

Analytic Ax-Schanuel for semi-abelian varieties and Nevanlinna theory

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Abstract

Let A be a semi-abelian variety with an exponential map $\exp : \text{Lie}(A) \rightarrow A$. The purpose of this paper is to explore Nevanlinna theory of the entire curve $\widehat{\exp} f := (\exp f, f) : \mathbf{C} \rightarrow A \times \text{Lie}(A)$ associated with an entire curve $f : \mathbf{C} \rightarrow \text{Lie}(A)$. Firstly we give a Nevanlinna theoretic proof to the *analytic Ax-Schanuel Theorem* for semi-abelian varieties, which was proved by J. Ax 1972 in the case of formal power series. (Ax-Schanuel Theorem). We assume some non-degeneracy condition for f such that the elements of the vector-valued function $f(z) - f(0) \in \text{Lie}(A) \cong \mathbf{C}^n$ are \mathbf{Q} -linearly independent in the case of $A = (\mathbf{C}^*)^n$. Our proof is based on the Log Bloch-Ochiai Theorem and a key estimate which we show.

Our next aim is to establish a *2nd Main Theorem* for $\widehat{\exp} f$ and its k -jet lifts with truncated counting functions at level one. We give some applications to a problem of a type raised by S. Lang and the unicity. The results clarify a relationship between the problems of Ax-Schanuel type and Nevanlinna theory.

Keywords: analytic Ax-Schanuel; Nevanlinna theory; semi-abelian Schanuel; Log Bloch-Ochiai; value distribution theory.

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1. Introduction

Let A be a semi-abelian variety of dimension $n (> 0)$ with Lie algebra $\text{Lie}(A)$ and an exponential map $\exp : \text{Lie}(A) \rightarrow A$. Let $f : \mathbf{C} \rightarrow \text{Lie}(A)$ be an entire curve (i.e., a holomorphic map from \mathbf{C} into $\text{Lie}(A)$), and set

$$(1.1) \quad \exp f := \exp \circ f : \mathbf{C} \rightarrow A, \quad \widehat{\exp} f := (\exp f, f) : \mathbf{C} \rightarrow \widehat{A} := A \times \text{Lie}(A).$$

We denote by $\overline{\widehat{\exp} f(\mathbf{C})}^{\text{Zar}}$ the Zariski closure of $\widehat{\exp} f(\mathbf{C})$ in \widehat{A} . Put

$$\text{tr. deg}_{\mathbf{C}} \widehat{\exp} f := \dim_{\mathbf{C}} \overline{\widehat{\exp} f(\mathbf{C})}^{\text{Zar}}.$$

We say that f is “ A -(resp. non)degenerate” if

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1.2 (subgroup condition) there is a (resp. no) connected algebraic proper subgroup $G \subsetneq A$ such that $G + \exp f(0) \supset \exp f(\mathbf{C})$.

(1) It is our first aim to give a Nevanlinna theoretic proof to the analytic version of the Ax-Schanuel Theorem for semi-abelian varieties.

Theorem 1.3 *Let A be a semi-abelian variety with $\text{Lie}(A)$ as above. If an entire curve $f : z \in \mathbf{C} \rightarrow \text{Lie}(A)$ is A -nondegenerate, then $\text{tr. deg}_{\mathbf{C}} \widehat{\exp} f \geq n + 1$.*

The present study is motivated by S. Lang [10], and J. Ax [1], [2], where Ax proved the above statement for *formally analytic* maps $f : \mathbf{C} \rightarrow \text{Lie}(A)$, i.e., f represented by a vector of *formal power series* (**Ax-Schanuel Theorem**), where in the case of $f : \mathbf{C} \rightarrow \mathbf{C}^n$ and $\exp_{(\mathbf{C}^*)^n} : \mathbf{C}^n \ni (w_j) \mapsto (e^{w_j}) \in (\mathbf{C}^*)^n$, $f_j(z) - f_j(0)$ ($1 \leq j \leq n$) are assumed to be \mathbf{Q} -linearly independent; he dealt with the case of formal power series of several variables. The proof of Ax [1], [2] is based on Kolchin's theory of differential algebra. It is noticed that the above \mathbf{Q} -linear independence of $f_j(z) - f_j(0)$ ($1 \leq j \leq n$) is equivalent to subgroup condition 1.2 with "no".

Because of the nature of the statement, it is natural and interesting to look for an analytic proof in the analytic setting. The present proof relies on the Log Bloch-Ochiai Theorem [12], [13], and Nevanlinna theory of entire curves into semi-abelian varieties (cf. Noguchi-Winkelmann [18] in general).

