GALOIS REPRESENTATIONS IN ARITHMETIC GEOMETRY II

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The étale cohomology of an algebraic variety defined over the rational number field $\mathbf{Q}$ gives rise to an ℓ-adic representation of the absolute Galois group $G_{\mathbf{Q}}$ of $\mathbf{Q}$. To study such an ℓ-adic representation, it is common to investigate its restriction to the decomposition group at each prime number p. The purpose of this article is to survey our knowledge on the restrictions at various primes. When the variety has bad reduction at p, we find an interesting phenomenon called *ramification*. In this article we also report on recent discoveries on ramification.

This article is closely related to but independent of [26].

1. Local-Global relation in number theory

1.1. **The Beta function and the Jacobi sums.** Consider the Jacobi sum

$$J(a,b) = \sum_{x \in \mathbf{F}_p \setminus \{0,1\}} \chi^a(x)\chi^b(1-x),$$

where χ is a multiplicative character of $\mathbf{F}_p$, that is, a character of the multiplicative group $\mathbf{F}_p^\times$. This is an arithmetic analog of the Beta function

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\,dx.$$

Their formal similarity comes from the fact that the function $x \mapsto x^s$ is a multiplicative character of $\mathbf{R}$, and the summation in the Jacobi sum is replaced by an integral in the Beta function. The Beta function and the Jacobi sums share a number of similar properties.

The similarity is not superficial but is rooted in geometry. Let C_n be the algebraic curve defined by the equation $X^n + Y^n = 1$. The curve C_n is called the Fermat curve. Both the Beta function and the Jacobi sum indeed come from the cohomology of C_n.

Let us first consider the case of the Beta function. Let X be an algebraic variety over $\mathbf{Q}$. Between the singular cohomology and the de Rham cohomology, the comparison isomorphism $H^q(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \to H^q_{dR}(X/\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$ is defined by the so-called period integrals. The Beta function $B(s,t)$ is a period integral for $H^1(C_n)$ if s and t are rational numbers with denominator n.

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On the other hand, if p does not divide n, the number of $\mathbf{F}_p$-rational points on C_n is expressed in terms of the Jacobi sums $J(a,b)$ (see [41]). The number of rational points on a variety over a finite field is related to its cohomology. As many readers may already know, the étale cohomology was defined by Grothendieck[2] in order to prove the Weil conjecture on the congruence zeta-function, and it was indeed proved using étale cohomology, see [12]. The congruence zeta-function is defined as a generating function of the number of $\mathbf{F}_{p^m}$-rational points on the variety. The main point of the proof is to express the congruence zeta-function as the characteristic polynomial of the Frobenius operator acting on the étale cohomology. This is how the Jacobi sum $J(a,b)$ is related to the cohomology of the Fermat curve C_n.

The cohomology of the Fermat curve C_n is thus related to the Beta function when we regard $\mathbf{Q}$ as a subfield of the complex number field $\mathbf{C}$, and to the Jacobi sums when we take a reduction modulo a prime number p. This picture is the prototype of the local-global relation in modern number theory.

$$\begin{array}{ccccc} & & & & \text{Beta function} \\ & & & \overset{\mathbf{C}}{\nearrow} & \\ \text{Fermat curve } C_n & \Longrightarrow & H^1(C_n) & & \\ & & & \underset{p}{\searrow} & \\ & & & & \text{Jacobi sum} \\ & & \text{(Global)} & & \text{(Local)} \end{array}$$

1.2. **ℓ-adic representations of the Galois group.** Let S be the set of all prime numbers together with the infinity symbol "∞"; i.e., $S = \{\text{prime numbers}\} \amalg \{\infty\} = \{2, 3, 5, 7, \ldots, \infty\}$. Our standard point of view is to regard S as something analogous to an algebraic curve, and $\mathbf{Q}$ as its function field. As each prime number p gives rise to an embedding of $\mathbf{Q}$ into the p-adic number field $\mathbf{Q}_p$, we associate ∞ to the embedding of $\mathbf{Q}$ into $\mathbf{R} = \mathbf{Q}_\infty$. The embeddings corresponding to prime numbers are called finite places, and the embedding of $\mathbf{Q}$ into $\mathbf{R}$ is called the infinite place. A principle in modern number theory is that the finite places and the infinite place should be treated completely equally. According to this point of view, a locally constant sheaf, or a local system on S (more precisely on an open set of S) is an ℓ-adic representation of the absolute Galois group $G_{\mathbf{Q}}$.

In Topology a local system on a topological space S is nothing but a representation of its fundamental group $\pi_1(S)$. The fundamental group $\pi_1(S)$ controls the coverings of the space S. Since the absolute Galois group $G_{\mathbf{Q}}$ controls the coverings of (an open set of) $S = \{2, 3, 5, 7, \ldots, \infty\}$, we consider a representation of $G_{\mathbf{Q}}$ as a local system on (an open set of) S.

