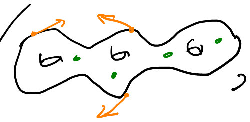
Tokyo 2017.05.25

Ihara Curves

Today: a more arithmetic flavor

Recall:

(1) $\Gamma_{g,n+\vec{r}} = \pi_0 \, \mathrm{Diff}^+\big($  , fix pts & tangent vecs $\big)$

genus g, n points, r tangent vectors

Always assume:

$$2g-2+r+n>0$$

(2) 

real oriented blow up replaces tangent vector by a boundary cpt.

(3) Relative unipotent completion of $\Gamma_{g,n+\vec{r}}$:

$$1 \to \mathcal{U}_{g,n+\vec{r}} \to \mathcal{G}_{g,n+\vec{r}} \to Sp_g \to 1$$

prounipotent

$$T^{un}_{g,n+\vec{r}} \to \mathcal{U}_{g,n+\vec{r}}$$

Surjective all $g \geq 2$

kernel is $\mathbb{Q}$ all $g \geq 3$.

(kernel infinite dimensional $g=2$)

Lie algebras

$$\mathfrak{g}_{g,n+\vec{r}}, \ \mathfrak{u}_{g,n+\vec{r}}, \ \ldots$$

Geometric monodromy:

$(g, n+\vec{r}) = (g, \vec{1})$

C, $P \in C$, $\vec{v} \in T_pC$, $C' = C - \{P\}$

$\pi = \pi_1(C', \vec{v})$, $\mathfrak{p} = \mathrm{Lie}\, \pi^{un}$

$\mathrm{Gr}^{\bullet}_{LCS}\, \mathfrak{p} \cong \mathbb{L}(H)$, $H = H_1(C)$

$$\Gamma_{g,\vec{1}} \to \mathrm{Aut}\, \pi$$

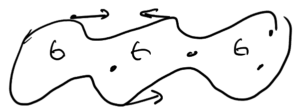
induces

$$\mathfrak{g}_{g,\vec{1}} \to \mathrm{Der}\, \mathfrak{p}$$

Motivic structures

For each choice of a point in

$\mathcal{M}_{g,n+\vec{r}}$:= moduli stack of complex structures on



There is a canonical MHS on

$$\mathcal{O}_{g,n+\vec{r}},\ \mathfrak{u}_{g,n+\vec{r}},\ \dots$$

Also have MHS on $\mathfrak{p}$ above.
and so on $\mathrm{Der}\,\mathfrak{p}$.

Thus have homom

$$\begin{tikzcd} \pi_1(MHS) \arrow[r, "\varphi"] & \mathrm{Aut}(\mathrm{Der}\,\mathfrak{p}) \otimes \mathbb{Q}_\ell \\ & G_K \arrow[u, "\varphi_\ell"'] \end{tikzcd}$$

when $(C, P, \vec{v})$ defined $/K$.

Conjecturally: $(\operatorname{im}\varphi)\otimes\mathbb{Q}_\ell = \left\{\begin{array}{l}\text{Zariski closure of}\\ \operatorname{im}\varphi_\ell .\end{array}\right.$

"Mumford–Tate group"

Today, interested in C where the Mumford–Tate group is

$$\pi_1(MTM(\mathbb{Z}), DR) = \mathbb{G}_m \ltimes \mathcal{K}$$

where "the motivic Lie algebra"

$$\operatorname{Lie}\mathcal{K} =: \mathfrak{k} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \dots)$$

Here $t \in \mathbb{G}_m$ acts on σ_{2m-1} by

$$t \cdot \sigma_{2m-1} = t^{2m-1}\sigma_{2m-1}$$

Def: $\mathbb{Q}(n)$ is the 1-dimensional rep

$$\pi_1(MTM) \to \mathbb{G}_m \to \mathbb{G}_m, \quad t \mapsto t^n$$

cyclotomic char

ℓ-adic realiz^n: $\chi_\ell^{\otimes n}$; Hodge realiz^n: type $(-n,-n)$

Remark: The proof of Oda's Conj (Takao; Ihara, Nakamura, Matsumoto, ...) and Brown's fundamental result imply that this is the smallest possible.

