Some transcendance problems in arithmetic and ananlytic functions.

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• Explore more applications and related subjects of the Nevanlinna theory.

1. Problems and some background

In Lang's Monog. (Transcendental numbers ....) 1966:

Schanuel Conj. Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ be linearly indep. over $\mathbf{Q}$. Then

$$\mathrm{tr.}\deg_{\mathbf{Q}}\{\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\} \geq n.$$

Lang's Conj.: Let $f : \mathbf{C} \to A$ be an entire curve (non-constant; originally, 1-parameter subgroup) into an abelian variety A with Zariski dense image.
Let D be a hyperplane cut of A.

(i) Then, $f(\mathbf{C}) \cap D \neq \emptyset$?
(ii) $\# f(\mathbf{C}) \cap D = \infty$?

We discuss these problems for semi-abelian varieties from the viewpoint of Nevanlinna theory:

E. Borel. An entire curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ omitting $n+2$ hyperplanes in general position. Then, f is linearly degenerate.

Bloch-Ochiai. Let $f : \mathbf{C} \to A$ be an entire curve into an abelian variety.

Then the Zariski closure $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a translate of an algebraic subgroup.

A unified form (Borel+Bloch-Ochiai+N. 1977–'81):

Log Bloch-Ochiai. Let $f : \mathbf{C} \to A$ be an entire curve into a semi-abelian variety.

Then the Zariski closure $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a translate of an algebraic subgroup.

Remarks. Lang's Conjecture (i) was proved by Siu-Yeung (abelian case, '96), N. (semi-abelian case, '98), ..., but (ii) was unanswered for long time, and proved by by Corvaja-N. in generalized form for semi-abelian varieties in 2011.

Big Picard type: For $f : \Delta^*(R) \to A$ (semi-abelian) with essential singularity at 0, we have similar results (N. '81, ...), but *not always*.

Big Picard type: Let V, W be algebraic varieties ($/\mathbf{C}$).
(1) $f : V \to W \Longrightarrow$ Extendable over $\bar{V}$.
(2) $f : V \to W$, transcendental (e.g., universal covering) $\Longrightarrow$ "algebraic — algebraic".

2. Ax-Schanuel

Schanuel Conjecture:

(i) $n = 1$: Gel'fond-Schneider (1934; Hilbert's 7th Problem).

(ii) $\mathsf{n} > 1$: Open. Even in $\mathsf{n} = 2$: With $(\alpha_1, \alpha_2) = (1, \pi \mathsf{i})$ it implies the Folklore: alg. indep. of e and π.

(iii) $\mathsf{e}, \mathsf{e}^{\pi}$ are alg. indep. (Nesterenko, '96). The elliptic modular function $\mathsf{j}(\tau)$ was used.

Formal Functional Analogue: J. Ax ('71, '72) proved the analogue:

Thm. 2.1 (**Ax-Schanuel**). Let $\mathsf{f}(\mathsf{t}) = (\mathsf{f}_\mathsf{j}(\mathsf{t})) \in (\mathbf{C}[[\mathsf{t}]])^\mathsf{n}$. If $\mathsf{f}_\mathsf{j}(\mathsf{t}) - \mathsf{f}_\mathsf{j}(0)$, $1 \le \mathsf{j} \le \mathsf{n}$, are linearly independent over $\mathbf{Q}$, then

$$\operatorname{tr.deg}_{\mathbf{C}}\{\mathsf{f}_1(\mathsf{t}), \ldots, \mathsf{f}_\mathsf{n}(\mathsf{t}), \mathsf{e}^{\mathsf{f}_1(\mathsf{t})}, \ldots, \mathsf{e}^{\mathsf{f}_\mathsf{n}(\mathsf{t})}\} \ge \mathsf{n} + 1.$$

More generally, he proved it for semi-ableian varieties, and dealt with t of several variables.

Ax's proof : By means of Kolchin's theory of differential algebra.

