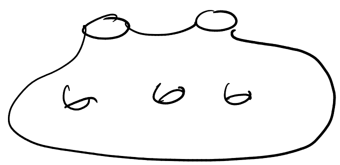
Tokyo 2017.05.23

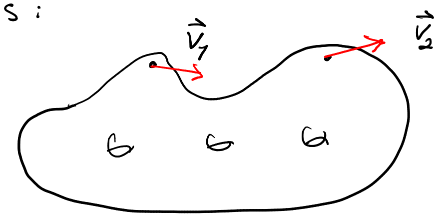
Johnson homomorphisms, stable and unstable

§1 Mapping class groups

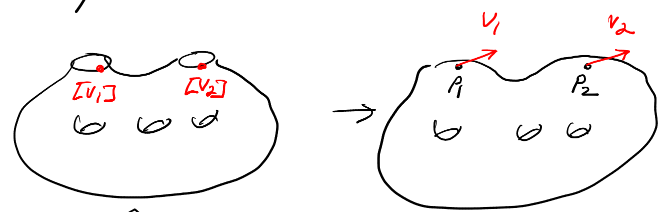
- We want to use tools from algebraic geometry (eg: Hodge theory, Galois actions)
- Topologists typically use surfaces with boundary:

$S =$ 

But these are not algebraic varieties. Instead they use non-zero tangent vectors:

$C =$ 

They are related by the "real oriented blow up":



$\hat{C} = \widehat{Bl}_{P_1,P_2} C$ $\qquad$ C

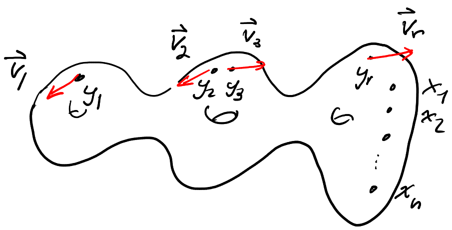
This allows us to define, for example,

$$\pi_1(C', \vec{v}) = \pi_1(\hat{C}, [\vec{v}])$$

where $\begin{cases} \vec{v} \in T_P C \text{ and } \vec{v} \neq 0. \\ C' = C - P \end{cases}$

Mapping class groups:

- S = compact, oriented surface of genus g
- $x_1, \ldots, x_n, y_1, \ldots, y_r$ $n+r$ distinct points
- $\vec{v}_1, \ldots, \vec{v}_r$ non zero tangent vectors $\vec{v}_j \in T_{y_j} S$



Assume that $\chi(S - \{x_1,\dots,x_n,y_1,\dots,y_r\}) < 0$

$$2g-2+r+n>0$$

Define:

$$\Gamma_{g,n+\vec r} := \pi_0 \operatorname{Diff}^+(S; x_1,\dots,x_n, \vec v_1,\dots,\vec v_r)$$

$$= \pi_0 \operatorname{Diff}^+(\widehat{Bl}_{\{y_1,\dots,y_r\}} S, \partial S \cup \{x_1,\dots,x_n\})$$

and

$$\mathcal{M}_{g,n+\vec r} := \left\{ \begin{array}{l} \text{complex structures on} \\ (S, \{x_1,\dots,x_n\}, \{\vec v_1,\dots,\vec v_r\}) \end{array} \right\}$$

= complex analytic orbifold

= "complex points" of a stack over $\mathbb{Z}$.

Have $\pi_1(\mathcal{M}_{g,n+\vec r}, *) \cong \Gamma_{g,n+\vec r}$

orbifold fundamental group

Torelli groups:

Have surjective homomorphism

$$\rho: \Gamma_{g,n+\vec r} \to \operatorname{Aut}^+ H^1(S;\mathbb{Z}) \cong Sp_g(\mathbb{Z})$$

$$T_{g,n+\vec r} := \ker \rho.$$

Monodromy action:

Have, for $g \geq 1$,

$$\Gamma_{g,\vec 1} \to \operatorname{Aut} \pi_1(S', \vec v)$$

free on 2g generators.