Why work with tangent vectors?

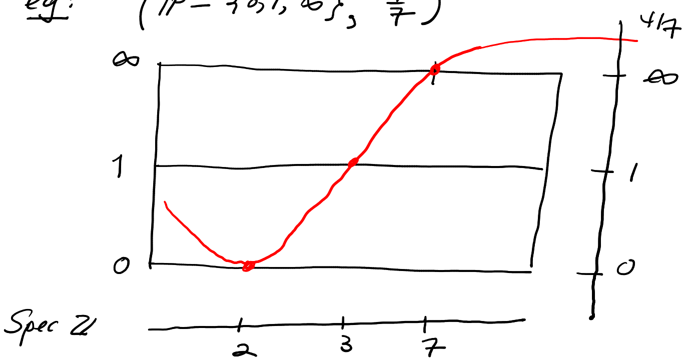
(1) varieties/stacks over $\mathbb{Z}$ with everywhere good reduction give motives, unramified $/\mathbb{Z}$. Such varieties/stacks seem to be rare:

Examples: • $\mathbb{P}^N_{/\mathbb{Z}}$, flag varieties

• $\mathcal{M}_{g,n+\vec{r}}$

(2) we're interested in motives in completions of fundamental groups of pairs (X,x) defined $/\mathbb{Z}$ with everywhere good reduction.

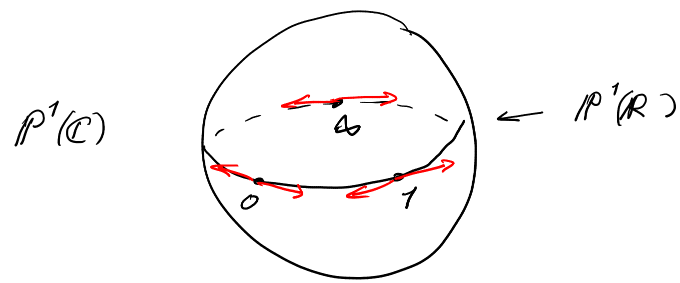
eg: $(\mathbb{P}^1 - \{0,1,\infty\}, \frac{4}{7})$



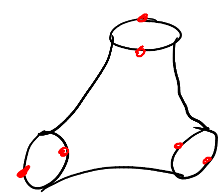
bad reduction at $\{2, 3, 7\}$.

Forced to use tangential base points:

- w natural coord on $\mathbb{P}^1$
- $\pm \partial/\partial w \in T_1 \mathbb{P}^1$ and its Σ_3 conjugates have everywhere good reduction:



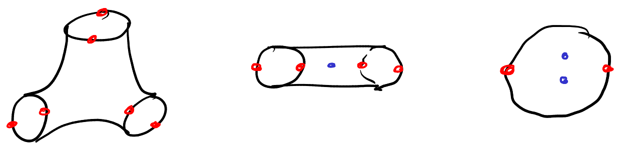
Real oriented blow up:



"indexed pants"

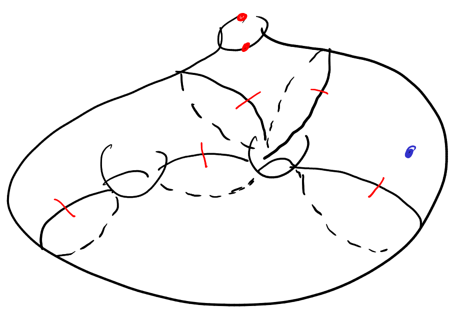
Ihara Curve of type $(g, n+\vec{r})$

Rough idea: These are curves built up from the following pieces



But the red dots have to match.

