A Remark to a Division Algorithm in the Proof
of Oka's First Coherence Theorem

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Abstract

The problem is the locally finite generation of a relation sheaf \( \mathcal{R}(\tau_1, \ldots, \tau_q) \) in \( \mathcal{O}_{\mathbb{C}^n} \). After \( \tau_j \) reduced to Weierstrass’ polynomials in \( z_n \), it is the key for applying an induction on \( n \) to show that elements of \( \mathcal{R}(\tau_1, \ldots, \tau_q) \) are expressed as a finite linear sum of \( z_n \)-polynomial-like elements of degree at most \( p = \max_j \deg z_n \tau_j \) over \( \mathcal{O}_{\mathbb{C}^n} \). In that proof one is used to use a division by \( \tau_j \) of the maximum degree, \( \deg z_n \tau_j = p \) (Oka ’48, Cartan ’50, L. Hörmander ’66, R. Narasimhan ’66, T. Nishino ’96, ....). Here we shall confirm that the division above works by making use of \( \tau_k \) of the minimum degree, \( \min_j \deg z_n \tau_j \). This proof is naturally compatible with the simple case when some \( \tau_j \) is a unit, and gives some improvement in the degree estimate of generators.

1 Introduction and results

It will be of no necessity to mention the importance of Oka’s First Coherence Theorem that the sheaf \( \mathcal{O}_{\mathbb{C}^n} \) (also denoted simply by \( \mathcal{O}_n \)) of germs of holomorphic functions over \( n \)-dimensional complex vector space \( \mathbb{C}^n \) (Oka [7], [8])\(^2\). Let \( \Omega \subset \mathbb{C}^n \) be an open set and let \( \tau_j \in \mathcal{O}(\Omega) := \Gamma(\Omega, \mathcal{O}_n), 1 \leq j \leq q \). Oka’s First Coherence Theorem claims that the relation sheaf \( \mathcal{R}(\tau_1, \ldots, \tau_q) \) defined by

\[
 f_1 \tau_{1z} + \cdots + f_q \tau_{qz} = 0, \quad f_j \in \mathcal{O}_{n,z}, \ z \in \Omega.
\]

is locally finite in \( \Omega \), where \( \tau_{jz} \) stands for the germ at \( z \). The problem is local, so that we consider in a neighborhood of a point \( a \in \Omega \); further we may assume \( a = 0 \) with complex coordinate system \( (z_1, \ldots, z_n) \).

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\(^2\)There are some differences in these two versions of Oka VII.
By Weierstrass’ Preparation Theorem \( \tau_j \) are reduced to Weierstrass’ polynomials \( P_j \in \mathcal{O}(P\Delta_{n-1})[z_n] \) about 0, where \( P\Delta_{n-1} \) is a small polydisk in \( z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \). Set

\[
\mathcal{R} = \mathcal{R}(P_1, \ldots, P_q),
\]

\[
p = \max_{1 \leq j \leq q} \deg_{z_n} P_j,
\]

\[
p' = \min_{1 \leq j \leq q} \deg_{z_n} P_j.
\]

We call \( f \in \mathcal{O}_{n-1,b'[z_n]} \) (resp. \( f \in \mathcal{O}(P\Delta_{n-1})[z_n] \)) a \( z_n \)-polynomial-like germ (resp. function) and denote by \( \deg_{z_n} f \) its degree in variable \( z_n \); for convention, “\( \deg_{z_n} f < 0 \)” means “\( f = 0 \)”. We also call an element \( (f_j) \in (\mathcal{O}_{n,(b',b_n)})^q \) (resp. \( (f_j) \in (\mathcal{O}(P\Delta_{n-1} \times \mathbb{C}))^q \)) with \( f_j \in \mathcal{O}_{P\Delta_{n-1},b'[z_n]} \) (resp. \( f_j \in \mathcal{O}(P\Delta_{n-1})[z_n] \)) a \( z_n \)-polynomial-like element (resp. section), and \( \deg_{z_n}(f_j) = \max_j \deg_{z_n} f_j \) the degree of \( (f_j) \).