Set $\pi_{g,\vec 1} = \pi_1(S', \vec v)$. Its LCS by

$$\pi_{g,\vec 1} = \underbrace{L^1 \geq L^2 \geq L^3 \geq \cdots}_{\text{characteristic subgroups}}$$

Set

$$J^m \Gamma_{g,\vec 1} = \ker\{\pi_{g,\vec 1} \to \pi_{g,\vec 1}/L^{m+1}\}.$$

Johnson filtration: $J^0 \geq J^1 \geq J^2 \geq \cdots$

Then

① $J^0/J^1 \cong Sp_g(\mathbb{Z})$ as $J^1 = T_{g,\vec 1}$

② $J^m/J^{m+1} \hookrightarrow \operatorname{Hom}(H, L^{m+1}/L^{m+2})$

is $Sp(H)$ invariant, where $H = L^1/L^2$, when $m > 1$

③ $J^1/J^2 \cong \Lambda^3 H$ Johnson $g \geq 3$

④ $(J^2/J^3)\otimes\mathbb{Q} \cong V_{[2,2]} \quad g \geq 3.$ ⊞ "tambo"

(Morita, Hain)

irreducible $Sp(H)$ module corresponding to partition $[2,2]$

⑤ $(J^3/J^4)\otimes\mathbb{Q}$ was determined by Asada–Nakamura, Hain and $(J^4/J^5)\otimes\mathbb{Q}$ by Morita

$(J^5/J^6)\otimes\mathbb{Q}$ Morita–Suzuki–Sakasai

⑥ $\bigcap_{m\geq 1} J^m T_{g,\vec{1}}$ is trivial

So have a Lie algebra homomorphism

fixes $\Sigma[a_i,b_i]$

$$Gr_J^{\bullet} T_{g,\vec{1}} \to \mathrm{Der}^{\theta} Gr_{LCS}^{\bullet} \pi_{g,1}$$

A Lie algebra

canonically isomorphic to $\mathbb{L}(H)$

Questions: ① Are the topologies on $T_{g,\vec{1}}$ defined by LCS and J equivalent?

② Is the image generated by $Gr_J^1 T_{g,\vec{1}}$ after tensoring with $\mathbb{Q}$?

THM (Hain, 1997) When $g\geq 3$

① No ② Yes.

Remark: There are many similar monodromy homomorphisms, such as,

$$\Gamma_{g,n} \to \mathrm{Out}\,\pi_1(S-\{x_1,\dots,x_n\})$$

There are similar statements in these cases.

The goal of the rest of this talk is to explain this, and more.

Seek "cosmic" explanation.

§2 Tannakian categories and completions of groups

F = field of char 0

Γ = any group.

$\mathrm{Rep}_F(\Gamma)$ = category of f.dim reps of Γ in F vect-spaces.

Have faithful $\omega: \mathrm{Rep}_F(\Gamma) \to \mathrm{Vec}_F$.

"fiber functor"

Axiomatize (carefully) to get axioms of a neutral F-linear tannakian category.

<u>Theorem</u> (Tannaka duality: cf Deligne)
If $\mathcal{C}$ is an F-linear tannakian category & $\omega : \mathcal{C} \to \mathrm{Vec}_F$ is a fiber functor, then $\mathcal{C}$ is equivalent to the category of representations of

$$\pi_1(\mathcal{C}, \omega) := \mathrm{Aut}^{\otimes} \omega$$

This is an affine group / F. Equiv, it is a proalgebraic group. (Inverse limit of affine algebraic groups.)

<u>FACT</u>: Every affine F group is an extension

$$1 \to \mathcal{U} \to G \to R \to 1$$

prounipotent   pro reductive.

$\mathrm{Rep}(R)$ = semi-simple objects of $\mathcal{C}$.

$\mathcal{U}$ controls extensions of these.

Homological property:

$$\mathrm{Ext}^j_{\mathcal{C}}(F, V) = H^j(\pi_1(\mathcal{C}), V) \quad \text{Hochshild}$$
$$= H^j(\mathcal{U}, V)^R \quad \text{Serre}$$
$$= H^j(\mathfrak{u}, V)^R$$

Lie algebra of $\mathcal{U}$.