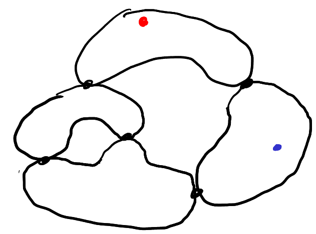
After that, erase them!



"indexed" pants decomposition

Ihara curve of type $(2, 1+\vec{1})$

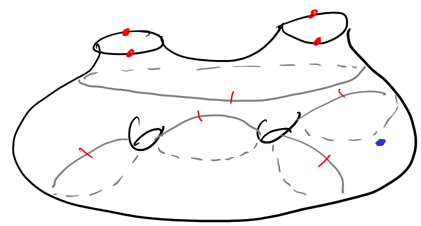
The curves in the pants decomposⁿ can be contracted to get a stable nodal curve:



$\in \overline{\mathcal{M}}_{2,2}$

It is maximally degenerate.

Path torsor of an Ihara curve



Objects: remaining tangent vectors + blue points.

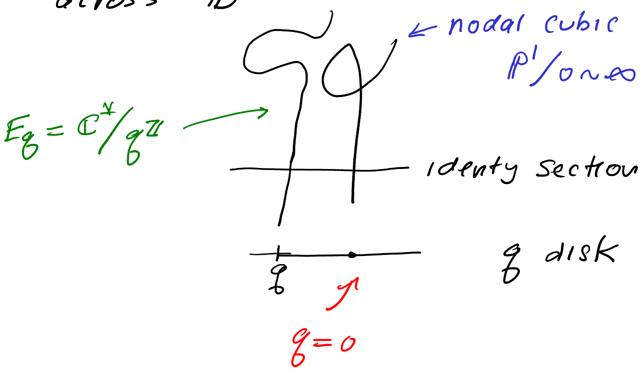
Arithmetic version :

(1) Tate curve: $\mathcal{M}_{1,1}^{an} = SL_2(\mathbb{Z})\backslash \mathfrak{h}$ ← upper ½ plane

$\overline{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup_{\mathbb{D}^*} \mathbb{D}$ ← q disk

$\mathbb{D}^* = \begin{pmatrix}\text{punctured} \\ q\text{-disk}\end{pmatrix} = \begin{pmatrix}1 & \mathbb{Z} \\ 0 & 1\end{pmatrix}\backslash \mathfrak{h}, \quad q = e^{2\pi i \tau}$

Universal elliptic curve extends across $\mathbb{D}$



In a formal neighbourhood of $q=0$, this is defined $/\mathbb{Z}$

$$\begin{array}{c}\mathcal{E}\\ \Big| \\ \mathbb{Z}[\![q]\!]\end{array}$$

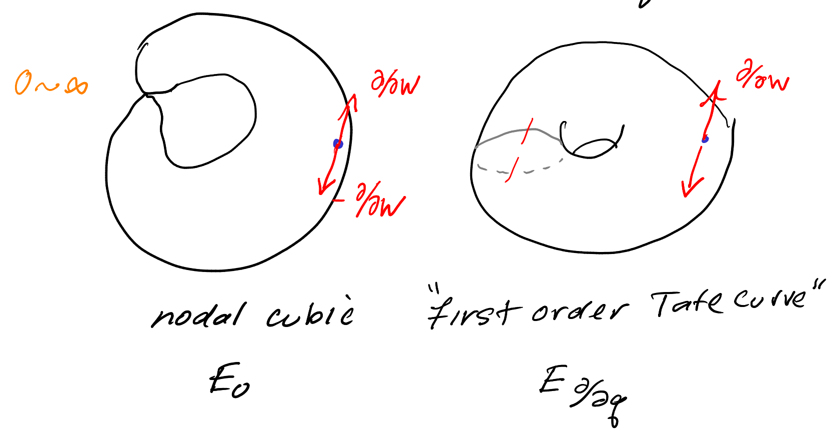
$$\text{Discriminant} = q\prod_{n\geq 1}(1-q^n)^{24}$$

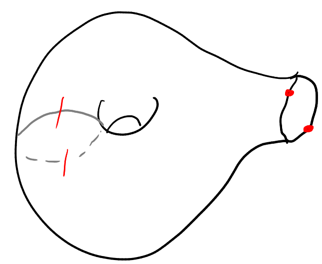
$$= \text{cusp form of wt } 12$$

This is non-zero in $\mathbb{F}_p[q]/(q^2)$ for all primes p.

