Analytic singular homology of non-Archimedean analytic spaces and Shnirel'man integral of differential forms along analytic cycles

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1 Introduction

Let C be a complete algebraically closed non-Archimedean field with characteristic 0 and residual characteristic $p > 0$, $\mu_n \subset C^\times$ the subgroup of n-th roots of unit for each $n \in \mathbb{N}$, and $\mu \subset C^\times$ the subgroup $\bigcup_{n \in \mathbb{N}} \mu_n$ of roots of unit. Fix points $a \in C$ and $b \in C^\times$, and take an arbitrary subset $D \subset C$ containing the "circle" $\gamma(a,b) \equiv a + b\mu = \{a + b\xi \mid \xi \in \mu\}$ with a centre at a and radius b. For a map $f \colon D \to C$, Shnirel'man defined the integral of f along $\gamma(a,b)$ by the following infinite sum:

$$\int_{\gamma(a,b)} f(z)dz \equiv \lim_{\substack{n\to\infty \\ p \nmid n}} \sum_{\xi \in \mu_n} f(a + b\xi) b\xi^{-1} \in C.$$

Note that this limit does not necessarily converge for an arbitrary map f even if f is a locally analytic function such as a locally constant function, and the integrals on $\gamma(a,b')$ and $\gamma(a,b'')$ differ even if $|b'| = |b''|$.

Theorem 1.1 (Shnirel'man)**.** *If $f \colon D \to C$ is analytic on $\gamma(a,b) \subset D$, i.e. there exists some $g \in C\{|b|^{-1}T, |b|T^{-1}\}$ such that the restriction of f on $\gamma(a,b)$ coincides with g, then the integral of f along $\gamma(a,b)$ converges and is independent of the choice of $b \in C^\times$. Furthermore the integral induces a continuous linear functional*

$$\int_{\gamma(a,b)} dz \colon k\{|b|^{-1}T, |b|T^{-1}\} \to k$$

$$f = \sum_{i > -\infty} a_i T^i \mapsto \int_{\gamma(a,b)} f(z)dz = Res(f,a) \equiv a_{-1}$$

for each closed subfield $k \subset C$, where $Res(f,a)$ is the residue of f at a. Therefore Shnirel'man integral can be applied to an analytic function also for a not necessarily algebraically closed base field k such as $\mathbb{Q}_p$.

This is an obvious analogue of the complex integral of a meromorphic function. The counterpart of a cycle in the complex integral is a non-oriented "circle" $\gamma(a,b) = a + b\mu$ each of whose point is presented as a linear expression of a root of unit. Comparing it with the complex residue formula

$$\int_{\theta=0}^{2\pi} f(z(\theta))dz(\theta) = \pm 2\pi i \mathrm{Res}(f,a),$$

where $z(\theta) \equiv a + re^{\pm 2\pi i \theta}$ for $a \in \mathbb{C}$ and $r \in [0,\infty)$, it is remarkable that no counterparts of the signature $\pm$ corresponding to the orientation of the cycle and the period $2\pi i$ appear in Shnirel'man integral. The reason of these absence is because the analogue of $\pm 2\pi i$ with an orientation $\pm 1 \in \mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}^\times$ does not belong to the base field k, and therefore the definition of Shnirel'man integral is normalised so that such a constant disappears. In fact, the constant is the period $\log \epsilon$ lying in Fontaine's p-adic period ring B_{dR} with an orientation in $\mathbb{Z}_p(1) \subset B_{\mathrm{dR}}^\times$. Our generalisation of Shnirel'man integral is obrained by regarding a linear expression of power root systems of elements in $\overline{k}^\times$ as an analytic path and using a period in B_{dR}.

	over $\mathbb{C}$	over $\mathbb{Q}_p$
unit circle	S^1	μ
orientation	$\pm 1 \in \mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}^\times$	$\epsilon \in \mathbb{Z}_p(1) \subset B_{\mathrm{dR}}^\times$
period	$\pm 2\pi i \in \mathbb{C}^\times$	$\log \epsilon \in B_{\mathrm{dR}}^\times$

