon Y^\circ_L \to L$. Then, we have*

$$-\dim \mathrm{tot}\phi_u(i^*\mathcal{K}, g) = (i^!\mathrm{Char}\ \mathcal{K}, dg)_u. \tag{4.17}$$

To prove (4.17), first we prove it on a dense open subset in the universal family by applying Proposition 4.7. Then, we deduce the general case by using Lemma 1.2, similarly as in the proof of Proposition 4.3.

Proof. We prove the equality (4.17) by using the universal family. We consider the commutative diagram

$$\begin{array}{ccccccc} X \times_{\mathbf{P}} \mathbf{H} & \longleftarrow & Y_{\mathbf{B}} & \longleftarrow & Y'_{\mathbf{B}} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^\vee & \longleftarrow & \mathbf{B} & \longleftarrow & \mathbf{C} & \longrightarrow & \mathbf{G} \end{array} \tag{3.16}$$

of the universal family of morphisms defined by pencils for hyperplane sections. We use the notation $Y^\triangledown_{\mathbf{B}} \subset Y^\circ_{\mathbf{B}}$ etc. as in Lemma 3.14. We consider the universal family

$$\begin{array}{ccccc} \mathbf{W}(S) & \xrightarrow{\subset} & Y^\triangledown_{\mathbf{B}} & \xrightarrow{\tilde{g}} & \mathbf{C} \\ & & & \searrow^{p^\triangledown} \quad \swarrow & \\ & & & \mathbf{B} & \end{array}$$

of $p_L\colon Y_L^\circ \to L$. Recall that we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{W}(S) & \longrightarrow & Y_{\mathbf{B}}^{\triangledown} & \xrightarrow{\tilde g} & \mathbf{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Q}(\widetilde{S})^{\square} & \longrightarrow & (X \times_{\mathbf{P}} \mathbf{A})^{\square} & \longrightarrow & \mathbf{G} \end{array}$$

where the left square is cartesian and the composition $\mathbf{Q}(\widetilde{S})^{\square} \to \mathbf{G}$ is quasi-finite.

An isolated characteristic point u of $p_L^\circ\colon Y_L^\circ \to L$ corresponds to a point (u, H, L) of $\mathbf{W}(S) \subset Y_{\mathbf{B}}^{\triangledown}$ such that $Y = X \cap H$, by Lemma 3.13. We regard the equality (4.17) as an equality of functions defined on $\mathbf{W}(S)$. In order to apply Theorem 4.11 for surfaces proved in [18, Proposition 3.20], we construct the diagram (4.18) below.

Let (H, L) be the pair of a hyperplane plane $H \subset \mathbf{P} = \mathbf{P}(E^\vee)$ and a line $L \subset H^\vee$. Let $L_H \subset E$ be the line corresponding to H and let $W = W_L$ be the subspace of E of dimension 3 containing L_H corresponding to the line $L \subset H^\vee = \mathbf{P}(E/L_H)$. Let $P = \mathbf{P}(W^\vee) = \mathrm{Gr}(2, W) \subset \mathbf{G} = \mathrm{Gr}(2, E)$ denote the projective plane parametrizing planes in $W = W_L \subset E$. The line L is canonically identified with the line in P consisting of planes in W containing $L' = L_H$.

We define $f\colon X_P \to P$ by the cartesian diagram

$$\begin{array}{ccccc} Y_L & \xrightarrow{i} & X_P & \xrightarrow{j} & X \times_{\mathbf{P}} \mathbf{A} \\ {\scriptstyle g=p_L}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{h} & P & \longrightarrow & \mathbf{G} \end{array} \tag{4.18}$$

extending (3.15). The composition $X_P \to X \times_{\mathbf{P}} \mathbf{A} \to X$ with the projection is an isomorphism on the complement $X_P^\circ = X - (X \cap A_P)$ of the intersection with the intersection A_P of linear subspaces of codimension 2 of $\mathbf{P}^\vee$ parametrized by P. The restriction of f to X_P° maps a point x to the unique linear subspaces of codimension 2 parametrized by P and containing x. We have $Y_L^\circ = Y_L \cap X_P^\circ$. The composition of the lower line is the restriction to L of the canonical morphism $\mathbf{C} \to \mathbf{G}$ where L is identified with the fiber of the morphism $\mathbf{C} \to \mathbf{B}$ at (H, L).