The proof of the local finiteness of \( \mathcal{R} \) relies on the induction on \( n \), and the key which makes the induction to work is:

**Lemma A.** Every element of \( \mathcal{R}_b \) at \( b = (b',b_n) \in P\Delta_{n-1} \) is expressed as a finite linear sum of \( z_n \)-polynomial-like elements of \( \mathcal{R}_b \) of degree at most \( p \) with coefficients in \( \mathcal{O}_b \).

There is some structure in the generator system with respect to the degree in \( z_n \). For \( 1 \leq i < j \leq q \) there are sections of \( \mathcal{R} \) given by

\[
T_{i,j} = (0, \ldots, 0, P_j, 0, \ldots, 0, -P_i, 0, \ldots, 0),
\]

which we call the **trivial solutions**, and are \( z_n \)-polynomial-like sections of \( \deg_{z_n} T_{i,j} \leq p \). Without loss of generality we may assume that

\[
p_1 = p',
\]

\[
p_q = p,
\]

and set

\[
T_j = T_{1,j}, \quad 2 \leq j \leq q.
\]
In the proof of Lemma A a division algorithm is applied; in the original proof of Oka as well as in many references such as H. Cartan [1], R. Narasimhan [4], L. Hörmander [3], T. Nishino [5], J. Noguchi [6], etc., the division algorithm by $P_q$ of the maximum degree is used to conclude the existence of a finite generator system consisting of $T_{i,q}$ of degree $\leq p$, $1 \leq i \leq q - 1$, and a finite number of $z_n$-polynomial-like elements $\alpha$ of degree $< p$. In case $p' = 0$, it is immediate that the trivial solutions $T_j$ with $2 \leq j \leq q$ form already a generator system, while by the original proof one still needs elements $\alpha$ of degree $< p$.

The aim of this note is to confirm that Oka’s original proof still works with the division algorithm by $P_1$ of the minimum degree in $z_n$:

**Lemma 1.1.** Let the notation be as above. Then an element of $\mathcal{R}_b$ is written as a finite linear sum of the trivial solutions, $T_j$, $2 \leq j \leq q$, and $z_n$-polynomial-like elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ of $\mathcal{R}_b$ with coefficients in $\mathcal{O}_{n,b}$ such that

\begin{align*}
\text{(1.2)} & \quad \deg_{z_n} \alpha_1 \leq p - 1, \\
& \quad \deg_{z_n} \alpha_j \leq p' - 1, \quad 2 \leq j \leq q.
\end{align*}

**N.B.** If $p' = 0$, then there is no term of $\alpha$, and if $p' = 1$, $\alpha_j$ are constants for $2 \leq j \leq q$.

To decrease $p - 1$ in (1.2) one needs to transform the relation sheaf $\mathcal{R}(P_1, P_2, \ldots, P_q)$ with dividing $P_j$ ($2 \leq j \leq q$) by $P_1$ (here we use an idea from Hironaka’s proof, cf. [2]). Set

\begin{align*}
P_j &= Q_j P_1 + R_j, \quad Q_j, R_j \in \mathcal{O}_{n-1}(P\Delta_{n-1})[z_n], \\
\deg_{z_n} R_j &\leq p' - 1, \quad 2 \leq j \leq q.
\end{align*}

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Then for \((f_j) \in (\mathcal{O}_{n,z})^q\) we have

\[
\sum_{j=1}^q f_j P_j \mathcal{z} = \left( f_1 + \sum_{j=2}^q f_j Q_j \right) P_{1z} + \sum_{j=2}^q f_j R_j \mathcal{z},
\]

where \(h_1 = f_1 + \sum_{j=2}^q f_j Q_j \mathcal{z}\). Thus the locally finite generation of \(\mathcal{R}(P_1, \ldots, P_q)\) is equivalent to that of \(\mathcal{R}(P_1, R_2, \ldots, R_q)\). Let