Recall that the absolute Galois group $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ is the automorphism group $\mathrm{Aut}(\bar{\mathbf{Q}})$ of an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. If ℓ is a prime number, an ℓ-adic representation of $G_{\mathbf{Q}}$ is a continuous representation $G_{\mathbf{Q}} \to$

$GL_{\mathbf{Q}_\ell}(V) \simeq GL_n(\mathbf{Q}_\ell)$, where V is an n-dimensional vector space over the ℓ-adic number field $\mathbf{Q}_\ell$. It is customary to use p for a prime number regarded as a point in S, and ℓ for the coefficient field of the local system. We need to consider Hodge structures together with ℓ-adic representations in order to treat the infinite place equally (see [33]).

To an algebraic variety or a modular form over $\mathbf{Q}$, we can associate an ℓ-adic representation (and a Hodge structure):

$$\left.\begin{array}{c}\text{algebraic variety,}\\ \text{modular form}\end{array}\right\} \Longrightarrow \begin{array}{c}\ell\text{-adic representation}\\ (+\text{ Hodge structure})\end{array}$$

Such a correspondence enables us to study algebraic geometric objects or representation theoretic objects using ℓ-adic representations, which is a linear algebraic object. Conversely, we can study Galois representation, a highly arithmetic object, using geometric or representation theoretic methods. If we take the Fermat curve as the algebraic variety above, we obtain the example given at the beginning. In practice, étale cohomology is a principal tool to construct Galois representations. Though there are some other methods, including that using fundamental groups or congruence, we do not touch these here.

Let X be an algebraic variety over $\mathbf{Q}$, and ℓ a prime number. Then, the natural action of $G_{\mathbf{Q}}$ on $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ gives rise to an ℓ-adic representation of $G_{\mathbf{Q}}$. As a $\mathbf{Q}_\ell$-vector space, $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is identified with the coefficient extension $H^q(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$, if we fix an embedding of $\bar{\mathbf{Q}}$ into $\mathbf{C}$. The corresponding Hodge structure is defined by the comparison isomorphism $H^q(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \to H^q_{dR}(X/\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$ of the singular cohomology with the algebraic de Rham cohomology.

Example 1. Let E be an elliptic curve over $\mathbf{Q}$. The Tate module $T_\ell E = \varprojlim_n \operatorname{Ker}(\ell^n : E(\bar{\mathbf{Q}}) \to E(\bar{\mathbf{Q}}))$ is a free $\mathbf{Z}_\ell$-module of rank 2, and it admits a natural representation of the Galois group $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. The ℓ-adic representation $H^1(E_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is naturally identified with the dual $\operatorname{Hom}(T_\ell E, \mathbf{Q}_\ell)$. If we regard $E(\mathbf{C})$ as the quotient $\mathbf{C}/T$ by a lattice $T \subset \mathbf{C}$, then we have $T_\ell E \simeq T \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$, and $H^1(E_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell) \simeq \operatorname{Hom}(T, \mathbf{Q}_\ell)$.

Example 2. Let $f = \sum_{q=1}^{\infty} a_n q^n$ be a cusp form that is a simultaneous eigenvector of all the Hecke operators T_n. Then we can construct a two-dimensional ℓ-adic representation V_f of the Galois group $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ such that at almost all prime numbers p, it is unramified and $\operatorname{Tr}(\operatorname{Fr}_p : V_f) = a_p$ (see [7]). This is called the ℓ-adic representation associated to the modular form f.

Many readers may know that Fermat's Last Theorem was proved recently, using such Galois representations (see [42]). It was in fact proved by showing that the ℓ-adic representation in Example 1 is isomorphic to the ℓ-adic representation associated to a modular form as in Example 2 (cf.[29]).

1.3. **The congruence zeta-function and Galois representations.** A standard method for studying a local system on $S = \{2, 3, 5, 7, \dots , \infty\}$ is to study the restriction at each point of S. We may consider each point of S as something similar to a circle S^1, which has a nontrivial fundamental group. The reason is that a point of S is a prime number p and its fundamental group is the absolute Galois group $G_{\mathbf{F}_p}$. The absolute Galois group $G_{\mathbf{F}_p} = \mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$ is the automorphism group $\mathrm{Aut}(\bar{\mathbf{F}}_p)$ of an algebraic closure $\bar{\mathbf{F}}_p$ of $\mathbf{F}_p$. The p-th power map $\varphi_p : \bar{\mathbf{F}}_p \to \bar{\mathbf{F}}_p$ is a topological generator of $G_{\mathbf{F}_p}$. We call its inverse the geometric Frobenius isomorphism, and denote it by Fr_p. From now on we identify the absolute Galois group $G_{\mathbf{F}_p}$ with the profinite completion $\hat{\mathbf{Z}}$ of $\mathbf{Z}$ via the isomorphism $\hat{\mathbf{Z}} \to G_{\mathbf{F}_p} : 1 \mapsto \mathrm{Fr}_p$. Each point of S has the completion of $\mathbf{Z}$ as its fundamental group, and it resembles S^1. By Chebotarev's density theorem a local system of S is more or less determined by its restriction at each point of S. Thus, to determine the restrictions of a local system on S at points of S is an important step in its study.

Let X be an n-dimensional non-singular projective variety over $\mathbf{Q}$. In order to determine the ℓ-adic cohomology of X at each prime p, it often suffices to study the reduction, $X \bmod p$. As we will see soon, for almost all p the congruence zeta-function of $X \bmod p$ determines the ℓ-adic representation at p. The reduction $X \bmod p$ is a projective variety over $\mathbf{F}_p$ that is defined by the reduction modulo p of the equations of X with integral coefficients. X is said to have good reduction at p if $X \bmod p$ is non-singular. X has good reduction modulo p except for the finite number of primes p.