Our Aim : 1) Prove Ax-Schanule for entire $f_j(z)$ and a semi-abelian variety A by means of Nevanlina theory,

2) Study and prove a 2nd Main Theorem for the "extended exponential map"

$$\widehat{\exp}_A f : z \in \mathbf{C} \to (\exp_A f(z), f(z)) \in A \times \mathrm{Lie}(A).$$

N.B. There is no "value" in formal analytic functions, but there is for analytic functions: We may think of more problems.

We mainly follow the developments of the theory for entire curves into A since Lang's Questions '66, and Log Bloch-Ochiai's Theorem.

Arithmetic Thry.–O-minimal Thry.–Nevnalinna Thry.:

(i) **Raynaud's Theorem** (1983, Manin-Mumford Conj.):
$X \subset A$ subvariety $(/K)$. $\Rightarrow X_{\mathrm{tor}} = \bigcup_{\mathrm{finite}} (a + B_{\mathrm{tor}})$,
where $a \in X_{\mathrm{tor}}$ and alg. subgrp's. B.
Proof: By method of char. $p > 0$.

(ii) Later, many proofs by Coleman, Hindary, Hrushovski, ... I was personally motivated by Pila-Zannier (2008) with O-minimal method.

(iii) Yet another proof by Nevanliina thry. (Log Bloch–Ochiai of Big Picard type) + 'O-minimal' (N., Atti Accad. Naz. Rend. Lincei Mat. Appl. **29** (2018)).

(iv) For Ax-Schanuel by "O-minimal", other proofs due Tsimerman 2015, Peterzil-Starchenko 2018, ...

(v) Yet Another Proof of Analy. Ax-Schanuel by Nevanlinna thry. : Today, 1st topic.

More on the Value Distribution: Today, 2nd topic:

N—, *Analytic Ax-Schanuel for semi-abelian varieties and Nevanlinna theory*, DOI:10.2969/jmsj/89588958, J. Math. Soc. Jpn., 2023.

······· Yet without "O-minimal".

(vi) **Expectation**: Analy. Ax-Schanuel + O-minimal + Arithmetic $\Longrightarrow$??

Application of Ax-Shanuel:(e.g.) W.D. Brownawell and K.K. Kubota, The algebraic independence of Weierstrass functions and some related numbers, Acta Arith. **33** (1977), 111–149.

This is covered by the present result of analy. case.

3. Main Results

Jet Spaces. Let A be a semi-abelian variey of dim n:

$$0 \to (\mathbf{C}^*)^{\mathsf{t}} \to \mathsf{A} \to \mathsf{A}_0 \to 0 \quad \text{(with } \mathsf{A}_0 \text{ abelian var.)},$$

$\exp = \exp_{\mathsf{A}} : \mathrm{Lie}(\mathsf{A}) \to \mathsf{A}$ be an exponential map;

$f : \mathbf{C} \to \mathrm{Lie}(A) \cong \mathbf{C}^n$ be an entire curve.

Set

$$\widehat{\exp} f : z \in \mathbf{C} \to (\exp f(z), f(z)) \in A \times \mathrm{Lie}(A).$$

Take its k-jet lift:

$$J_k(\widehat{\exp} f) : z \in \mathbf{C} \to (J_k(\exp f(z)), J_k(f(z))) \in J_k(A \times \mathrm{Lie}(A)) \cong J_k(A) \times J_k(\mathrm{Lie}(A)).$$

Speciality:

$$\begin{aligned} J_k(A) &\cong A \times J_{k,A}, & J_k(\mathrm{Lie}(A)) &\cong \mathrm{Lie}(A) \times J_{k,\mathrm{Lie}(A)} \\ J_k(A \times \mathrm{Lie}(A)) &= A \times J_{k,A} \times \mathrm{Lie}(A) \times J_{k,\mathrm{Lie}(A)}, & J_{k,A} &= J_{k,\mathrm{Lie}(A)}, \\ J_k(\widehat{\exp} f)(z) &= (\exp f(z), J_{k,\exp f}(z), f(z), J_{k,f}(z)), & J_{k,\exp f}(z) &= J_{k,f}(z) \ (k \geq 1). \end{aligned}$$