\[
T'_j = (R_j, 0, \ldots, 0, -P_1, 0, \ldots, 0), \quad 2 \leq j \leq q
\]

be the trivial solutions of \(\mathcal{R}(P_1, R_2, \ldots, R_q)\), which are \(z_n\)-polynomial-like sections of degree \(p'_1\) and \(z_n\)-polynomial-like elements \(\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_q)\) of \(\mathcal{R}'_b\) with coefficients in \(\mathcal{O}_{n,b}\) such that

\[
\text{deg}_{z_n} \alpha'_1 \leq p' - 2, \quad \text{deg}_{z_n} \alpha'_j \leq p' - 1, \quad 2 \leq j \leq q.
\]

**N.B.** If \(p' = 0\), then there is no term of \(\alpha'\), and if \(p' = 1\), then \(\alpha'_1 = 0\) and \(\alpha'_j\) are constants for \(2 \leq j \leq q\).

### 2 Proofs of Lemmas

(1) (Lemma 1.1) By making use of Weierstrass’ Preparation Theorem at \(b = (b', b_n)\) with \(b' \in P\Delta_{n-1}\) we decompose \(P_1\) to a unit \(u\) and a Weierstrass polynomial \(Q\):

\[
P_1(z', z_n) = u \cdot Q(z', z_n - b_n), \quad \text{deg}_{z_n} Q = d \leq p_1.
\]
Here and in the sequel we abbreviate $Q_z$ to $Q$ for the sake of notational simplicity; there will be no confusion.

It follows that $u \in \mathcal{O}_{n-1,b'}[z_n]$, and then

\begin{equation}
\text{deg}_{z_n} u = p_1 - d.
\end{equation}

Take an arbitrary $f = (f_1, \ldots, f_q) \in \mathcal{R}_b$. By Weierstrass’ Preparation Theorem we divide $f_i$ by $Q$:

\[
f_i = c_i Q + \beta_i, \quad 1 \leq i \leq q;
\]

\[
c_i \in \mathcal{O}_{n,b}, \quad \beta_i \in \mathcal{O}_{n-1,b'}[z_n],
\]

\begin{equation}
\text{deg}_{z_n} \beta_i < d.
\end{equation}

Since $u \in \mathcal{O}_{n,b}$ is a unit, with $\tilde{c}_i := c_i u^{-1}$ we get the division of $f_i$ by $P_1$:

\begin{equation}
\text{deg}_{z_n} \beta_i < d.
\end{equation}

By making use of this we have

\begin{equation}
(f_1, \ldots, f_q) + \tilde{c}_2 P_2 + \cdots + \tilde{c}_q P_q
= (\tilde{c}_1 P_1 + \beta_1, \tilde{c}_2 P_1 + \beta_2, \ldots, \tilde{c}_q P_1 + \beta_q)
+ (\tilde{c}_2 P_2, -\tilde{c}_2 P_1, 0, \ldots, 0)
+ \cdots
+ (\tilde{c}_q P_q, 0, \ldots, 0, -\tilde{c}_q P_1)
= \left( \sum_{i=1}^{q} \tilde{c}_i P_i + \beta_1, \beta_2, \ldots, \beta_q \right)
= (g_1, \beta_2, \ldots, \beta_q).
\end{equation}

Here we put $g_1 = \sum_{i=1}^{q} \tilde{c}_i P_i + \beta_1 \in \mathcal{O}_{n,b}$. Note that $\beta_i \in \mathcal{O}_{n-1,b'}[z_n], 2 \leq i \leq q$.

Since $(g_1, \beta_2, \ldots, \beta_q) \in \mathcal{R}_b$,

\begin{equation}
g_1 P_1 = -\beta_2 P_2 - \cdots - \beta_q P_q \in \mathcal{O}_{n-1,b'}[z_n].
\end{equation}
It should be noticed that if \( p_1 = 0 \), then \( P_1 = 1, \beta_i = 0, 1 \leq i \leq q \), and hence \( g_1 = 0 \); the proof is finished in this case.