Let $v = g(u) \in L$ be the image of u and t be the local coordinate of L at v. The restriction of the second Radon transform $R_E^{(2)}\mathcal{K} = R\Psi_p q^*\mathcal{K}$ on $(X \times_{\mathbf{P}} \mathbf{A})^{\square} \overleftarrow{\times}_{\mathbf{G}} \mathbf{G}$ is constructible and its formation commutes with base change by Lemma 3.12. Since the image of the point $(u, H, L) \in \mathbf{W}(S)$ is contained in $\mathbf{Q}(\widetilde{S})^{\square} \subset (X \times_{\mathbf{P}} \mathbf{A})^{\square}$, the complex $R\Psi_f\mathcal{K}$ is constructible on a neighborhood of $u \leftarrow v$ in $X_P \overleftarrow{\times}_P P$ and the characteristic cycle $\mathrm{Char}\ R\Psi_f\mathcal{K}|_{u \overleftarrow{\times}_P P}$ is defined. We show that the second equality in

$$-\dim \mathrm{tot}\phi_u(i^*\mathcal{K}, g) = (h^!\mathrm{Char}\ R\Psi_f\mathcal{K}|_{u \overleftarrow{\times}_P P}, dt)_v = (i^!\mathrm{Char}\ \mathcal{K}, dg)_u. \tag{4.19}$$

always holds and that the first holds on a dense open of $\mathbf{W}(S)$.

We show the second equality by applying Proposition 4.7 to the morphism $f'\colon X'_{P'} \to P'$ in the diagram (4.20) below. Since $\mathbf{Q}(\widetilde{S})^{\square} \to \mathbf{G}$ is quasi-finite, by taking an étale neighborhood $V \to \mathbf{G}$ of v and an open neighborhood $U \subset (X \times_{\mathbf{P}} \mathbf{A})^{\square} \times_{\mathbf{G}} V$ of u, we may assume that $U \times_{(X \times_{\mathbf{P}} \mathbf{A})^{\square}} \mathbf{Q}(\widetilde{S}) \to V$ is finite and that u is the unique point of the inverse image of v. Define

$$(4.20)\qquad \begin{array}{ccc} Y'_L & \xrightarrow{i'} & X'_{P'} = X_P \times_{X\times_{\mathbf P}\mathbf A} U \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ L' & \xrightarrow{h'} & P' \qquad = P \times_{\mathbf G} V \end{array}$$

from the left square of (4.18) by taking the base change by $V \to \mathbf{G}$ for the lower line and further taking the base change by $U \subset (X \times_{\mathbf P} \mathbf{A}) \times_{\mathbf G} V$ for the upper line.

For the right hand side of (4.19), we have $(i^! \mathrm{Char}\ \mathcal{K}, dg)_u = (g'_! i'^! \mathrm{Char}\ \mathcal{K}, dt)_v$ since u is the unique point of the inverse image. By (4.20), we have

$$(4.21)\qquad g'_! i'^! \mathrm{Char}\ \mathcal{K} = h'^! f'_! \mathrm{Char}\ \mathcal{K}.$$

Since $Y^{\triangledown}_{\mathbf B}$ does not meet $\mathbf{T}_{\geqq 2}(S)$, for T_i such that $\dim T_i \geqq 2$, the morphism $f' \colon X'_{P'} \to P'$ satisfies the condition (Q1) in (Q$'$) before Lemma 4.6 on a neighborhood of u. Since $U \times_{Y^{\triangledown}_{\mathbf B}} \mathbf{W}(S) \to V$ is finite and u is the unique point of the inverse image of v, it also satisfies the condition (Q2$'$) in (Q$'$). Hence by Proposition 4.7, we have

$$(4.22)\qquad (f'_! \mathrm{Char}\ \mathcal{K})|_{u \overset{\leftarrow}{\times}_{P'} P'} \equiv \mathrm{Char}\ R\Psi_f \mathcal{K}|_{u \overset{\leftarrow}{\times}_P P} \mod \langle T^*_{P_{(v)}} P_{(v)} \rangle.$$

Since $h^![T^*_{P_{(v)}} P_{(v)}] = -T^*_{L_{(v)}} L_{(v)}$ and the section dt does not meet the 0-section, the equality (4.21) and the congruence (4.22) imply the second equality in (4.19).