<u>Examples</u>

① Γ discrete group

$\mathcal{C}$ = category of unipotent reps of Γ / F

$$\Gamma^{un}_{/F} := \pi_1(\mathcal{C}, \omega)$$

Also: $\mathcal{O}(\Gamma^{un}) = \varinjlim \mathrm{Hom}(F\Gamma / I^n, F)$

Concrete:

$$\Gamma = \langle x_0, x_1 \rangle \cong F_2$$

$$\theta : \langle x_0, x_1 \rangle \to F\langle\langle X_0, X_1 \rangle\rangle$$
$$x_j \longmapsto e^{X_j}$$

$$\Gamma^{un} \cong \exp \mathbb{L}(X_0, X_1)^{\wedge}$$

$$\mathrm{Lie}\, \Gamma^{un} = \mathbb{L}(X_0, X_1)^{\wedge}.$$

② Relative Unipotent Completion

Input:

Γ = discrete group

F = field of char 0 (default $F = \mathbb{Q}$)

R = reductive F-group

$\rho : \Gamma \to R(F)$ Zariski dense repⁿ.

Eg: $\Gamma = \Gamma_{g, n+\vec{r}}$

$F = \mathbb{Q}$

$R = Sp_g$

$\rho : \Gamma_{g,n+\vec{r}} \to Sp_g(\mathbb{Q})$ //

$\mathcal{C} = \mathcal{C}(\Gamma, R)$

= category of Γ-modules V that admit a filtration

$$V = V^0 \supseteq V^1 \supseteq V^2 \supseteq \cdots \supseteq V^N = 0$$

by Γ-submodules where Γ acts on each V^j / V^{j+1} via an action of R:

$$\Gamma \to R \curvearrowright V^j / V^{j+1}$$

This is tannakian. Fiber functor

$$\omega : \mathcal{C} \to Vec_F$$

underlying vector space.

Def: The completion of Γ w.r.t. ρ is $\mathcal{G} := \pi_1(\mathcal{C}(\Gamma, R), \omega)$.

It is an extension

Cohomological properties: $H^\bullet(\mathcal{G}, V) = H^\bullet(\mathcal{U}, V)^R$

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

$\mathcal{U}$: prounipotent.

Construction gives a homom $\Gamma \to \mathcal{G}(F)$

Example: $\mathcal{G}_{g,n+\vec{r}}$:= relative completion of $\Gamma_{g,n+\vec{r}}$

$\mathcal{U}_{g,n+\vec{r}}$:= its pro unipotent radical.

Remark: Presentations of $\mathfrak{u}_{g,n+\vec{r}}$ known when $g \geq 3$ and $g = 0, 1$. More later.

The completed monodromy map

Let $\pi = \pi_{g,\vec{1}}$ $\qquad H = H_1(\pi)$

$\mathfrak{p} = \operatorname{Lie} \pi^{un}$ $\qquad = H_1(\mathfrak{p})$

This has the LCS filtration

$$\mathfrak{p} = L^1\mathfrak{p} \supseteq L^2\mathfrak{p} \supseteq \cdots$$

$$\operatorname{Gr}_L^\bullet \mathfrak{p} \cong \mathbb{L}(H) \circlearrowleft Sp(H)$$

↑ canonical isomorphism

Now

- $\Gamma_{g,\vec{1}} \circlearrowright \mathfrak{p}$, preserves $L^\bullet$
- action on $\operatorname{Gr}_L^\bullet \mathfrak{p}$ factors through $\Gamma_{g,\vec{1}} \to Sp(H)$

So get representations

$$\begin{tikzcd} \mathcal{G}_{g,\vec{1}} \arrow[r] & \operatorname{Aut} \mathfrak{p}. \\ \Gamma_{g,\vec{1}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Be brief!!

Mixed Hodge structures (MHS)

A (pure) Hodge structure of weight $m \in \mathbb{Z}$ is a finite dimensional $\mathbb{Q}$-vector space $V_\mathbb{Q}$ and a bigrading

$$V_\mathbb{C} = \bigoplus_{p+q=m} V^{p,q}$$

of $V_\mathbb{Q} \otimes \mathbb{C}$, where $\overline{V^{p,q}} = V^{q,p}$.