In general, it follows from the expression of the above right-hand side of (2.5) that \( g_1 P_1 \in \mathcal{O}_{n-1, \nu}[z_n] \) and

\[
\deg_{z_n} g_1 P_1 \leq \max_{2 \leq i \leq q} \deg_{z_n} \beta_i + \max_{2 \leq i \leq q} \deg_{z_n} P_1 \leq d + p - 1.
\]

On the other hand, \( g_1 P_1 = g_1 uQ \) and \( Q \) is a Weierstrass' polynomial at \( b \). We see that

\[
\alpha_1 := g_1 u \in \mathcal{O}_{n-1, \nu}[z_n],
\]

(2.6)

\[
\deg_{z_n} \alpha_1 = \deg_{z_n} g_1 P_1 - \deg_{z_n} Q \leq d + p - 1 - d = p - 1.
\]

Set \( \alpha_i = u \beta_i \) for \( 2 \leq i \leq q \). Then, by (2.1) and (2.2) we have

(2.7) \quad \deg_{z_n} \alpha_i \leq p_1 - d + d - 1 = p_1 - 1 = p' - 1, \quad 2 \leq i \leq q,

and by (2.9) that

(2.8) \quad f = - \sum_{i=2}^{q} \tilde{c}_i T_i + u^{-1}(\alpha_1, \alpha_2, \ldots, \alpha_q).

\( \square \)

(2)(Lemma 1.4) First note that \((f_1, \ldots, f_q)\) and \((h_1, f_2, \ldots, f_q)\) with \( h_1 = f_1 + \sum_{j=2}^{q} f_j Q_j \) as defined in (1.3) are related by

\[
\begin{pmatrix}
  h_1 \\
  f_2 \\
  \vdots \\
  f_q
\end{pmatrix}
\begin{pmatrix}
  1 & Q_2 & \cdots & Q_q \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & 0 \\
  0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_q
\end{pmatrix} =
\begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_q
\end{pmatrix}
\begin{pmatrix}
  1 & -Q_2 & \cdots & -Q_q \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & 0 \\
  0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
  h_1 \\
  f_2 \\
  \vdots \\
  f_q
\end{pmatrix}.
\]
Therefore, the locally finite generation of $\mathcal{R}$ is equivalent to that of $\mathcal{R}'$.

The proof is similar to the above except for some degree estimates. Now we have for $(f_j) \in (\mathcal{O}_{n,b})^q$

\begin{align*}
(f_1, \ldots, f_q) + \ddot{c}_2 T'_2 + \cdots + \ddot{c}_q T'_q & = \left(\ddot{c}_1 P_1 + \beta_1 + \sum_{i=2}^q \ddot{c}_i R_i, \beta_2, \ldots, \beta_q \right) \\
& = (h_1, \beta_2, \ldots, \beta_q).
\end{align*}

Here we put $h_1 = \ddot{c}_1 P_1 + \beta_1 + \sum_{i=2}^q \ddot{c}_i R_i \in \mathcal{O}_{n,b}$. In stead of (2.5) we have

\begin{align*}
(2.10) \quad h_1 P_1 = -\beta_2 R_2 - \cdots - \beta_q R_q \in \mathcal{O}_{n-1,b}[z_n].
\end{align*}

From this we obtain

$$\deg_{z_n} h_1 P_1 \leq d - 1 + p' - 1 = d + p' - 2.$$ 

With $\alpha'_1 := h_1 u$ we have $h_1 P_1 = h_1 u Q = \alpha'_1 Q$ and so

$$\deg_{z_n} \alpha'_1 \leq d + p' - 2 - d = p' - 2.$$ 

For $\alpha_i = u \beta_i$, $2 \leq i \leq q$ we have the same estimate:

$$\deg_{z_n} \alpha_i \leq p' - 1.$$ 

With the above defined we have

$$f = -\sum_{i=2}^q \ddot{c}_i T'_i + u^{-1}(\alpha'_1, \alpha_2, \ldots, \alpha_q).$$

\begin{flushright}
$\square$
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References


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