We consider:

$$J_k(\widehat{\exp} f)(z) \in A \times \mathrm{Lie}(A) \times J_{k,A} \hookrightarrow J_k(A \times \mathrm{Lie}(A)). \tag{3.1}$$

$\widehat{J}_{k,A} = \mathrm{Lie}(A) \times J_{k,A} \cong \mathbf{C}^n \times \mathbf{C}^{nk}$ is called the **extended jet part**.
$X_k(\widehat{\exp} f) = \overline{J_k(\widehat{\exp} f)(\mathbf{C})}^{\mathrm{Zar}}$ is the Zariski closure of the image:

$$\mathrm{tr.deg}_{\mathbf{C}} \widehat{\exp} f := \dim_{\mathbf{C}} X_0(\widehat{\exp} f) = \dim_{\mathbf{C}} \overline{\widehat{\exp} f(\mathbf{C})}^{\mathrm{Zar}}.$$

Def. 3.2. $f : \mathbf{C} \to \mathrm{Lie}(A)$ is <u>A-degnerate</u> if $\exists$ alg. subgroup $G \subsetneqq A$ s.t. $\exp f(\mathbf{C}) \subset \exp f(0) + G$ (coset type).

N.B. $A = (\mathbf{C}^*)^t$: $f = (f_j)$ is $(\mathbf{C}^*)^n$-degenerate $\Longleftrightarrow$ $f_j, 1 \leq j \leq n$, are lin. dep./$\mathbf{Q}$.

Thm. 3.1 (Analy. Ax-Schanuel). If an entire curve $f : \mathbf{C} \to \mathrm{Lie}(A)$ is A-nondeg., then $\mathrm{tr.deg}_{\mathbf{C}}\, \widehat{\exp} f \geq n + 1$.

Order Functions.

$f = (f_1, \dots, f_n) : z \in \mathbf{C} \to f(z) \in \mathbf{C}^n \cong \mathrm{Lie}(A)$, an entire curve.

Nevanlinna-Shimizu-Ahlfors order function:

$$T(r, f_j) = T_{f_j}(r, \omega_{\mathrm{FS}}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* \omega_{\mathrm{FS}}.$$

Roughly, $T(r, f_j) \sim \log \max_{|z|=r} |f_j(z)|$.

$T_f(r) := \max_{1 \leq j \leq n} T(r, f_j)$.

$T_{\exp f}(r) = T_{\exp f}(r, \omega_L)$ with the curvature form ω_L of a big l.b. $L \to \bar{A}$.

$T_{\widehat{\exp} f}(r) := T_{\exp f}(r) + T_f(r)$ for $\widehat{\exp} f : \mathbf{C} \to A \times \mathrm{Lie}(A)$.

$S(r) = O\left(\log^+ T_{\exp f}(r)\right) + O(\log r) + O(1)|| = o(T_{\exp f}(r))||$ (with except'l intervals of total finite length).

Lem. 3.3 (Key). (i) $T_f(r) = S(r)$.

(ii) $T_{\widehat{\exp} f}(r) = T_{\exp f}(r) + S(r)$.

Proof. Use the complex Poisson integral + Borel's technic.

Proof of Analytic Ax-Schanuel Thm.3.1.