Set $F^p V_\mathbb{C} = \bigoplus_{s \geq p} V^{s,m-s}$. Then

$$\cdots \supseteq F^p V \supseteq F^{p+1} V \supseteq \cdots$$

Example: H^m(smooth proj variety)

This is the Hodge filtration. Note

$$V^{p,q} = F^p V \cap \overline{F}^q V \qquad p+q=m$$

Def: A mixed Hodge structure is a finite dimensional vector space $V_\mathbb{Q}$ endowed with 2 filtrations:

$$0 \subseteq W_r V \subseteq \cdots \subseteq W_j V \subseteq W_{j+1} V \subseteq \cdots \subseteq W_s V = V$$

of $V_\mathbb{Q}$ (the weight filtration) and

$$V_\mathbb{C} = F^a V \supseteq F^{a+1} V \supseteq \cdots \supseteq F^b V = 0$$

(the Hodge filtration) such that

$$Gr^W_m V_{\mathbb{Q}} \quad + \text{ induced Hodge filtration}$$

is a Hodge structure of weight m for all $m \in \mathbb{Z}$.

Theorem (Deligne)

① The category of (graded polarizable) $\mathbb{Q}$-MHS is a $\mathbb{Q}$-linear tannakian category. (Deligne)

② the cohomology of every complex algebraic has a MHS, functorial w.r.t. morphisms of varieties. It agrees with the usual Hodge structure for smooth projective varieties.

KEY POINT: A, B Hodge strs of weights $a, b \in \mathbb{Z}$. Then $a \le b$ implies

$$Ext^1_{MHS}(A, B) = 0$$

Distill essential features of $\pi_1(MHS, \omega)$:

① MHS^{ss} = semi-simple MHS are $\oplus$'s of pure Hodge strs.

These are graded by weight, so have central cocharacter

$$\chi : \mathbb{G}_m \to \pi_1(MHS^{ss})$$

$t \in \mathbb{G}_m$ acts on a HS of wt m by mult by t^m.

② If V is a HS of wt m, then

$$Ext^1_{MHS}(\mathbb{Q}, V) = \Big[\underbrace{H^1(\underline{u})}_{\text{wt } -m.} \otimes \underbrace{V}_{\text{wt } m}\Big]^{\pi_1(MHS)^{ss}} \qquad \mathbb{G}_m \xrightarrow{\chi}$$

This vanishes when $m \geq 0$. So

(i) $H^1(\underline{u})$ has $\mathbb{G}_m$ weights > 0

(ii) $H_1(\underline{u})$ " " " < 0

So have:

$$0 \to \mathfrak{u}^{MHS} \to \pi_1(MHS) \to \pi_1(MHS^{ss}) \to 1$$

$$\mathbb{G}_m \xrightarrow{x} \pi_1(MHS^{ss}) \quad \text{central cochar}$$

$H_1(\mathfrak{u}^{MHS})$ has negative weights.

This motivates ...

<u>Negatively weighted extensions</u>:

<u>Ref</u>: Hain – Matsumoto

$$1 \to \underset{\text{prounipotent}}{U} \to G \to \underset{\text{reductive}}{R} \to 1$$

$$\mathbb{G}_m \xrightarrow{x} R \quad \text{central}$$

$H_1(U)$ negative $\mathbb{G}_m$ weights

<u>FACT</u>: (Levi) $G \to R$ is split and any 2 splittings are conjugate by an element of U:

$$G \cong R \ltimes U \quad \text{(not canonically)}$$

So have lift $\tilde{x}: \mathbb{G}_m \to G$ (no longer central, in general)

$$1 \to U \to G \to R \to 1$$

$$\mathbb{G}_m \xrightarrow{\tilde{x}} G, \quad \mathbb{G}_m \xrightarrow{x} R$$

$\tilde{x}$ unique up to conjugation by U.

