HOW TO BOUND THE SUCCESSIVE MINIMA ON ARITHMETIC VARIETIES

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Abstract

I would like to explain a new method to bound the last successive minima from above that are associated to high powers of a hermitian line bundle $\overline{L}$ on a normal projective arithmetic variety X. As applications, we can prove the following results.

1) The last successive minima are generally bounded from above.
2) The sequence defining the sectional capacity of $\overline{L}$ converges.
3) The sectional capacity of $\overline{L}$ is Lipschitz continuous and birationally invariant.
4) Necessary and sufficient conditions for $H^0(X, mL)$ to have free basis consisting of strictly small sections for sufficiently large m.
5) A generalization of the theorem of successive minima of S. Zhang [Zha95b]. In particular, we can reprove the general equidistribution theorem for rational points of small heights, which was first proved by Berman-Boucksom [BB10] by using the Monge-Ampere operators.

What is ... Arakelov Geometry?

- *Arithmetic varities*: A *projective arithmetic variety* is a reduced irreducible scheme X with flat projective structure morphism $X \to \mathrm{Spec}(\mathbf{Z})$. We always suppose that X is normal and that the generic fiber $X_{\mathbf{Q}} := X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{Q})$ is smooth over $\mathrm{Spec}(\mathbf{Q})$.
- *Arithmetic divisors*: A *continuous hermitian line bundle* is a couple $\overline{L} = (L, |\cdot|_{\overline{L}})$ of a line bundle L on X and a continuous hermitian metric $|\cdot|_{\overline{L}}$ on $L(\mathbf{C})$ that is invariant under the complex conjugation. The *supremum norm* of a section $s \in H^0(X, L) \otimes_{\mathbf{Z}} \mathbf{R}$ is defined as
$$\|s\|_{\sup}^{\overline{L}} := \sup\left\{|s(x)|_{\overline{L}} \,|\, x \in X(\mathbf{C})\right\}.$$
- *Last successive minima* $\lambda_{\max}(\overline{L})$: The *last successive minimum* of $\overline{L}$ is the least positive real number $\lambda > 0$ such that the set
$$\{s \in H^0(X, L) \,|\, \|s\|_{\sup}^{\overline{L}} \leq \lambda\}$$
generates the vector space $H^0(X, L) \otimes_{\mathbf{Z}} \mathbf{Q}$ over $\mathbf{Q}$ and denoted by $\lambda_{\max}(\overline{L})$.
- *Height functions*: For a continuous hermitian line bundle $\overline{L}$ (or an arithmetic $\mathbf{R}$-divisor, or more generally an arithmetic $\mathbf{R}$-divisor with log log-singularity along a snc-divisor... etc), we can associate a height function
$$h_{\overline{L}} : X(\overline{\mathbf{Q}}) \to \mathbf{R}.$$
We would like to produce many *useful* height functions to study distribution of rational points.

A New Technique

In [Zha95a], S. Zhang proved the famous arithmetic Nakai-Moishezon criterion by using the following techniques.

Zhang's Techniques

① The L^2-method (extension of sections with norm estimates). This technique works only when $L_{\mathbf{Q}}$ is ample over $X_{\mathbf{Q}}$. We can finally conclude that this is not neccessary to prove the existence of free basis of strictly small sections.

② Choose a suitable filtration
$$-L = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = O_X$$
and bound $\lambda_{\max}$ by successive quotients. In bounding the successive minima of the quotients $mL \otimes I_i/I_{i-1}$, the *finitely-generatedness* of the section ring of L plays a vital role.

Inspired by Moriwaki's work, the author [Iko12] found out a new technique that can be generally used to bound the successive minima.