The A-nondegeneracy and the Log Bloch–Ochiai imply $\overline{\exp f(\mathbf{C})}^{\mathrm{Zar}} = A$:

$$\mathrm{tr.deg}_{\mathbf{C}}\, \exp f = n. \tag{3.4}$$

Lem. 3.5. $\mathrm{tr.deg}_{\mathbf{C}(f)} \mathbf{C}(f, (\exp f)^* \mathbf{C}(A)) \geq 1.$

Pf. If "$= 0$", $(\exp f)^*\mathbf{C}(A)$ is alg. over (f_j), so that $T_{\exp f}(r) = O(T_f(r)) = o(T_{\exp f}(r))$ by Key Lem. 3.3; Contradiction! △

(3.4) $\Rightarrow \mathrm{tr.deg}_{\mathbf{C}} \widehat{\exp} f \geq n$. Suppose $\mathrm{tr.deg}_{\mathbf{C}} \widehat{\exp} f = n$. $\Rightarrow$ $\underline{f_j \text{ are alg. } /(\exp f)^* \mathbf{C}(A)}$.
$\Longrightarrow \exists$ non-trivial alg. relations

$$P_j(f_j, \hat{\phi}) = P_j(f_j, \hat{\phi}_1, \ldots, \hat{\phi}_n) = 0, \quad 1 \leq j \leq n, \tag{3.6}$$

where $\{\phi_j\}_{j=1}^n$ is a transcendental basis of $\mathbf{C}(A)$, and $\hat{\phi}_j := \phi_j \circ \exp f$.
Lem. 3.5 $\Rightarrow \mathrm{tr.deg}_{\mathbf{C}}\{f_j\}_{j=1}^n < n$: That is, $\exists$ a non-trivial alg. relation

$$Q(f_1, \ldots, f_n) = 0. \tag{3.7}$$

Eliminate f_j $(1 \leq j \leq n)$ in (3.6) and (3.7). $\Rightarrow$ Alg. rel. in $\hat{\phi}_j$'s $\Rightarrow$ f is A-degnerate: Contradiction! □

Example. (Brownawell-Kubota) A product of elliptic curves, $A := \prod^n E_j$ and $\underline{\text{alg. indep.}}$ $f = (f_j) : \mathbf{C} \to \mathrm{Lie}(A)$:

$$\mathrm{tr.deg}_{\mathbf{C}}\{f_1, \ldots, f_n, \wp_1(f_1), \ldots, \wp_m(f_n)\} \geq n + 1.$$

Here one may claim the same for more generally $\underline{A\text{-nondegenerate}}$ $f = (f_j)$: e.g., with $f_1(z) = z, f_2(z) = z$ and $\underline{\text{non-isogenious}}$ E_j $(j = 1, 2)$,

$$\mathrm{tr.deg}_{\mathbf{C}}\{z, \wp_1(z), \wp_2(z)\} = 3.$$

$\overline{\mathrm{Lie}(\mathsf{E}_1) \times \mathrm{Lie}(\mathsf{E}_2) \times \mathsf{A}} = \mathbf{P}^2(\mathbf{C}) \times \mathsf{E}_1 \times \mathsf{E}_2$,

$$\mathsf{T}_{\widehat{\exp}\,\mathsf{f}}(\mathsf{r}) = \frac{\pi \mathsf{r}^2}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \mathsf{o}(1)\right),$$

wherer λ_j are the areas of the fundamental parallelograms of $\wp_j$ $(j = 1, 2)$.
Let $\mathsf{P}(\mathsf{z}_1, \mathsf{z}_2, \mathsf{w}_1, \mathsf{w}_2)$ be a polynomials of degrees $\mathsf{d}_1, \mathsf{d}_2$ in $\mathsf{w}_1, \mathsf{w}_2$ respectively, and $\Xi_{\mathsf{P}} = \{\mathsf{P}(\mathsf{z}, \mathsf{z}, \wp_1(\mathsf{z}), \wp_2(\mathsf{z})) = 0\}$. Then

$$\mathsf{N}_\infty(\mathsf{r}, \Xi_{\mathsf{P}}) = \int_1^r \frac{\#\left(\Xi_P \cap \Delta(t)\right)}{t} dt = N_1(r, \Xi_P) + o(r^2)$$
$$= \pi r^2 \left(\frac{d_1}{\lambda_1} + \frac{d_2}{\lambda_2} + o(1)\right).$$

4. Nevanlinna thry. for $\widehat{\exp}\,\mathsf{f}$

Thm. 4.1 (2nd Main Thm.). Let $\mathsf{f} : \mathbf{C} \to \mathrm{Lie}(\mathsf{A})$ be A-nondegenerate.