So the first order Tate curve $E_{\partial/\partial q}$ has everywhere good reduction.

Smoothing of node over $\partial/\partial q$:



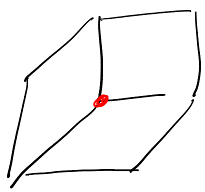


$$E'_{\partial/\partial q} = E_{\partial/\partial q} - \{id\}$$

Higher genus: Have

$\overline{\mathcal{M}}_{g,n}$ = Deligne – Mumford compactification of $\mathcal{M}_{g,n}$

- smooth orbifold
- $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ = divisor with normal crossings



0-dimensional strata of boundary are maximally degenerate stable curves

- They correspond to pants decompositions of $(S, \{x_1, \dots, x_n\})$.
- each component is $\mathbb{P}^1$ with 3 marked points.
- Ihara – Nakamura constructed a smooth curve

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \mathbb{Z}[[q_1, \dots, q_N]] \end{array} \qquad N = 3g-3+n$$

q_j smooth j^{th} node.

For $\partial/\partial q = \pm \partial/\partial q_1 \pm \dots \pm \partial/\partial q_n$

$C_{\partial/\partial q}$ = fiber over $\partial/\partial q$ is smooth, everywhere good reduction.

We use such $\partial/\partial q$ as basepoints for $\pi_1(\underbrace{\mathcal{M}^{an}_{g,n}, \partial/\partial q}_{\text{everywhere good reduction}})$.

$\Downarrow$

motives unramified $/\mathbb{Z}$

Theorem: The unipotent completion of the path torsor of an Ihara curve is an object of $\mathrm{MTM}(\mathbb{Z})$.

In particular, if $(C, \vec{v})$ is an Ihara curve of type $(g, \vec{1})$, then

$$\mathfrak{p} = \mathrm{Lie}\, \pi_1^{un}(C', \vec{v})$$

is a pro-object of $\mathrm{MTM}(\mathbb{Z})$.

There is therefore an action of

$$\pi_1(\mathrm{MTM}) \circlearrowleft \mathfrak{p}$$

$$\| $$

$$\mathbb{G}_m \ltimes \mathcal{K}$$

So we have a homomorphism

$$\varphi_P = \underline{\mathfrak{k}} = \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \dots) \longrightarrow \mathrm{Der}\, \mathfrak{p}$$

pants decomp. not yet graded!

This depends on, and is determined by, the "indexed pants decomposition" P corresponding to the Ihara curve C.

depends on P

Problem: Compute φ_P, and compute $\mathrm{im}\, \varphi_P$ / image of $\mathfrak{u}_{g,\vec{1}}$.

independent of P Cf. Morita's Conjecture

Have joint project with Francis Brown to do this. It is "bottom up", in contrast to GRT, which is "top down".

NOTE THAT WE HAVE NOT APPLIED $Gr^W_\bullet$ YET.

The wrinkle: The MHS on $\mathfrak{p}$ is a "limit MHS." because of this, it acquires a second weight filtration $M_\bullet$, called the "relative weight filtration". This requires some explanation.