For example, if an elliptic curve E is defined by the equation $y^2 = 4x^3 - g_2x - g_3$ with $g_2, g_3 \in \mathbf{Z}$, then E has good reduction at p if p does not divide $6\Delta = 6(g_2^3 - 27g_3^2)$. Fermat curve C_n has good reduction at p if p does not divide n. The modular curve $X_0(N)$ of level N has good reduction at p if p does not divide N.

If X has good reduction at p and ℓ is different from p, then the restriction of the ℓ-adic representation $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ at p is described by the congruence zeta-function of $X \bmod p$ as follows (see [11]).

As we will see in §2.1, the geometric Frobenius isomorphism Fr_p at p acts on the space $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$. We denote its characteristic polynomial by $P_q(t) = \det(1 - \mathrm{Fr}_p\, t : H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell))$. The $P_q(t)$'s for various q satisfy the following equality

$$\frac{P_1(t) \cdot \cdots \cdot P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdot \cdots \cdot P_{2n}(t)} = \exp\Big(\sum_{m=1}^{\infty} \frac{N_m(X \bmod p)}{m} t^m\Big), \tag{1}$$

where $N_m(X \bmod p)$ is the number of $\mathbf{F}_{p^m}$-rational points on $X \bmod p$. The right-hand side of the equality (1) is the definition of the congruence zeta-function $Z(X \bmod p, t)$ of the variety $X \bmod p$ over $\mathbf{F}_p$.

As a consequence of the Weil conjecture (see §2.4), the polynomials $P_q(t)$ do not have a common root, and the decomposition of the left-hand side of the equality (1) is uniquely determined. In other words the characteristic polynomial $P_q(t) = \det(1 - \mathrm{Fr}_p\, t : H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell))$ is obtained by counting the number of rational points on X mod p.

$$\begin{aligned} &\text{number of } \mathbf{F}_{p^m}\text{-rational points on } X \bmod p \\ \Longrightarrow\quad &\text{congruence zeta-function } Z(X \bmod p, t) \\ \Longrightarrow\quad &\text{characteristic polynomial of } \mathrm{Fr}_p \text{ on } H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell). \end{aligned}$$

Example 3. (a) Let E be an elliptic curve over $\mathbf{Q}$. Suppose that E has good reduction at p. Define $a_p(E)$ by the equation

$$(\text{the number of } \mathbf{F}_p\text{-rational points on } E \bmod p) = p + 1 - a_p(E).$$

Then we have $P_1(t) = 1 - a_p(E)t + pt^2$.

(b) If X is the Fermat curve C_n, then for any multiplicative character χ of $\mathbf{F}_p$ of exact order n we have

$$P_1(t) = \prod_{a,b,a+b\not\equiv 0 \bmod n} (1 - J(a,b)t).$$

This shows that the ℓ-adic representation $H^1(C_{n,\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is described by the Jacobi sums.

(c) If p does not divide N, the numerator $P_1(t)$ of the congruence zeta-function of $X_0(N) \bmod p$ is the product of $1 - a_p(f_i)t + pt^2$, where $f_i = \sum_{n=1}^{\infty} a_n(f_i)q^n$ are modular forms of weight 2 and level dividing N (see [36]).

In this way, the restriction of a Galois representation at a prime p is described by its congruence zeta-function. However, in order to apply this general theory to a prime p, it must satisfy the following condition:

(0) X has good reduction at p, and $p \neq \ell$.

For fixed X and ℓ, there are only finitely many points in $S = \{2, 3, 5, \dots, \infty\}$ that do not satisfy the condition (0), and they fall into one of the following cases:

(1) the infinite place ∞,
(2) $p = \ell$, or
(3) p at which X does not have good reduction and $p \neq \ell$.

Thus, the points in S are divided into four classes according to (0) to (3):

$$S = \{\text{good primes}\} \amalg \{\infty\} \amalg \{\ell\} \amalg \{\text{bad primes}\}.$$

The restriction of an ℓ-adic representation at a good prime is described by the congruence zeta-function. Consequently, only finitely many points requires more detailed study. Case (1) is related to Hodge theory, and we will not touch it in this article. Case (2) is the subject of p-adic Hodge theory. A satisfactory general theory has been established on this, thanks to many mathematicians including Fontaine, Messing, Faltings, Hyodo, Kato, Tsuji, et al. (see [14], [40]). Ramification theory deals with Case (3). At such primes, each variety shows its own distinguishing properties. In the sequel we explain recent developments in ramification theory.

2. Galois representations of local fields

2.1. **The Galois group of a local field.** In Topology to study the degeneration of a local system along the boundary is to study its monodromy action along it. Similarly, to study the ℓ-adic representation $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ of $G_{\mathbf{Q}}$ at a bad prime p is to study the restriction to the absolute Galois group $G_{\mathbf{Q}_p} = \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p) \subset G_{\mathbf{Q}}$ of the p-adic field $\mathbf{Q}_p$. The structure of the absolute Galois group $G_{\mathbf{Q}_p}$ is described as follows (see [34]).