We show that the first equality in (4.19) holds on a dense open subset of $\mathbf{W}(S)$ by applying Lemma 2.31 to the cartesian diagram obtained by taking the base change of the quasi-finite morphism $\mathbf{Q}(S)^{\square} \to \mathbf{G}$ by the lower line $L \to P \to \mathbf{G}$ of (4.18). Recall that the complex $R\Psi_{\bar f} \mathcal{K}$ restricted on $(X \times_{\mathbf P} \mathbf{A})^{\square} \overset{\leftarrow}{\times}_{\mathbf G} \mathbf{G}$ is constructible and its formation commutes with base change by Lemma 3.12. By Corollary 3.17.2, the closure D of the image of $\mathbf{Q}(S)^{\square} \to \mathbf{G}$ is a Cartier divisor and each irreducible component of $\mathbf{Q}(S)^{\square}$ dominates its irreducible component. Since the non-characteristicity for immersions of smooth curves is an open condition on the tangent vectors, Lemma 3.7 implies that the assumption in Lemma 2.31 that the composition $L \to \mathbf{G}$ is non-characteristic with respect to $R\Psi_{\bar f} \mathcal{K}$ on a neighborhood of $u \leftarrow v$ is satisfied on a dense open subset of $\mathbf{W}(S)$. Therefore, on the same dense open subset, the ramification of the pull-back of $R\Psi_{\bar f} \mathcal{K}$ along the pull-back of D is non-degenerate and we have an equality

$$(4.23)\qquad -\dim \mathrm{tot}_u \phi(i^* \mathcal{K}, p_L) = (h^! \mathrm{Char}^{\circ} R\Psi_f \mathcal{K}|_{u \overset{\leftarrow}{\times}_P P}, dt)_v.$$

We have $\mathrm{Char} R\Psi_f \mathcal{K}|_{u \overset{\leftarrow}{\times}_P P} = \mathrm{Char}^{\circ} R\Psi_f \mathcal{K}|_{u \overset{\leftarrow}{\times}_P P}$ on the same dense open subset, since Theorem 4.11 is proved in dimension 2 in [18, Proposition 3.20]. Thus, we obtain the first equality in (4.19) and hence (4.17) on a dense open subset of $\mathbf{W}(S)$.

In order to complete the proof of the equality (4.17), we introduce two functions on $\mathbf{W}(S)$ flat over $\mathbf{B}$. Let $\varphi_{i^*\mathcal{K}}$ be the function defined in Lemma 3.14.3 by

$$(4.24)\qquad \varphi_{i^*\mathcal{K}}(z) = \dim \mathrm{tot} \phi_u(\mathcal{K}|_{Y^{\circ}_L}, p_L)$$

for the point $z \in \mathbf{W}(S)$ corresponding to the triple (H, L, u) of a hyperplane H, a line $L \subset H^{\vee}$ and a point $u \in Y^{\circ}_L$. By Lemma 3.14.3, the function $\varphi_{i^*\mathcal{K}}$ is flat over $\mathbf{B}$.

We define another flat function $\varphi_{i^! A}$ on $\mathbf{W}(S)$. Let A denote Char $\mathcal{K}$ and define a cycle $\mathbf{P}(\widetilde{A})$ of $\mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H}$ supported on $\mathbf{P}(\widetilde{S})$ as in the proof of Proposition 4.3. As in Lemma 3.6, define a cycle $\mathbf{Q}(\widetilde{A})$ supported on $\mathbf{Q}(\widetilde{S})$ by the upper line

$$X \times_{\mathbf{P}} \mathbf{H} \longleftarrow X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} \longrightarrow X \times_{\mathbf{P}} \mathbf{A} \tag{3.13}$$

of (3.10) regarded as a correspondence. Since $\mathbf{B}$ is regular, a coherent $\mathcal{O}_{Y^\circ_{\mathbf{B}}}$-module is of finite tor-dimension as an $\mathcal{O}_{\mathbf{B}}$-module. Hence the pull-back of $\mathbf{Q}(\widetilde{A})$ by the upper right horizontal arrow $Y^\circ_{\mathbf{B}} \to X \times_{\mathbf{P}} \mathbf{A}$ in (3.16) defines a function $\varphi_{i^! A}$ on $\mathbf{W}(S)$ flat over $\mathbf{B}$ by Lemma 1.3.