(i) For a reduced alg. subset $\mathsf{Z} \subset \mathsf{X}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})$ $(\subset \mathsf{A} \times \widehat{\mathsf{J}}_{\mathsf{k},\mathsf{A}})$ $(\mathsf{k} \geqq 0)$, $\exists\, \bar{\mathsf{A}} \times \bar{\widehat{\mathsf{J}}}_{\mathsf{k},\mathsf{A}}$, a proj. compactification with closures $\bar{\mathsf{X}}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})$ and $\bar{\mathsf{Z}}$ such that

(4.1)
$$\mathsf{T}_{\mathsf{J}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})}(\mathsf{r}, \omega_{\bar{\mathsf{Z}}}) = \mathsf{N}_1(\mathsf{r}, \mathsf{J}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})^*\mathsf{Z}) + \mathsf{S}_\varepsilon(\mathsf{r}),$$

where $\mathsf{S}_\varepsilon(\mathsf{r}) \leq \varepsilon \mathsf{T}_{\exp \mathsf{f}}(\mathsf{r})\ ||_\varepsilon\ (\forall\, \varepsilon > 0)$,
and $\omega_{\bar{\mathsf{Z}}}$ is a sort of curvature form associated with $\bar{\mathsf{Z}}$.

(ii) If $\mathrm{codim}_{\mathsf{X}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})}\mathsf{Z} \geqq 2$, then

(4.2)
$$\mathsf{T}_{\widehat{\exp}\,\mathsf{f}}(\mathsf{r}, \omega_{\bar{\mathsf{Z}}}) = \mathsf{S}_\varepsilon(\mathsf{r}).$$

(iii) $(\mathsf{k}=0)$ If D is a reduced divisor on $\mathsf{A}\times \mathrm{Lie}(\mathsf{A})$ and $D \not\supset X_0(\widehat{\exp} f)$, then

$$\mathsf{T}_{\widehat{\exp}\,\mathsf{f}}(\mathsf{r},\omega_{\bar{\mathsf{D}}}) = \mathsf{N}_1(\mathsf{r},(\widehat{\exp}\,\mathsf{f})^*\mathsf{D}) + \mathsf{S}_\varepsilon(\mathsf{r}). \tag{4.3}$$

where $\bar{\mathsf{D}} \subset \bar{\mathsf{A}} \times \overline{\mathrm{Lie}(\mathsf{A})}$.

Pf. $\exists\, \ell \in \mathbf{N}$ such that

$$\mathsf{T}_{\mathsf{J}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})}(\mathsf{r},\omega_{\bar{\mathsf{Z}}}) = \mathsf{N}_\ell(\mathsf{r},\mathsf{J}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})^*\mathsf{Z}) + \mathsf{S}(\mathsf{r}).$$

Here, using this and codim $\mathsf{Z} \geq 2$, we prove (ii).

Using (ii), we deduce

$$\mathsf{N}_\ell(\mathsf{r},\mathsf{J}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})^*\mathsf{Z}) - \mathsf{N}_1(\mathsf{r},\mathsf{J}_\mathsf{k}(\widehat{\exp}\,\mathsf{f})^*\mathsf{Z}) = \mathsf{S}_\varepsilon(\mathsf{r}),$$

$\Longrightarrow$ (i). □

As an aplication we have:

Thm. 4.2. Let $\widehat{\exp}\,\mathsf{f} : \mathbf{C} \to \mathsf{A}\times \mathrm{Lie}(\mathsf{A})$ and $\bar{\mathsf{D}} \subset \bar{\mathsf{A}} \times \overline{\mathrm{Lie}(\mathsf{A})}$ be as in (iii) above. Assume that some positive multiple $\nu\bar{\mathsf{D}}$ contains a big divisor coming from $\bar{\mathsf{A}}$.