Let $\mathbf{Q}_p(\zeta_m, p \nmid m)$ be the extension of $\mathbf{Q}_p$ obtained by adjoining all the m-th roots of unity, where m is prime to p. It is the maximal unramified extension of $\mathbf{Q}_p$, and we denote it by $\mathbf{Q}_p^{\mathrm{ur}}$. Also, let $\mathbf{Q}_p^{\mathrm{ur}}(p^{1/m}, p \nmid m)$ be the extension of $\mathbf{Q}_p^{\mathrm{ur}}$ obtained by adjoining all the m-th roots of p, where m is prime to p. It is the maximal tamely ramified extension of $\mathbf{Q}_p$, and we denote it by $\mathbf{Q}_p^{\mathrm{tr}}$. The normal subgroups of $G_{\mathbf{Q}_p} = \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ corresponding to these extensions via Galois theory are customarily written as follows:

$$\begin{array}{ccccccc} \mathbf{Q}_p & \subset & \mathbf{Q}_p^{\mathrm{ur}} & \subset & \mathbf{Q}_p^{\mathrm{tr}} & \subset & \bar{\mathbf{Q}}_p \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ G_{\mathbf{Q}_p} & \supset & I & \supset & P & \supset & 1 \end{array}$$

The subgroup I is called the inertia group, and P the wild inertia group.

The quotient group $G_{\mathbf{Q}_p}/I$ is canonically identified with the absolute Galois group $G_{\mathbf{F}_p} = \mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$ of the finite field $\mathbf{F}_p$. Recall that the group $G_{\mathbf{F}_p}$ is identified with the profinite completion $\hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$ of $\mathbf{Z}$ via the map $1 \mapsto \mathrm{Fr}_p$. The quotient group I/P is identified with the inverse limit $\varprojlim_{p\nmid m} \mu_m$, where $\mu_m = \{\zeta \in \bar{\mathbf{F}}_p | \zeta^m = 1\}$ is the subgroup of $\bar{\mathbf{F}}_p$ consisting of all the m-th roots of unity. Hence it is noncanonically isomorphic to the subgroup $\hat{\mathbf{Z}}'$ of $\hat{\mathbf{Z}}$ obtained by removing the p-component $\mathbf{Z}_p$. The quotient I/P is an algebraic analog of the monodromy of the punctured disk. The group P is particular to number theory, and it is a fairly large pro-p group. It is as large as the countably many power of the Pontryagin dual of the discrete group $\bar{\mathbf{F}}_p$.

$$\begin{array}{lll} G_{\mathbf{Q}_p}/I & = G_{\mathbf{F}_p} = \hat{\mathbf{Z}} & \text{topologically generated by } \mathrm{Fr}_p, \\ I/P & = \varprojlim_{p\nmid m} \mu_m \simeq \hat{\mathbf{Z}}' & \text{corresponds to topological monodromy,} \\ P & & \text{a large pro-}p\text{ group.} \end{array}$$

An ℓ-adic representation of the absolute Galois group $G_{\mathbf{Q}_p}$ is said to be unramified if its restriction to the inertia group I is trivial. An unramified representation is identified with a representation of $G_{\mathbf{F}_p}$ through the canonical isomorphism $G_{\mathbf{Q}_p}/I \to G_{\mathbf{F}_p}$, and is determined by the action of the generator Fr_p. If a variety X has good reduction at $p \neq \ell$, then the ℓ-adic representation $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is unramified at p, and the action of $G_{\mathbf{Q}_p}$ is determined by the action of Fr_p. Furthermore, the characteristic polynomial of Fr_p is determined by the congruence zeta-function as we have seen in the previous section.

¿From now on we consider the case where the variety X does not have good reduction at p. In this case we need to deal with ramified representations of $G_{\mathbf{Q}_p}$. Two problems arise regarding this:

Problem A. Determine the isomorphism class of $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ as an ℓ-adic representation of $G_{\mathbf{Q}_p}$.

Problem B. Compute the invariants of $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ arising from its ramification.

In the sequel we will formulate each of these problems more precisely, and we will discuss recent progresses. As for Problem A, we would like to determine the isomorphism class at all p as precisely as in the case of good reduction, where it is described by the congruence zeta-function. As for Problem B, we would like to understand phenomena particular to the bad reduction case, as all the invariants related to ramification are trivial at the primes of good reduction. It turns out that the relation of Galois representations with the differential forms provides solutions to Problem B.