We show

$$\varphi_{i^! A}(z) = (i^!\text{Char } \mathcal{K}, dg)_u \tag{4.25}$$

for a point $z \in \mathbf{W}(S)$ corresponding to (H, L, u). For the right hand side, we have $(\mathbf{P}(i^! A), Y^\circ_L)_u = (i^!\text{Char } \mathcal{K}, dg)_u$ by Lemma 3.3. The morphism $Y^\circ_{\mathbf{B}} \to X \times_{\mathbf{P}} \mathbf{A}$ pulling-back $\mathbf{Q}(\widetilde{A})$ is the composition of the upper line in the cartesian diagram

$$\begin{array}{ccccc} X \times_{\mathbf{P}} \mathbf{A} & \longleftarrow & X \times_{\mathbf{P}} \mathbf{A} \times_{\mathbf{G}} \mathbf{D} & \longleftarrow & Y'_{\mathbf{B}} \supset Y^\circ_{\mathbf{B}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{G} & \longleftarrow & \mathbf{D} & \longleftarrow & \mathbf{C}. \end{array}$$

Hence we have $\varphi_{i^! A}(z) = (\mathbf{P}(i^! A), Y^\circ_L)_u$ by Lemma 3.6.1 and (4.25) is proved.

By (4.25), the equality (4.17) is equivalent to the equality

$$-\varphi_{i^*\mathcal{K}} = \varphi_{i^! A} \tag{4.26}$$

of functions on $\mathbf{W}(S)$. Since the functions in both sides of (4.26) are flat over $\mathbf{B}$ and the equality has been proved to hold on a dense open, we have a equality of the functions on $\mathbf{W}(S)$ by Lemma 1.2. □

We complete the proof of Theorem 4.8. We show that the equality (4.17) implies (4.16) choosing L appropriately. Let S_i be an irreducible component of the singular support $i^! SS\mathcal{K}$ and let $\Delta_i \subset H^\vee$ be the closure of the image of $\mathbf{P}(\widetilde{S}_i)$ as in the proof of Proposition 4.3. Since $E \subset \Gamma(X, \mathcal{L})$ is assumed to satisfy the condition (E) at the beginning of Section 3.5, the subspace $E' \subset \Gamma(Y, \mathcal{L} \otimes \mathcal{O}_Y)$ also satisfies the condition (E). Hence $\mathbf{P}(\widetilde{S}_i) \to \Delta_i$ is generically finite by Corollary 3.17.1, there exists a dense open subscheme $\Delta^\circ_i \subset \Delta_i$ smooth over k such that $\Delta^\circ_i \cap \Delta_{i'} = \varnothing$ for $i \neq i'$ and $\mathbf{P}(\widetilde{S}_i) \to \Delta_i$ is finite over Δ°_i.

Let $L \subset H^\vee$ be a line meeting Δ°_i at a closed point v. Let u be a point of $\mathbf{P}(\widetilde{S}_i)$ above v and regard it as a point of X°_L. The assumptions that v is an isolated point of $L \cap \Delta^\circ_i$ and that $\mathbf{P}(\widetilde{S}_i) \times_{\mathbf{P}^\vee} \Delta^\circ_i \to \Delta^\circ_i$ is finite imply that u is an isolated point of $X^\circ_L \cap \mathbf{P}(\widetilde{S}_i)$. Since $\mathbf{P}(\widetilde{S}_i) \times_{\mathbf{P}^\vee} \Delta^\circ_i$ does not meet $\mathbf{P}(\widetilde{S}_{i'})$ for $i' \neq i$, the point u is not contained in $\mathbf{P}(\widetilde{S}_{i'})$ for $i' \neq i$ and is an isolated characteristic point of $p^\circ_L \colon X^\circ_L \to L$. Hence the equality (4.17) implies that the coefficients of S_i in Char $i^*\mathcal{K}$ and $i^!$Char $\mathcal{K}$ are equal. Since this holds for every component S_i of $i^! SS\mathcal{K}$, we obtain (4.16). □

Corollary 4.10. *Assume that a singular support $S = SS\mathcal{K}$ satisfies the condition* (SS2). *Let $f \colon Y \to X$ be a smooth morphism. Then we have*

$$\text{Char} f^*\mathcal{K} = f^*\text{Char}\mathcal{K} \tag{4.27}$$

The right hand side denotes the image by the correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$.