Then $\exists$ irred. comp. $\mathsf{D}' \subset \mathsf{D} \cap \mathsf{X}_0(\widehat{\exp}\,\mathsf{f})$ such that $\widehat{\exp}\,\mathsf{f}(\mathbf{C}) \cap \mathsf{D}'$ is Zariski dense in D'; in particular, $|\widehat{\exp}\,\mathsf{f}(\mathbf{C}) \cap \mathsf{D}| = \infty$.

This follows from the estimates in 2nd Main Thm. with $N_1(r,*)$ (essential).

Remark. For $\exp f : \mathbf{C} \to \mathsf{A}$, by Corvaja-N. ('12), answering a question in Lang's Monog. '66.

The proof of the 2nd Main Thm. 4.1 is rather long but we carry out the proof along the way as for $\exp f : \mathbf{C} \to \mathsf{A}$ (N.-Winkelmann-Yamanoi) by making use of Key Lem 3.3.

The next theorem says that the distribution $(\widehat{\exp} f)^* D$ on $\mathbf{C}$ contains an ample information of $\widehat{A}$, D and f; we have the following unicity theorem of H. Cartan–P. Erdös–K. Yamanoi type (cf. Yamanoi Forum Math. 2004, Corvaja-N. Math. Ann. 2012)

Thm. 4.3 (Unicity). Let A_j $(j = 1, 2)$ be two semi-abelian varieties and let D_j $(j = 1, 2)$ be effective reduced A_j-big divisors on $\widehat{A}_j$ with

$$\widehat{\mathrm{St}}(D_j) := \{x \in \widehat{A}_j : x + D_j = D_j\} = \{0\}.$$

Let $f_j : \mathbf{C} \to \mathrm{Lie}(A_j)$ be A_j-nondegenerate. Assume that

$$\mathrm{Supp}\,(\widehat{\exp}_{A_1} f_1)^* D_1 = \mathrm{Supp}\,(\widehat{\exp}_{A_2} f_2)^* D_2.$$

Then $\exists\, \alpha : A_1 \xrightarrow{\cong} A_2$ with $\hat{\alpha} : \widehat{A}_1 \to \widehat{A}_2$, such that

- $\hat{\alpha}^* D_2 = D_1$,
- $\widehat{\exp}_{A_2} f_2 = \hat{\alpha} \circ \widehat{\exp}_{A_1} f_1$, up to translations of $\widehat{A}_j$.

Remarks to some extensions:

(i) $\mathbf{C} \Rightarrow \Delta(r)^*$ (isolated essential singularity, Big Picard type).

(ii) $\mathbf{C} \Rightarrow$ affine alg. curve.

(iii) $\mathbf{C} \Rightarrow$ (parabolic) Riemann suface with involving a counting function of Euler numbers.

(iv) Hyperbolic case?

Hyperbolic Bloch–Ochiai by “O-minimal”, Pila, Ulmo, Mok, ...: How related?

In Thm 4.2 we have $|\widehat{\exp} \mathsf{f}(\mathbf{C}) \cap \mathsf{D}| = \infty$.

Question: What is the cluster set of $\widehat{\exp} \mathsf{f}(\mathbf{C}) \cap \mathsf{D}$, the set of accumulation points of $\widehat{\exp} \mathsf{f}(\mathbf{C}) \cap \mathsf{D}$?

This question makes sense for $\exp \mathsf{f}(\mathbf{C}) \cap \mathsf{D}$ with $\mathsf{D} \subset \mathsf{A}$, too.

Also recall:

$$\text{Analy. Ax-Schanuel} + \text{O-minimal} + \text{Arithmetic} \Longrightarrow ??$$

Thank you for your attention!!

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