The weight filtration $W_\bullet$ on $\mathfrak{p}$ is still its LCS:

$$W_{-m}\mathfrak{p} = L^m \mathfrak{p}$$

But there is the relative weight

filtration $M_\bullet$:

$$\cdots \subseteq M_j\,\mathfrak{p} \subseteq M_{j+1}\,\mathfrak{p} \subseteq \cdots$$

It induces a filtration on each $Gr^W_j\,\mathfrak{p}$. In particular, it induces one on

$$H = Gr^W_{-1}\,\mathfrak{p}.$$

To see what it is, note that every Ihara curve $C_{\partial/\partial q}$ naturally bounds a handle body U:

$$C_{\partial/\partial q} = \partial U$$

It is characterized by the fact that every vanishing cycle of $C_{\partial/\partial q}$ (ie every curve in the pants decomp) bounds in U. One then has the sequence

$$\begin{tikzcd}
0 \arrow[r] & M_{-2}H_1(C_{\partial/\partial q}) \arrow[r] \arrow[d, equal] & M_0 H_1(C_{\partial/\partial q}) \arrow[r] \arrow[d, equal] & Gr^M_0 H_1(C_{\partial/\partial q}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker j_* \arrow[r] & H_1(C_{\partial/\partial q}) \arrow[r, "j_*"] & H_1(U) \arrow[r] \arrow[d, "\cong"] & 0 \\
 & & & H_1(C_0) &
\end{tikzcd}$$

spanned by the vanishing cycles

the nodal curve

with this weight filtration, H is a split MHS isomorphic to

$$\mathbb{Q}(0)^g \oplus \mathbb{Q}(1)^g$$

M wt: 0 -2 average $= -1 =$ wt H

where $\mathbb{Q}(n)$ is the Hodge structure of type $(-n,-n)$.

The relative weight filtration $M_\bullet$ on H extends, by linear algebra, to the relative weight filtration on each irreducible representation $\mathbb{S}^{\langle\lambda\rangle}H$ of $Sp(H)$.

The local monodromy operator is

The product of the Dehn twists on the vanishing cycles. Let

$$N : \mathfrak{p} \to \mathfrak{p}$$

be its logarithm. Then the relative weight filtration of $\mathfrak{p}$ satisfies:

(1) $N M_k \mathfrak{p} \subseteq N M_{k-2} \mathfrak{p}$

(2) it is preserved by bracket:

$$[M_k \mathfrak{p}, M_l \mathfrak{p}] \subseteq M_{k+l} \mathfrak{p}$$

(3) The induced filtration on H is the one described above.

Ref: Exposition in my paper in Morita 60 volume.

The limit MHS on $\mathfrak{p}$ has weight filtration $M_\bullet$. It is filtered by $W_\bullet$.

$$g_P : \mathfrak{k} \to \mathrm{Der}\, \mathfrak{p}$$

is a morphism of MHS, where

$$\sigma_{2m-1} \in M_{-4m+2}\, \mathfrak{k}.$$

We have to show that we can split

$W_\bullet$ as well as $M_\bullet$.

THM: There are natural (though not canonical splittings of $M_\bullet$ and $W_\bullet$ of the Betti realization

The DR realization has a canonical bigrading that splits $M_\bullet, F^\bullet, W_\bullet$.

§ The elliptic case: Universal mixed elliptic motives

Collaborators: Makoto Matsumoto
Francis Brown — Mixed modular motives

Precursors:
Beilinson–Levin: elliptic polylogs
Nakamura: ℓ-adic case

Both: Meta-abelian quotient of $\mathfrak{p}$

Related: Work of Enriquez

$$\mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{G}_m - \{1\} \hookrightarrow E_{\partial/\partial q} - \{id\}$$

$$\mathfrak{p} = \mathrm{Lie}\, \pi_1^{un}(E'_{\partial/\partial q}, \partial/\partial w)$$

$$H = H_1(E_{\partial/\partial q})$$

$$= \mathbb{Q}T \oplus \mathbb{Q}A$$ A, T is $\mathbb{Q}$-DR basis

$$= \mathbb{Q}(0) \oplus \mathbb{Q}(1)$$

log monodromy: $A\, \partial/\partial T$

$$A, T \in Gr^W_{-1}\, \mathfrak{p}$$

$$A \in Gr^M_{-2} H, \quad T \in Gr^M_0 H$$

Canonical DR splitting:

$$\mathfrak{p} \cong (Gr^W_\bullet Gr^M_\bullet\, \mathfrak{p})^\wedge$$

$$Gr^W_\bullet Gr^M_\bullet\, \mathfrak{p} = \mathbb{L}(H)$$

$$= \mathbb{L}(A, T)$$

with obvious bigrading. //

$\mathbb{P}^1 - \{0, 1, \infty\} \hookrightarrow E_{\partial/\partial q}$ induces

k equivariant, M-graded

$$\left\{ \begin{aligned} \mathbb{L}(X_0, X_1) &\longmapsto \mathbb{L}(A, T)^\wedge \\ X_0 &\longmapsto \frac{T}{e^T - 1} \cdot A \\ X_1 &\longmapsto [T, A]. \end{aligned} \right.$$

$u \cdot v := ad_u(V)$

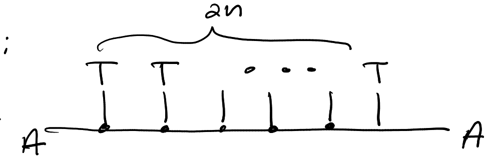
W wt \ M wt	0	−2	−4	−6	−8	
−1	T	A	0	0	0	
−2	0	T, A	0	0	0	
−3	0	T^2, A	A^2, T	0	0	
−4	0	$T^4 \cdot A$	*	$A^4 \cdot T$	0	
	0	*	*	*	*	0

θ = symplectic form

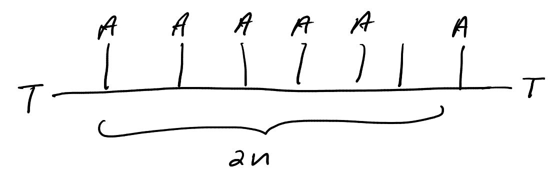
M=W line.

$$\mathrm{Der}^\theta \mathbb{L}(H) = Gr^M_\bullet Gr^W_\bullet \mathrm{Der}^\partial \mathfrak{p}$$

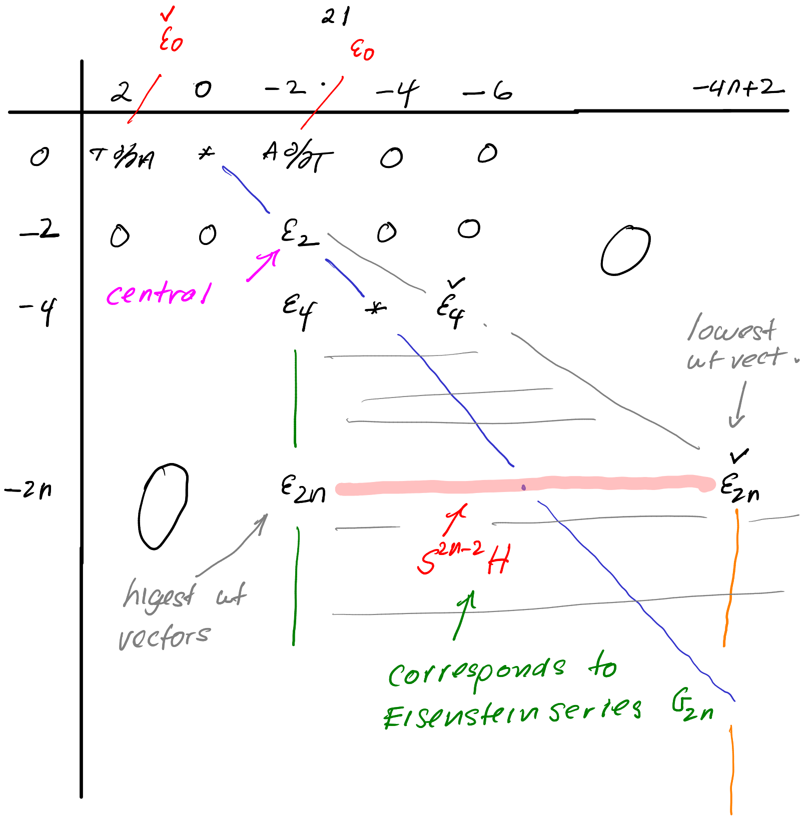
Derivations:

$\varepsilon_{2n} =$ 

$$\in Gr^W_{-2n} Gr^M_{-2} \mathrm{Der}^\theta \mathbb{L}(H)$$

$\check{\varepsilon}_{2n} =$ 

$$\in Gr^W_{-2n} Gr^M_{-4n+2} \mathrm{Der}^\theta \mathbb{L}(H)$$



The $Gr^M_\bullet Gr^W_\bullet$ bigrading of the image of

$$\mathfrak{g}_{1,\vec{1}} \longrightarrow \mathrm{Der}^\partial \mathfrak{p}$$

Genus 1 Johnson homom!

The image of $\mathfrak{u}_{1,\vec{1}}$ is generated by the ε_{2n} $n \geq 0$.

What about $\underline{k} \longrightarrow \mathrm{Der}^\theta \mathbb{L}(H)$?

It is M-graded

$$\sigma_{2m-1} \in Gr^M_{-4m+2}\, \underline{k} \longrightarrow Gr^M_{-4m+2} \mathrm{Der}^\theta \mathbb{L}(H)^\wedge$$

Theorem:

all geometric

$$\sigma_{2m-1} \mapsto \check{\varepsilon}_{2m} + \underbrace{\zeta_{2m-1} + \cdots}$$

where $\zeta_{2m-1} \in \left[Gr^W_{-4m+2} \mathrm{Der}^\theta \mathbb{L}(H)\right]^{SL(H)}$

$$\cap\!|$$

$$Gr^M_{-4m+2}\, Gr^W_{-4m+2} \mathrm{Der}^\theta \mathbb{L}(H).$$

Various proofs:

(1) elliptic polylogs

(2) Nakamura's Galois computations

(3) Brown's period computations.

Cor Ihara–Takao congruence

pf Pollack relations.

<u>Rk</u>: The ζ_{2m-1} generate a free Lie algebra mod geometric derivations. (Consequence of proof of the Oda Conjecture by Takao, Ihara, Nakamura, Matsumoto, ... + Francis Brown's big injectivity result.

<u>Dehn twist</u>: acts on $\mathfrak{p}$. Is logarithm is

$$N := \sum_{n=0}^{\infty} (2n-1) \frac{B_{2n}}{(2n)!} \varepsilon_{2n} \quad \in Gr^{M}_{-2} Der^{\theta}$$

Constraint:

$$[N, \sigma_{2m-1}] = 0 \quad \text{all } m \geq 2$$

§ <u>Higher genus</u>: $(g \geq 2)$

Morita Conjecture:

(1) There is a unique copy of S^{2n-1} in $Gr^{W}_{-2n+1} \; Der^{\theta} \mathbb{L}(H)$

It is detected by the Morita

trace.

(2) The image of bracket

$$\left(\Lambda^2 S^{2n-1} H\right)^{Sp_g} \longrightarrow Gr^{W}_{-4n+2} Gr^{M}_{-4n-2} Der^{\theta} \mathbb{L}(H)$$

<u>Conjecture</u> (Morita): These elements generate the image of

$$\varphi_p : \mathfrak{k} \longrightarrow Der^{\theta} \mathbb{L}(H)$$

modulo the image of $\mathfrak{u}_{g,\vec{1}}$

geometric derivations

<u>Remarks</u>: Morita's conjecture, if true, is very strong as it would give canonical generators of $\mathfrak{k}$. More likely that image of $\sigma_{2m-1} \equiv$ Morita element mod $\text{Depth}^{\geq 2}$. This is what Brown and I expect to prove.