2.2. **ℓ-adic representations of local fields.** The Problem A may be answered using a certain type of trace formula, as long as we assume the Tate conjecture and the weight-monodromy conjecture. To explain this further, we recall the fact that an ℓ-adic representation $\rho : G_{\mathbf{Q}_p} \to GL(V)$ is determined by a pair consisting of a representation ρ' of the Weil group and a nilpotent endomorphism N (see [10]). The Weil group $W_{\mathbf{Q}_p}$ is the subgroup of all the elements of $G_{\mathbf{Q}_p}$ whose image in $G_{\mathbf{F}_p}$ is a power of Fr_p:

$$\begin{array}{ccccc} W_{\mathbf{Q}_p} & & \subset & G_{\mathbf{Q}_p} & \\ \downarrow & & & \downarrow & \\ \langle \mathrm{Fr}_p \rangle & (\simeq \mathbf{Z}) & \subset & G_{\mathbf{F}_p} & (\simeq \hat{\mathbf{Z}}) \end{array}$$

Take a lift F of the geometric Frobenius Fr_p to the Weil group $W_{\mathbf{Q}_p}$. By the monodromy theorem of Grothendieck ([35]), there exists a unique pair (ρ', N), where $\rho' : W_{\mathbf{Q}_p} \to GL_{\mathbf{Q}_l}(V)$ is a continuous representation of the Weil group $W_{\mathbf{Q}_p}$, and N is a nilpotent endomorphism of V characterized by the following property: The kernel of the restriction of ρ' to the inertia group $I \subset W_{\mathbf{Q}_p}$ is an open subgroup of I and we have

$$\rho(F^n\sigma) = \rho'(F^n\sigma)\exp(t_\ell(\sigma)N)$$

for all $n \in \mathbf{Z}$ and $\sigma \in I$. Here $t_\ell : I \to I/P \to \mathbf{Z}_\ell$ is the composition of a noncanonical isomorphism $I/P \to \hat{\mathbf{Z}}'$ and the projection of $\hat{\mathbf{Z}}'$ to the ℓ-component $\mathbf{Z}_\ell$. For $F^n\sigma \in W_{\mathbf{Q}_p}$ $(n \in \mathbf{Z}, \sigma \in I)$, we have an equality

$\operatorname{Tr}\rho(F^n\sigma) = \operatorname{Tr}\rho'(F^n\sigma)$. Conversely, since $W_{\mathbf{Q}_p}$ is a dense subgroup of $G_{\mathbf{Q}_p}$, an ℓ-adic representation ρ of $G_{\mathbf{Q}_p}$ is determined by a pair (ρ', N):

$$\begin{aligned} &\ell\text{-adic representation } \rho \text{ of } G_{\mathbf{Q}_p} \\ \iff &\begin{cases} \text{continuous representation } \rho' \text{ of } W_{\mathbf{Q}_p} \\ + \text{ nilpotent endomorphism } N. \end{cases} \end{aligned}$$

2.3. **The trace formula.** If an ℓ-adic representation $\rho : G_{\mathbf{Q}_p} \to V$ arises from $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ of an algebraic variety X, how can we determine the corresponding ρ' and N? First we consider ρ'. The fundamental framework to solve this problem is given by the Tate conjecture. For an ℓ-adic representation V of the absolute Galois group $G_F = \operatorname{Gal}(\bar{F}/F)$ of a field F, we denote by $V(q)$ the ℓ-adic representation obtained by tensoring the q-th power of the ℓ-adic cyclotomic character.

Conjecture (Tate Conjecture [37],[38])**.** *Let F be a finite field, a number field, or, more generally, a finitely generated field over a prime field. Let X be a non-singular projective variety over F. Let q be a nonnegative integer, and ℓ a prime number different from the characteristic of F. Then for an ℓ-adic representation $V = H^{2q}(X_{\bar{F}}, \mathbf{Q}_\ell(q))$ of G_F we have the following.*

1. *The invariant part V^{G_F} is generated by algebraic cycles.*
2. *The generalized eigenspace $\{v \in V \mid (\sigma - 1)^{\dim V} v = 0 \text{ for all } \sigma \in G_F\}$ is equal to the invariant part V^{G_F}.*

An algebraic cycle in $H^{2q}(X_{\bar{F}}, \mathbf{Q}_\ell(q))$ is a linear combination of cycle classes $[Z]$, where Z is a subvariety of codimension q. The only general class of varieties over number fields for which the Tate conjecture is proved is the following:

- X is an abelian variety, and $q = 1$ ([13]).

Apart from this, the conjecture has been verified for a number of varieties by constructing algebraic cycles case by case.

The second part of the Tate conjecture is often referred to as the semi-simplicity conjecture. The semi-simplicity conjecture for varieties over the residue field implies that the representation ρ' of $W_{\mathbf{Q}_p}$ is semi-simple. Since the isomorphism class of a semi-simple representation is determined by its trace, in order to determine ρ', it suffices to know the traces $\operatorname{Tr}\rho'(F^n\sigma)$ $(n \in \mathbf{Z}, \sigma \in I)$. In fact, it suffices to know them for $n \geq 0$.

The Tate conjecture also implies that the Künneth projectors are given by algebraic correspondences on X. An algebraic correspondence on X is a linear combination Γ (with coefficients in $\mathbf{Q}$) of the classes of subvarieties in $X \times X$ of dimension $d = \dim X$. An algebraic correspondence Γ induces a G_F-equivariant endomorphism Γ^* on $H^q(X_{\bar{F}}, \mathbf{Q}_\ell)$. The statement that the Künneth projectors are given by algebraic correspondences means that for any non-negative integer q, there exists an algebraic correspondence Γ_q such that the endomorphism Γ_q^* of $H^{q'}(X_{\bar{F}}, \mathbf{Q}_\ell)$ is the identity for $q = q'$ and 0 for $q \neq q'$. All in all, to determine ρ', it suffices to know the alternating sum

$$\sum_{q=0}^{2d} (-1)^q \mathrm{Tr}(\Gamma^* \circ F^n \sigma : H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell))$$

for an algebraic correspondence Γ.