A New Technique

③ Effective use of restricted cohomology and nilpotent sections. Let Y be a closed arithmetic subvariety of X. A multiple $s^{\otimes n}$ of a nonzero global section s (with many good properties) defines a NON-reduced subscheme $nY' := \mathrm{div}\left((s|_Y)^{\otimes n}\right)$ of X. Then the sequence
$$\begin{aligned} 0 &\to H^0(X|Y', mL - (k+n)A) \\ &\to H^0(X|(n+1)Y', mL - kA) \\ &\to H^0(X|nY', mL - kA) \to 0 \end{aligned}$$
is EXACT. By choosing suitable s, we can deduce many strong results on the successive minima.

Main Theorems

Main Theorem 1 [Iko12]

Let $\|\cdot\|^{(m)}$ be a norm on $H^0(X, mL)_{\mathbf{R}}$. Suppose that,

- for any non-zero $s \in H^0(X, m_0L)$, there are two positive constants $\sigma_0(s), \tau_0(s) > 0$ such that the inequalities
$$\tau_0(s)^{m+k} \cdot \|u\|^{(m)} \leq \|s^{\otimes k} \otimes u\|^{(m+m_0k)} \leq \sigma_0(s)^{m+k} \cdot \|u\|^{(m)}$$
hold for any $m, k \geq 0$ and for any $u \in H^0(X, mL)$.

Then there are positive constants m_0, σ, and a non-zero polynomial P with non-negative real coefficients such that the inequality
$$\lambda_{\max}(H^0(X, mL), \|\cdot\|^{(m)}) \leq P(m) \cdot \sigma^m$$
holds for any $m \geq m_0$.

The supposition in the above theorem is sufficiently general. In fact
$$\log \lambda_{\max}(m_1\overline{L}_1 + \cdots + m_r\overline{L}_r) \leq O(|m_1| + \cdots + |m_r|)$$
follows from the above theorem.

Main Theorem 2 [CI12]

Let $\delta > 0$ be any positive real number. Suppose that

- $L_{\mathbf{Q}}$ is big on $X_{\mathbf{Q}}$ and
- there are positive integers $n_1, \ldots, n_N$ and sections $s_1 \in H^0(X, n_1L), \ldots, s_N \in H^0(X, n_NL)$ having the properties that
$$\{x \in X_{\mathbf{Q}} \,|\, s_1(x) = \cdots = s_N(x) = 0\} = \mathrm{SBs}(L_{\mathbf{Q}})$$
and that
$$\max\left\{\left(\|s_j\|_{\sup}^{n_j\overline{L}}\right)^{1/n_j}\right\} < \exp(-\delta).$$

Then there is a positive constant $C > 0$ such that
$$\lambda_{\max}(m\overline{L}) \leq Cm^{\dim X(\dim X - 1)/2}\left(\exp(\delta)\max\left\{\left(\|s_j\|_{\sup}^{n_j\overline{L}}\right)^{1/n_j}\right\}\right)^m$$
for all sufficiently large $m \geq 1$.

Theorem of successive minima

Let $\overline{\xi}$ be an arithmetic $\mathbf{R}$-divisor with big generic fiber $\xi_{\mathbf{Q}}$. Then
$$\begin{aligned} \frac{1}{\dim X}\sum_{i=1}^{\dim X} \sup_{Y \in (X\setminus \mathrm{Bs}_+(\xi))_{\mathbf{Q}}^{(i)}} \left\{\inf_{x \in (X\setminus(\mathrm{Bs}_+(\xi)\cup Y))(\overline{\mathbf{Q}})} h_{\overline{\xi}}(x)\right\} &\leq \frac{\widehat{\mathrm{vol}}^{\hat{\chi}}(\overline{\xi})}{\dim X \mathrm{vol}(\xi_{\mathbf{Q}})} \\ &\leq \sup_{Y \in (X\setminus \mathrm{Bs}_+(\xi))_{\mathbf{Q}}^{(1)}} \left\{\inf_{x \in (X\setminus(\mathrm{Bs}_+(\xi)\cup Y))(\overline{\mathbf{Q}})} h_{\overline{\xi}}(x)\right\}. \end{aligned}$$

References

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