<u>PROP^{N}</u> (Hain – Matsumoto). Suppose that

$$1 \to U \to G \to R \to 0$$

$$\mathbb{G}_m \xrightarrow{x} R$$

is a negatively weighted extension

(1) Every $V \in Rep(G)$ has a natural weight filtration $W_\bullet$

(2) There is a natural (though not canonical, in general) isomorphism

$$V \xrightarrow{\varphi_V} \bigoplus_m Gr^W_m V$$

that is compatible with $\otimes$, duals, etc:

$$\begin{tikzcd} V_1 \arrow[r, "\varphi_{V_1}"] \arrow[d, "f"'] & \bigoplus_m Gr_m^W V_1 \arrow[d, "Gr f"] \\ V_2 \arrow[r, "\varphi_{V_2}"] & \bigoplus Gr_m^W V_2 \end{tikzcd}$$

commutes all $f: V_1 \to V_2$ in $Rep(G)$.

(3) The functor

$$Gr_\bullet^W : Rep(G) \to \text{Graded-}Rep(R)$$

is exact. In particular

$$Gr(\ker f) = \ker Gr f$$

$$Gr(\operatorname{im} f) = \operatorname{im} Gr f$$

all $f: V_1 \to V_2$ in $Rep(G)$.

proof (sketch)

(a) choose a lift $\tilde{\chi}: \mathbb{G}_m \to G$ of χ. Every G-module V splits

$$V = \bigoplus_m V^{(m)}$$

under $\tilde{\chi}: \mathbb{G}_m \to G$.

Set $\quad W_r V = \bigoplus_{m \le r} V^{(m)}$

Then $\quad V^{(m)} \cong Gr_m^W V$

and $\quad V = \bigoplus V^{(m)} \cong \bigoplus Gr_m^W V.$

This is natural identification as every $f: V_1 \to V_2$ is $\mathbb{G}_m$-equivar.

(b) We can decompose

$$\underline{u} = Lie(U)$$

as $\quad \underline{u} = \bigoplus \underline{u}^{(m)}$

under adjoint action. Now $\underline{u}$ (pro) nilpotent + $H_1(\underline{u})$ negatively weighted implies that $\underline{u}^{(m)} = 0$, when $m \geq 0$.

(c) The weight filtration is well-defined. That is, it is independent of choice of $\tilde{\chi}$:

Every other lift is $u\tilde{\chi}u^{-1}$, where $u \in U$. Its decomp is

$$V = \bigoplus_{m} u V^{(m)} u^{-1}$$

Now use (b) to see that

$$\bigoplus_{m \le r} u V^{(m)} u^{-1} = \bigoplus_{m \le r} V^{(m)}$$

all $r \in \mathbb{Z}$. $\square$

CONCLUSION

Every $\mathbb{Q}$-MHS V is naturally, though not canonically, isomorphic to its weight graded $Gr^W_\bullet V$. This isomorphism is compatible with $\otimes$, duals.

Remark: The Holy Grail of the theory of motives is to construct tannakian categories of mixed motives. The corresponding group

will be a negatively weighted extension. So all motives should have a natural weight filtration, and there should be natural (though not canonical) isomorphisms

$$V \to \bigoplus_{m \in \mathbb{Z}} Gr^W_m V .$$

Applications:

① Existence of symplectic Magnus expansions (cf Massuyeau)

C = smooth projective curve $/ \mathbb{C}$ genus ≥ 1

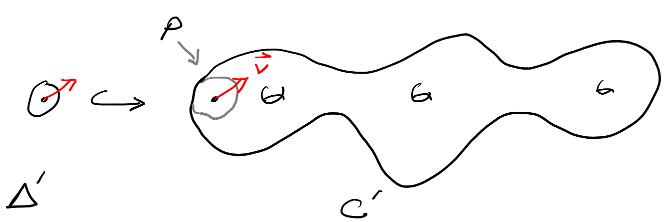
$P \in C$

$C' = C - \{P\}$

$\vec{v} \in T_P C$, non-zero

Δ = "formal" neighbourhood of P in C

$\Delta' = \Delta - \{P\}$ = punctured disk.