Proof. Since $f^*SS\mathcal{K}$ is a singular support of $f^*\mathcal{K}$ by Proposition 2.22, it suffices to show the equalities of the coefficients for each irreducible component. By induction on the relative dimension, we may assume that $Y \to X$ is smooth of relative dimension 1. Further, we may assume that $Y \to X$ admits a section $i\colon X \to Y$. Since $i\colon X \to Y$ is non-characteristic with respect to f^*S, the assertion follows by Theorem 4.8. □

Theorem 4.11. *Let $j\colon U \to X$ be the open immersion of the complement $U = X - D$ of a divisor D with simple normal crossings and $\mathcal{F}$ be a locally constant constructible sheaf on U such that the ramification along D is* strongly non-degenerate. *Then the characteristic cycle* Char $j_!\mathcal{F}$ *equals to* $\mathrm{Char}^\circ j_!\mathcal{F}$ *defined by ramification theory recalled in Section* 2.4. *In other words,* $\mathrm{Char}^\circ j_!\mathcal{F}$ *satisfies the Milnor formula* (4.1).

Recall that Theorem 4.11 is proved in dimension 2 in [18, Proposition 3.20] using a global argument, as in [6]. Theorem 4.11 gives an affirmative answer to [17, Conjecture 3.16].

Proof. It suffices to show the equalities of the coefficients for each irreducible component of $S = SS\mathcal{K}$. By the additivity of characteristic cycles and the compatibility with étale pull-back, it suffices to show the tamely ramified case and the totally wildly ramified case separately. By Proposition 2.26, the singular support satisfies (SSd).

First, we prove the totally wildly ramified case. We prove it by induction on the dimension of X. Let $i\colon Y \to X$ be the immersion of a smooth divisor non-characteristic with respect to S. Then, by the induction hypothesis the assertion holds for $i^*\mathcal{K}$. Hence it follows by Theorem 4.8.

Next, we prove the tamely ramified case. Since the singular support satisfies (SSd), the characteristic cycle is compatible with smooth pull-back by Corollary 4.10 and it is reduced to the case where $X = \mathbf{A}^d$ and $U = \mathbf{G}_m^d$. By the induction on d and further by the compatibility with smooth pull-back, it suffices to show that the coefficient of the fiber T^*_xX at the origin $x \in X = \mathbf{A}^d = \mathrm{Spec}\ k[T_1, \ldots, T_d]$ is $(-1)^d \cdot \mathrm{rank}\ \mathcal{F}$. Since the morphism $f\colon X \to C = \mathrm{Spec}\ k[T]$ defined by by $T \mapsto T_1 + \cdots + T_d$ has an isolated characteristic point at x, it is reduced to the following.

Lemma 4.12. *Let $S = \mathrm{Spec}\ \mathcal{O}_K$ be the spectrum of a henselian discrete valuation ring with algebraically residue field, X be a smooth scheme of finite type of relative dimension $d-1$ over S and D be a divisor of X with simple normal crossings. Let x be a closed point of the closed fiber of X contained in D. Assume that $t_1, \ldots, t_d$ is a regular system of local parameters at x such that D is defined by $t_1 \cdots t_d$ and that the class of a uniformizer π of S in $\mathfrak{m}_x/\mathfrak{m}_x^2$ is not contained in any subspace generated by $d-1$ elements of the basis $\bar{t}_1, \ldots, \bar{t}_d$.*

Let Λ be a finite field of characteristic ℓ invertible on S and let $\mathcal{F}$ be a locally constant constructible sheaf on the complement $U = X - D$ of Λ-modules tamely ramified along D. Let $j\colon U \to X$ denote the open immersion. Then, on a neighborhood of X, the complex $\phi(j_!\mathcal{F})$ is acyclic except at x and at degree $d-1$ and $\phi_x^{d-1}(j_!\mathcal{F})$ is of dimension rank $\mathcal{F}$ *with tamely ramified action of the inertia group $I_K = \mathrm{Gal}(\bar{K}/K)$.*

Proof. By the assumption, we may write $\pi = \sum_{i=1}^d a_it_i$ in $\mathfrak{m}_u$ and a_i are invertible. Hence by replacing t_i by a_it_i, we may assume that X is étale over $\mathrm{Spec}\ \mathcal{O}_K[t_1, \ldots, t_d]/(\pi - (t_1 +$