2 Singular homology and integration

Let k be a local field with characteristic 0 and residual characteristic $p > 0$. Set $\mathbb{Q}^* \equiv \mathrm{Hom}(\mathbb{Q}, \overline{k}^\times)$, and regarding a character $x \in \mathbb{Q}^*$ as an exponential-like analytic function from $[0,1] \cap \mathbb{Q} \to k$, define the ring $k_{[0,1]}$ of analytic functions $[0,1] \to k$ as the completion of the algebra $k[\mathbb{Q}^*]$ with respect to a certain norm. For a k-affinoid space $X = \mathcal{M}(\mathcal{A})$, we define an analytic path $[0,1] \to X$ as a bounded homomorphism $\mathcal{A} \to k_{[0,1]}$. Similarly, define the ring k_I of analytic functions $I \to k$ for a rectangle or a standard simplex $I \subset \mathbb{R}^n$ of an arbitrary dimension, and the set of analytic path $I \to X$. By the well-accustomed way, one obtains the chain complices of analytic singular simplices and analytic cubic singular simplices, and the analytic singular homologies. Also for a non-affinoid analytic space X, we defined the analytic singular homologies of X by some complicated gluing argument.

Definition 2.1.

$$C_n(X,R) \equiv R^{\oplus\{n\text{-(cubic) singular simplex}\}} \qquad (R;\ \textit{commutative ring})$$
$$H_*(X,R) \equiv H_*(0 \to C_0(X,R) \xrightarrow{d} C_1(X,R) \to \cdots)$$

We introduced the notion of an overconvergent analytic function on I. For example, an element $f = \sum_{x \in \mathbb{Q}^*} f_x x \in k[\mathbb{Q}^*] \subset k_{[0,1]}$ is said to be overconvergent on $[0,1]$ if $f_x(x(1)-1) = 0$ for any $x \in \mathbb{Q}^*$ whose restriction on $\mathbb{Z}[p^{-1}]$ is torsion. Denote by $k_I^\dagger$ the ring of an overconvergent analytic functions on I. We defined the integral of an element of $k_I^\dagger$. For an element $f \in k[\mathbb{Q}^*] \cap k_{[0,1]}^\dagger$, the integral of f on $[0,1]$ is given by the following way:

Definition 2.2.

$$\int_0^1 f dt \equiv f_1 + \sum_{x \in \mathbb{Q}^* \backslash \{1\}} f_x \frac{x(1)-1}{\log x} \in B_{dR},$$

where $\log x \in B_{dR}$ is the logarithm

$$\log x \equiv -\log\left(1 - \left(1 - \frac{[x|_{\mathbb{Z}[p^{-1}]}]}{x(1)}\right)\right) = \sum_{i=1}^\infty \frac{1}{i}\left(1 - \frac{[x|_{\mathbb{Z}[p^{-1}]}]}{x(1)}\right)^i$$

and $[x|_{\mathbb{Z}[p^{-1}]}]$ is the Teichmüller embedding of $x|_{\mathbb{Z}[p^{-1}]} \in Hom(\mathbb{Z}[p^{-1}], \overline{k}^\times)$ into B_{dR}, identifying $Hom(\mathbb{Z}[p^{-1}], \overline{k}^\times) \cong \varprojlim_{z \mapsto z^p} \overline{k}^\times$.

Pulling back to I, we defined the integral of a differential form on X along a path $I \to X$, and it induces the pairing of the space of holomorphic i-forms on X and i-th analytic singular homology group of X. This pairing satisfies Stokes' theorem, and hence it is useful to show whether a given analytic cycle is homologically trivial or not. For example, it is easy to verify that the analytic singular homology reflects the central hall of the open torus $\mathbb{G}_{mk}$, which is topologically simply connected, and the two specific loops of a Tate curve $\mathbb{G}_{mk}/q^{\mathbb{Z}}$, which is homotopic to S^1. In general the analytic singular homology has much more information of the analytic structure of an analytic space than the singular homology of the underlying topological space does.

3 Relation to cohomology

If X is a Stein space or an affinoid space, then the Hodge to de Rham spectral sequence induces the isomorphism between the space of holomorphic differential forms and the de Rham cohomology, and hence we obtained the canonical pairing

$$H_*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^*_{\mathrm{dR}}(X) \to B_{\mathrm{dR}}.$$

It also induces a pairing with the étale cohomology and the analytic singular homology in the case X is an analytification of an affine variety by the p-adic Hodge theory for an open algebraic variety. We are now trying to generalise these pairing to wider class.

4 References

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