$\mathfrak{p} = \mathrm{Lie}\,\pi_1^{un}(C', \vec v)$ } have natural "limit" MHS (Hain)

$\mathcal{U}_\mathfrak{p} = \mathbb{Q}\pi_1(C', \vec v)^\wedge$

$H = H_1(C') = H_1(C)$ pure Hodge structure of weight -1

$a_1, \dots, a_g, b_1, \dots, b_g$ symplectic basis

Since H has weight -1 and since

$$[\ ,\ ] : \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$$

is a morphism of MHS,

$$LCS^m \mathfrak{p} = W_{-m}\mathfrak{p}$$

So there is a <u>natural</u> isomorphism

$$Gr^W_\bullet \mathfrak{p} \cong \mathbb{L}(H).$$

On the other hand

$\mathrm{Lie}\,\pi_1^{un}(\Delta', \vec v) \cong H_1(\Delta')$ Hurewicz

$\cong H_1(\mathbb{C}^*)$ $\Delta' \hookrightarrow \mathbb{C}^*$ htpy equiv

$= \mathbb{Q}(1)$ the Hodge struct of wt -2, type $(-1,-1)$

and $\mathrm{Lie}\,\pi_1^{un}(\Delta', \vec v) \to \mathfrak{p}$ induces morphisms

$$\mathbb{Q}(1) \to \mathfrak{p} \quad \text{and} \quad \mathbb{Q}(1) \to Gr^W_\bullet \mathfrak{p}$$

This is easily seen to have image

$$\Theta := \sum_{j=1}^{g} [a_j, b_j] \in Gr^W_{-2}\mathfrak{p} = \Lambda^2 H.$$

<u>Cor</u>: $\mathrm{Lie}\,\pi_1^{un}(C, P) \cong \mathbb{L}(H)^\wedge / (\Theta)$.

This is a special case of the formality result of Deligne-Griffiths-Morgan-Sullivan.

② The Johnson homomorphism revisited.

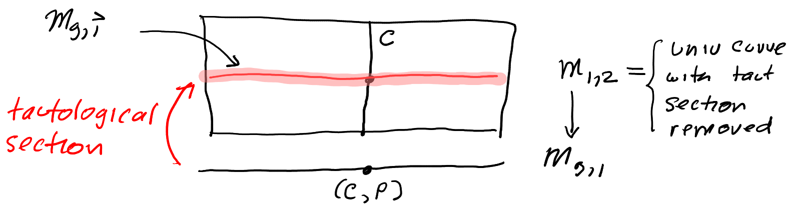
<u>Thm</u>: For every $(C, P, \vec v)$ as above There are natural MHSs on $\mathfrak{g}_{g,\vec 1}$ and

$\mathfrak{g}_{g,2}$. There are inclusions

$$\mathfrak{g}_{g,\vec{1}} \hookrightarrow \mathfrak{g}_{g,2}, \qquad \mathfrak{p} \hookrightarrow \mathfrak{g}_{g,2}$$

corresponding to

$$\Gamma_{g,\vec{1}} \hookrightarrow \Gamma_{g,2}, \qquad \pi_1(C', \vec{v}) \hookrightarrow \Gamma_{g,2}.$$



These are morphisms of MHS. So the adjoint action of $\mathfrak{g}_{g,\vec{1}}$ on $\mathfrak{g}_{g,2}$ induces the monodromy morphism

$$\mathfrak{g}_{g,\vec{1}} \longrightarrow \operatorname{Der} \mathfrak{p}$$

by restriction to the ideal $\mathfrak{p}$. It is therefore a morphism of MHS. By exactness of $Gr^W_\bullet$, we can replace it with

$$Gr\, \mathfrak{g}_{g,\vec{1}} \longrightarrow Gr^W_\bullet \operatorname{Der} \mathfrak{p} \cong \operatorname{Der} \mathbb{L}(H)$$

FACTS:

$$\begin{tikzcd} 1 \arrow[r] & T_{g,\vec{1}} \arrow[r] \arrow[d] & \Gamma_{g,\vec{1}} \arrow[r] \arrow[d] & Sp_g(\mathbb{Z}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathcal{U}_{g,1} \arrow[r] & \mathcal{G}_{g,\vec{1}} \arrow[r] & Sp_g \arrow[r] & 1 \end{tikzcd}$$