We have recently obtained a formula for this alternating sum.

Theorem 1 ([32]). *Let X be a non-singular proper algebraic variety of dimension d over $\mathbf{Q}_p$, Γ an algebraic correspondence on X, F a lift of the geometric Frobenius, n a nonnegative integer, and σ an element of the inertia group I. Then there exist a non-singular projective variety W over an algebraic closure $\bar{\mathbf{F}}_p$ and an algebraic correspondence Γ' on W such that for a prime number $\ell \neq p$ we have*

$$\sum_{q=0}^{2d} (-1)^q \, \mathrm{Tr}(\Gamma^* \circ F^n \sigma : H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell)) = (\Gamma', \Delta_W),$$

where the right-hand side is the intersection number of Γ' with the diagonal Δ_W of $W \times W$.

Theorem 1 means that the arithmetic of an ℓ-adic representation coming from an algebraic variety over a local field (the left-hand side) is reduced to the geometry of a variety over a finite field obtained by the reduction modulo p (the right-hand side). While elements of the Galois group appear in the left-hand side, it should be remarked that the right-hand side is purely geometric. If X has good reduction at p, $\Gamma = \Delta$, and $n = 1$, then we can take X mod p as W and the graph of the Frobenius endomorphism $F : W \to W$ as Γ'. This corresponds to the fact described in §1.3 that the action of the Frobenius is determined by the congruence zeta-function. To be precise, we need to assume the semi-simplicity conjecture to determine completely the action of the Galois group $G_{\mathbf{F}_p}$, which was omitted there for simplicity. If the variety X has a semi-stable model, then as W we can take the disjoint union of the irreducible components, together with their intersections, of the reduction of a semi-stable model.

It is a consequence of Theorem 1 that the alternating sum in the left-hand side is independent of ℓ. In the analog of positive characteristic, Terasoma [39] has shown that when $\Gamma = \Delta$, each term in the sum is independent of ℓ, using weight-monodromy conjecture, which is proved in this case. In the case $\ell = p$, we can formulate a similar conjecture using p-adic Hodge theory. Theorem 1 is proved in [32] by reducing to the Lefschetz trace formula using functorial properties of the weight spectral sequence ([25]) and alteration ([6]). Theorem 1 had already been known to hold for the following cases:

- X is an abelian variety ([16]).
- Γ is the diagonal Δ ([23]).
- σ is an element of the wild inertia group P ([20]).

- X is a Kuga-Sato variety, or its analog over a Shimura curve, and Γ is a certain type of the Hecke correspondences ([28], [31]).

A Kuga-Sato variety is a compactification of the fibered product of the universal elliptic curves over modular curves. As an application of the last case, we see that the restriction to the decomposition group $G_{\mathbf{Q}_p}$ of a p-adic representation associated to an elliptic modular form or a Hilbert modular form is compatible with the local Langlands correspondence in the sense of p-adic Hodge theory.

2.4. **The weight-monodromy conjecture.** While the representation ρ' is related to the Tate conjecture, the endomorphism N is related to the Weil conjecture.

Theorem (Weil Conjecture [12])**.** *Let X be a non-singular proper variety over a finite field $\mathbf{F}_p$, ℓ a prime number different from p, and q a nonnegative integer. Then, the eigenvalues of the geometric Frobenius Fr_p acting on $H^q(X_{\bar{\mathbf{F}}_p}, \mathbf{Q}_\ell)$ are algebraic integers all of whose complex conjugates have absolute value $p^{q/2}$.*

Algebraic integers satisfying the last property in the Weil conjecture are called Weil numbers of weight q. The Weil conjecture says that if we decompose the polynomial $P_q(t) = \det(1 - \mathrm{Fr}_p\, t : H^q(X_{\bar{\mathbf{F}}_p}, \mathbf{Q}_\ell))$ in §1.3 into $P_q(t) = \prod_i (1 - \alpha_i t)$, then each α_i is a Weil number of weight q.

The weight-monodromy conjecture is a generalization of the Weil conjecture to the varieties over local fields. In general, for a nilpotent endomorphism N of a vector space V, the increasing sequence of subspaces $M_i = \sum_{j-k=i} \mathrm{Ker} N^{j+1} \cap \mathrm{Im} N^k \subset V$ satisfies the following properties.

1. $M_i = V, M_{-i} = 0$ for a sufficiently large i.
2. $N M_i \subset M_{i-2}$.
3. For $i > 0$ the linear map N^i induces an isomorphism $N^i : Gr_i^M V = M_i/M_{i-1} \to Gr_{-i}^M V$.

For example, if $N^2 = 0$, then we have $M_{-2} = 0, M_{-1} = \mathrm{Im}\, N$, $M_0 = \mathrm{Ker}\, N, M_1 = V$. If V is an ℓ-adic representation of the Galois group of a local field, and N is the nilpotent endomorphism of V determined as in §2.2, the sequence $M = (M_i)_{i \in \mathbf{Z}}$ is called the monodromy filtration on V. In this case each M_i is a subrepresentaion.