$\cdots + t_d)$). Further, by Abhyankar's lemma, we may assume that $\mathcal{F}$ is trivialized by the abelian covering $s_i^m = t_i$ for an integer m invertible on S. Since the assertion is étale local on X, we may assume $X = \text{Spec } \mathcal{O}_K[t_1, \ldots, t_d]/(\pi - (t_1 + \cdots + t_d))$. Hence by [8, 1.3.3 (i)], the complex $\phi(j_!\mathcal{F})$ is acyclic outside x. Since the complex $\phi(j_!\mathcal{F})[d-1]$ is perverse by [9, Corollaire 4.6], the complex $\phi(j_!\mathcal{F})$ is acyclic except at x.

Let $p\colon X' \to X$ be the blow-up at x and $j'\colon U \to X'$ be the open immersion. Let D' be the proper transform of D and E be the exceptional divisor. Then, the union of D' with the closed fiber X'_s has simple normal crossings. Hence, the action of the inertia group I_K on $\phi(j'_!\mathcal{F})$ is tamely ramified by [16, Proposition 6] and $\phi_x(j_!\mathcal{F}) = R\Gamma(E, \phi(j'_!\mathcal{F}))$ is also tamely ramified.

We consider the stratification of E defined by the intersections with the intersections of irreducible components of D'. Then, on each stratum, the restriction of the cohomology sheaves $\phi^q(j_!\mathcal{F})$ are locally constant and is tamely ramified along the boundary by [16, Proposition 6]. Further, the alternating sum of the rank is 0 except for $E^\circ = E - (E \cap D')$ and equals rank $\mathcal{F}$ on E°. Hence, we have

$$\dim \phi_x(j_!\mathcal{F}) = \chi(E, \phi(j'_!\mathcal{F})) = \chi_c(E^\circ, \phi(j'_!\mathcal{F})) = \text{rank } \mathcal{F} \cdot \chi_c(E^\circ, \phi(j'_!\mathcal{F})).$$

Since $\chi_c(E^\circ) = (-1)^{d-1}$, the assertion follows. □

Theorem 4.13. *Let X be a projective smooth variety. Assume that a singular support $S = SS\mathcal{K}$ satisfies the condition* (SSd). *Then, we have*

$$\chi(X, \mathcal{K}) = (\text{Char } \mathcal{K}, T^*_X X)_{T^*X}. \tag{4.28}$$

Proof. We prove the assertion by induction on $\dim X$. If $\dim X = 1$, this is the Grothendieck-Ogg-Shafarevich formula.

Take a line $L \subset \mathbf{P}^\vee$ and a hyperplane $H \in L$ satisfying the conditions in Lemma 3.19. Then, since the immersions $i\colon Y = X \cap H \to X$ and $i'\colon Z = X \cap A_L \to Y$ are non-characteristic, the pull-backs $i^!SS\mathcal{K}$ and $i'^!SS\mathcal{K}$ satisfy the conditions (SS$(d-1)$) and (SS$(d-2)$) respectively, by Lemma 2.21. By Theorem 4.8, we have $i^!\text{Char } \mathcal{K} = \text{Char } i^*\mathcal{K}$ and $i'^!\text{Char } \mathcal{K} = \text{Char } i'^*\mathcal{K}$. By the induction hypothesis, we have

$$\chi(Y, \mathcal{K}) = (i^!\text{Char } \mathcal{K}, T^*_Y Y)_{T^*Y}, \quad \chi(Z, \mathcal{K}) = (i'^!\text{Char } \mathcal{K}, T^*_Z Z)_{T^*Z} \tag{4.29}$$

since $i^!SS\mathcal{K} = SSi^*\mathcal{K}$ and $i'^!SS\mathcal{K} = SSi'^*\mathcal{K}$ satisfy the conditions (SS$(d-1)$) and (SS$(d-2)$) respectively.

By the projection formula, we have

$$\chi(X, \mathcal{K}) = \chi(X_L, \mathcal{K}) - \chi(Z, \mathcal{K}) \tag{4.30}$$

since A_L meets X transversely. By applying the Grothendieck-Ogg-Shafarevich formula to $Rp_{L*}\mathcal{K}$, we have

$$\chi(X_L, \mathcal{K}) = \chi(L, Rp_{L*}\mathcal{K}) = 2\chi(Y, \mathcal{K}) - \sum_v \dim \text{tot}_v \phi(\mathcal{K}, p_L) \tag{4.31}$$

where v runs isolated characteristic points of $p_L\colon X_L \to L$ since $p_L\colon X_L \to L$ assumed to have at most isolated characteristic points.