UMP implies have Lie alg of $T^{un}_{g,\vec{1}}$.

$$T^{un}_{g,1} \to \mathcal{U}_{g,\vec{1}}, \qquad \mathfrak{t}_{g,\vec{1}} \to \mathfrak{u}_{g,\vec{1}}$$

① $g \geq 2$, $\mathfrak{t}_{g,\vec{1}} \to \mathfrak{u}_{g,\vec{1}}$ is surjective

② $g \geq 3$, $\ker\{\mathfrak{t}_{g,1} \to \mathfrak{u}_{g,1}\} = \mathbb{Q}(1)$.

③ When $g \geq 3$, Johnson implies that

$$Gr^W_{-1} \mathfrak{t}_{g,\vec{1}} = H_1(\mathfrak{t}_{g,\vec{1}}) \cong \Lambda^3 H$$

Exactness implies that

$$LCS^m \mathfrak{t}_{g,\vec{1}} = W_{-m} \mathfrak{t}_{g,\vec{1}}$$

④ Brendle–Margalit–Putman implies that

hyperelliptic Torelli group

$$Gr^W_j H_1(T\Delta_{g,\vec{1}}) = \begin{cases} H & j = -1 \\ V_{\boxplus} & j = -2 \\ 0 & \text{other} \end{cases} \quad . \ //$$

So when $g \geq 3$

$$Gr^W_{-2}\, \underline{t}_{g,\vec{1}} \longrightarrow Gr^W_{-2}\, \underline{u}_{g,\vec{1}}$$

is not injective. Exactness of $Gr^W_\bullet$ implies that when $g \geq 3$,

(1) $\ker\{\underline{t}_{g,\vec{1}} \longrightarrow \operatorname{Der} \mathfrak{p}\} \neq 0$

(2) $J^m t_{g,\vec{1}} = W_{-m} t_{g,\vec{1}} + \text{kernel}, \quad m \geq 2$

This implies that the topologies on $\underline{t}_{g,\vec{1}}$ and thus $T_{g,\vec{1}}$, are inequivalent.

(3) $\operatorname{im}\{Gr^\bullet_J\, \underline{t}_{g,\vec{1}} \longrightarrow \operatorname{Der} \mathbb{L}(H)\}$

is generated by $Gr^W_{-1}\, t_{g,\vec{1}} = \Lambda^3 H$.

Hyperelliptic case: (eg $g=2$)

Image of

$$Gr\, Lie\, (T\Delta^{un}_{g,\vec{1}}) \longrightarrow \operatorname{Der} \mathbb{L}(H)$$

almost inner derivns

is generated by $H \subseteq Gr^W_{-1} \operatorname{Der} \mathbb{L}(H)$ and by $V_{⊞}$ in $Gr^W_{-2} \operatorname{Der} \mathbb{L}(H)$.

Speculation: (Kashiwara–Vergne Problem)

$(C, P, \vec{v})$ as above

$\pi = \pi_1(C', \vec{v}) \qquad C' = C - \{P\}$

$\mathfrak{p} = Lie\, \pi^{un}$

$|\mathbb{Q}\pi| := \mathbb{Q}\pi/(uv - vu)$

$\mathbb{Q}\pi^\wedge$ = completed group algebra
$= \hat{U}\mathfrak{p}$

has natural MHS

$|\mathbb{Q}\pi^\wedge|$

Have (completed) Goldman bracket

$$\Delta : |\mathbb{Q}\pi^\wedge|^{\otimes 2} \longrightarrow |\mathbb{Q}\pi^\wedge|$$

and (completed) Turaev cobracket

$$\delta : I \longrightarrow I \wedge I \qquad I = \text{aug ideal}$$

Observation: if δ and Δ are morphisms of MHS, then isomorphism

$$|\mathbb{Q}\pi^\wedge| \cong |Gr^W_\bullet\, \mathbb{Q}\pi^\wedge|^\wedge$$

gives a homomorphic Magnus expansion.
The unipotent radical $\mathcal{U}^{MHS}$ of $\pi_1(MHS)$
acts on these.