Conjecture (Weight-monodromy Conjecture [9])**.** *Let X be a non-singular proper variety over the p-adic field $\mathbf{Q}_p$, ℓ a prime number different from p, and q a nonnegative integer. Suppose that $F \in W_{\mathbf{Q}_p}$ is a lift of the geometric Frobenius Fr_p, and M is the monodromy filtration of the ℓ-adic representation $V = H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell)$. Then, the eigenvalues of F acting on $Gr_i^M V$ are Weil numbers of weight $q + i$.*

Known cases of the weight-monodromy conjecture are as follows.

- When $q \leq 2$ ([25]).

- X is a Kuga-Sato variety, or its analog over a Shimura curve ([28], [30], [31]).
- An analog for local fields of positive characteristic ([12], [18]).

The Weil conjecture implies that the weight-monodromy conjecture is equivalent to that the filtration of $H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell)$ defined by the weight spectral sequence (see [25]) is the same as the monodromy filtration.

The semi-simplicity conjecture and the weight-monodromy conjecture imply that the isomorphism class of the pair (ρ', N) depends only on the isomorphism class of ρ', and hence is determined by $\operatorname{Tr}\rho'$. Since we can compute $\operatorname{Tr}\rho'$ using Theorem 1 under the Tate conjecture, the isomorphism class of the ℓ-adic representation $H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell)$ may be determined as in the following chart.

$$\left.\begin{array}{l}\text{Theorem 1 (= trace formula)}\\ +\text{ Tate conjecture}\\ +\text{ weight-monodromy conj.}\end{array}\right\} \Longrightarrow \begin{array}{c}\text{isom. class of}\\ (\rho', N)\end{array} \Longrightarrow \begin{array}{c}\text{isom. class of}\\ H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell)\end{array}$$

3. Ramification theory

Problem B (in §2.1) deals with the ramification theory in higher dimension. An impressive feature of the ramification theory is the deep relation of Galois representations with differential forms. While the comparison theorem between the singular (resp. p-adic) cohomology and the de Rham cohomology are built into Hodge (resp. p-adic Hodge) theory from the beginning, there is no functor directly relating the ℓ-adic cohomology to the de Rham cohomology. Nonetheless, it often happens that the invariants arising from ramifications of ℓ-adic cohomology are expressed in terms of differential forms or the de Rham cohomology. A typical example is the conductor formula discussed below.

3.1. **The conductor formula.** Among the invariants of the ramification of an ℓ-adic Galois representation of a local field the most fundamental one is the conductor. The wild ramification group $P \subset G_{\mathbf{Q}_p}$ has a decreasing sequence $(G^v)_{v\in\mathbf{Q}, v>0}$ of normal subgroups of P, called the filtration by the ramification groups (see [34]). For an ℓ-adic representation V of $G_{\mathbf{Q}_p}$, its Swan conductor is defined by

$$\operatorname{Sw} V = \sum_{v\in\mathbf{Q}, v>0} v \cdot \dim V^{G^{v+}}/V^{G^v},$$

where $G^{v+} = \overline{\bigcup_{v'>v} G^{v'}}$, and V^{G^v} and $V^{G^{v+}}$ indicate the invariant parts. The Swan conductor $\operatorname{Sw} V$ is a nonnegative integer. The action of P on V is trivial if and only if $\operatorname{Sw} V = 0$.

Let X be a non-singular proper variety over $\mathbf{Q}_p$ of dimension $n-1$. Bloch conjectured in [4] that the alternating sum

$$\mathrm{Sw}(X/\mathbf{Q}_p) = \sum_{q=0}^{2(n-1)} (-1)^q \,\mathrm{Sw}\, H^q(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_\ell)$$

can be calculated geometrically in the following way. Let $X_{\mathbf{Z}_p}$ be a proper regular model of X over the p-adic integer ring $\mathbf{Z}_p$. The localized Chern class of the sheaf of differential forms $\Omega^1_{X_{\mathbf{Z}_p}/\mathbf{Z}_p}$ on $X_{\mathbf{Z}_p}$ determines a 0-cycle class $c_n(\Omega^1_{X_{\mathbf{Z}_p}/\mathbf{Z}_p}) \in CH_0(X_{\mathbf{F}_p})$ supported on the closed fiber.

Conjecture (Bloch's conductor formula [4])**.** *With above notation we have*

$$\chi(X_{\bar{\mathbf{Q}}_p}) - \chi(X_{\bar{\mathbf{F}}_p}) + \mathrm{Sw}(X/\mathbf{Q}_p) = -\deg(-1)^n c_n(\Omega^1_{X_{\mathbf{Z}_p}/\mathbf{Z}_p}),$$

where $\chi = \sum_q (-1)^q \dim H^q$ *is the Euler number of the ℓ-adic cohomology, and* deg *is the degree of a* 0*-cycle class.*

If X is the spectrum of a finite extension of $\mathbf{Q}_p$, then the above formula is nothing but the classical conductor-discriminant formula. If E is an elliptic curve, then it is proved in ([27]) to be equivalent to the formula of Tate-Ogg ([24]). Bloch [4] proved the conductor formula when X is an algebraic curve. The formula asserts that the invariant arising from the ramification of ℓ-adic cohomology of X may be calculated geometrically using differential forms. This formula may also be considered as an arithmetic analog of the Lefschetz trace formula $\chi(V_{\bar{F}}) = (\Delta_V, \Delta_V) = (-1)^n \deg c_n(\Omega^1_{V/F})$, where V is a n-dimensional non-singular proper variety over F. As an application of the conductor formula, a relation between the sign of the functional equation of the Hasse-Weil L-function and the Galois module structure of the de Rham cohomology for a variety over $\mathbf{Q}$ is established by Chinburg-Pappas-Taylor in [5].