By the Milnor formula (4.1), we have

$$-\dim \mathrm{tot}_v \phi(\mathcal{K}, p_L) = (\mathrm{Char}\ \mathcal{K}, dp_L). \tag{4.32}$$

Substituting the first equality of (4.29) and (4.32) to (4.30), we obtain

$$\chi(X_L, \mathcal{K}) = 2(i^!\mathrm{Char}\ \mathcal{K}, T^*_Y Y)_{T^*Y} + \sum_v (\mathrm{Char}\ \mathcal{K}, dp_L)_v = (\mathrm{Char}\ \mathcal{K}, T^*_{X_L} X_L)_{T^*X_L}. \tag{4.33}$$

Substituting this and the second equality of (4.29) to (4.31), we obtain

$$\chi(X, \mathcal{K}) = (\mathrm{Char}\ \mathcal{K}, T^*_{X_L} X_L)_{T^*X_L} - (i'^!\mathrm{Char}\ \mathcal{K}, T^*_Z Z)_{T^*Z} = (\mathrm{Char}\ \mathcal{K}, T^*_X X)_{T^*X}$$

as required. □

References

[1] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields*, Amer. J. of Math. 124 (2002), 879–920.

[2] A. Beilinson, *email to K. Kato and T. Saito*, 5.12.13.

[3] N. Bourbaki, ÉLÉMENTS DE MATHÉMATIQUE, Algébre commutative, Springer.

[4] P. Deligne, *La formule de Milnor*, Groupes de Monodromie en Géométrie Algébrique, SGA 7II, Springer Lecture Notes in Math. 340 (1973), 197-211.

[5] ——, *Théorèmes de finitude en cohomologie ℓ-adique*, Cohomologie étale SGA $4\frac{1}{2}$, Springer Lecture Notes in Math. 569 (1977) 233–251.

[6] ——, *Notes sur Euler-Poincaré: brouillon project*, 8/2/2011.

[7] A. Grothendieck, ÉLÉMENTS DE GÉOMÉTRIE ALGÉBRIQUE IV, Étude locale des schémas et des morphismes de schémas, Publ. Math. IHES 20, 24, 28, 32 (1964-67).

[8] L. Illusie, *Appendice à Théorèmes de finitude en cohomologie ℓ-adique*, Cohomologie étale SGA $4\frac{1}{2}$, Springer Lecture Notes in Math. 569 (1977) 252–261.

[9] ——, *Autour du théorème de monodromie locale*, Périodes p-adiques, Astérisque 223 (1994) 9-57.

[10] ——, *Produits orientés*, Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, Astérisque 363-364 (2014) 213-234.

[11] ——, *Around the Thom-Sebastiani theorem*, preprint.

[12] M. Kashiwara, P. Schapira, SHEAVES ON MANIFOLDS, Springer-Verlag, Grundlehren der Math. Wissenschaften 292, (1990).

[13] K. Kato, *Vanishing cycles, ramification of valuations, and class field theory,* Duke Math. J., 55-3, (1987), 629-659.

[14] G. Laumon, *Semi-continuité du conducteur de Swan (d'après P. Deligne)*, Séminaire E.N.S. (1978-1979) Exposé 9, Astérisque 82-83 (1981) 173–219.

[15] F. Orgogozo, *Modifications et cycles évanescents sur une base de dimension supérieure à un*, Int. Math. Res. Notices, (2006) 1-38.

[16] T. Saito, *ϵ-factor of a tamely ramified sheaf on a variety*, Inventiones Math. 113 (1993) 389-417

[17] —–, *Wild Ramification and the Cotangent Bundle*, arXiv:1301.4632v5.

[18] —–, *Characteristic cycle and the Euler number of a constructible sheaf on a surface*, Kodaira Centennial issue of the Journal of Mathematical Sciences, the University of Tokyo, vol 22, (2015) pp. 387-442.

[19] E. Yang, *Logarithmic version of the Milnor formula,* in preparation.