Theorem 2 ([20])**.** *If the reduced closed fiber $X_{\mathbf{F}_p,red}$ is a divisor of $X_{\mathbf{Z}_p}$ with normal crossing, then Bloch's conductor formula holds.*

Theorem 2 implies that if the divisor $D = X_{\mathbf{F}_p,\mathrm{red}}$ has an embedded resolution of singularilities in the strong sense, then Bloch's conductor formula holds. A divisor D is said to have an embedded resolution of singularilities in the strong sense if D can be modified to a divisor with normal crossing by repeating blowing-ups $X_{\mathbf{Z}_p}$ at non-singular subschemes.

3.2. **Perspective.** We raise the following three topics as open problems in ramification theory.

1. The conductor formula with coefficient sheaf.
2. The ϵ-factor.
3. Analogy with integrable connections with irregular singularities.

(1) The problem is to find a ramification theoretic formula for the Swan conductor $\mathrm{Sw}(U,\mathcal{F}) = \sum_{q=0}^{2n}(-1)^q \mathrm{Sw} H^q(U_{\bar{K}},\mathcal{F})$ for a non-singular variety U of dimension n over a local field K and a smooth ℓ-adic sheaf $\mathcal{F}$ on U. In the case $n = 1$ and $\operatorname{rank}\mathcal{F} = 1$, such a formula is obtained by Kato ([19]). A formula seems within reach in higher dimension in the case $\operatorname{rank}\mathcal{F} = 1$ by virtue of the conductor formula for an automorphism proved in [20]. Thus the problem is to generalize it to higher rank case.

In order to formulate it, the rank 1 case suggests to study the filtration by ramification groups of the absolute Galois group of the local fields at the generic points of the boundary. The residue fields of such local fields are imperfect if $\dim U > 1$. In a joint work with Abbés ([1]), a filtration is defined for such local fields using rigid geometry. Fujiwara had also defined a filtration by a similar idea ([15]), which imspired our definition. We still do not know much about the properties of the filtration. There remain many tasks before formulating the conductor formula with coefficient sheaf. The conductor formula for an algebraic correspondence mentioned above should be a first step toward it.

(2) The ϵ-factor is an arithmetic invariant of ramification finer than the conductor ([10]). It plays important roles in the functional equation of Hasse-Weil L-functions and in the formulation of the local Langlands correspondence, which has been proved recently by Harris and Taylor ([17]). The problem is to find a formula for the ϵ-factor of the ℓ-adic cohomology defined by $\epsilon(U,\mathcal{F}) = \prod_{q=0}^{2n}\epsilon(K, H^q(U_{\bar{K}},\mathcal{F}))^{(-1)^q}$. The problem is open even in the case $\dim U = 1$ and $\operatorname{rank}\mathcal{F} = 1$. Related problems are studied by Yasuda [43] and by Kobayashi [21].

(3) Analogy between wild ramification of ℓ-adic sheaves and irregular singularities of integrable connections on varieties of characteristic 0 was already pointed out in [8]. Let K be a function field over a field k of characteristic 0, U a non-singular variety over K and $(\mathcal{E},\nabla)$ an integrable connection on U. Then the Gauss-Manin connection $\nabla_{GM} : H^q(U, DR_{U/K}(\mathcal{E})) = H^q(U,\mathcal{E}\otimes\Omega^\bullet_{U/K}) \to H^q(U, DR_{U/K}(\mathcal{E}))\otimes\Omega^1_{K/k}$ is defined on the relative de Rham cohomology. It is a counterpart of the ℓ-adic representation $H^q(U_{\bar{F}},\mathcal{F})$ of the absolute Galois group G_F for an ℓ-adic sheaf $\mathcal{F}$ on a smooth variety U over a field F of characteristic $p > 0$. For the latter, if F is finite and $\dim U = 1$, the product formula ([10],[22]) is the equality between the alternating product $\prod_q \det(-\mathrm{Fr} : H^q(U_{\bar{F}},\mathcal{F}))^{(-1)^q}$ and the product of the local ϵ-factors.

In the case where U is a smooth curve over a function field K over a field k of characteristic 0, Beilinson, Bloch and Esnault study the determinant $\det(R\Gamma(U, DR_{U/K}(\mathcal{E})),\nabla_{GM})$ of the Gauss-Manin connection as a connection on the K-vector space of dimension 1. Recently, in [3] they obtained a formula quite analogous to the product formula mentioned above. The analogy between wild ramification and irregular singularities is mysterious and quite interesting.

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