The above Ax-Schanuel Theorem was obtained as an analogue of the following attractive conjecture:

1.4 **Schanuel's Conjecture** (cf. Lang [10] p.30). Let $\alpha = (\alpha_j) \in \mathbf{C}^n$ be a vector such that α_j ($1 \leq j \leq n$) are \mathbf{Q} -linearly independent, and set $\widehat{\exp} \alpha = ((e^{\alpha_j}), \alpha) \in (\mathbf{C}^*)^n \times \mathbf{C}^n$. Then, $\text{tr. deg}_{\mathbf{Q}} \widehat{\exp} \alpha \geq n$.

This conjecture is known only for $n = 1$ (Gel'fond-Schneider ([10], [26]); Hilbert's 7th Problem), and even in $n = 2$ it implies the algebraic independence of e and π ; cf., e.g., Waldschmidt [26] §1, Jones-Wilkie [9] for more.

(2) Our second aim is to establish a 2nd Main Theorem for $\widehat{\exp} f : \mathbf{C} \rightarrow \widehat{A}$ and its jet lifts. Let $J_k(\widehat{\exp} f) : \mathbf{C} \rightarrow J_k(\widehat{A})$ be the k -jet lift of $\widehat{\exp} f$ and let $X_k(\widehat{\exp} f)$ be the Zariski closure of the image $J_k(\widehat{\exp} f)(\mathbf{C})$ in $J_k(\widehat{A})$. Because of the flat structures of $J_k(A) \cong A \times J_{k,A}$ and $J_k(\text{Lie}(A)) \cong \text{Lie}(A) \times J_{k,\text{Lie}(A)}$ we may write

$$(1.5) \quad X_k(\widehat{\exp} f) \subset A \times \widehat{J}_{k,A} \subset J_k(A \times \text{Lie}(A)), \quad \widehat{J}_{k,A} := \text{Lie}(A) \times J_{k,A} \cong \mathbf{C}^n \times \mathbf{C}^{nk}$$

after the natural identification $J_{k,A} \cong J_{k,\text{Lie}(A)}$ (see §4.4 for details). We have an A -action on $A \times \widehat{J}_{k,A}$ by the group operation of the first factor.

Let $\widetilde{J}_{k,A}$ be a projective compactification of $\widehat{J}_{k,A}$; e.g., $\widetilde{J}_{k,A} = \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^{nk}(\mathbf{C})$.

The main result is as follows (cf. §§2, 4 for notation):

Theorem 1.6 (2nd Main Theorem) *Let A be a semi-abelian variety. Let $\widehat{\exp} f : \mathbf{C} \rightarrow \widehat{A}$ be the entire curve associated with an A -nondegenerate entire curve $f : \mathbf{C} \rightarrow \text{Lie}(A)$. Then we have:*

- (i) Let Z be a reduced algebraic subset of $X_k(\widehat{\exp} f)$ ($\subset A \times \widehat{J}_{k,A}$) ($k \geq 0$). Then there exists an equivariant projective smooth compactification \bar{A} of A with the closure $\bar{X}_k(\widehat{\exp} f)$ (resp. \bar{Z}) of $X_k(\widehat{\exp} f)$ (resp. Z) in $\bar{A} \times \widehat{J}_{k,A}$ such that

$$(1.7) \quad T_{J_k(\widehat{\exp} f)}(r, \omega_{\bar{Z}}) = N_1(r, J_k(\widehat{\exp} f)^* Z) + S_{\varepsilon, \exp f}(r).$$

- (ii) Moreover, if $\text{codim}_{X_k(\widehat{\exp} f)} Z \geq 2$, then

$$(1.8) \quad T_{J_k(\widehat{\exp} f)}(r, \omega_{\bar{Z}}) = S_{\varepsilon, \exp f}(r).$$

- (iii) ($k = 0$) In particular, if D is a reduced effective algebraic divisor on \widehat{A} such that $D \not\subset X_0(\widehat{\exp} f)$, then there is an equivariant projective smooth compactification \bar{A} of A such that

$$(1.9) \quad T_{\widehat{\exp} f}(r, L(\bar{D})) = N_1(r, (\widehat{\exp} f)^* D) + S_{\varepsilon, \widehat{\exp} f}(r),$$

where $\bar{D} \subset \bar{\widehat{A}} = \bar{A} \times \overline{\text{Lie}(A)}$ with $\overline{\text{Lie}(A)} := \widehat{J}_{0,A}$.

As applications we obtain the following (cf. question 4.1 (ii), list 4.2 (6) in §4 below, and Corvaja-Noguchi [4] for entire curves into semi-abelian varieties):

Theorem 1.10 *Let $\widehat{\exp} f : \mathbf{C} \rightarrow \widehat{A}$ be as in Theorem 1.6 and let D be an A -big divisor on $X_0(\widehat{\exp} f)$. Then there is an irreducible component E of D such that $\widehat{\exp} f(\mathbf{C}) \cap E$ is Zariski dense in E ; in particular, the cardinality $|\widehat{\exp} f(\mathbf{C}) \cap D| = \infty$.*

Here, D being A -big means roughly that D contains a big divisor in A -factor (see Definition 4.17). The next theorem says that the distribution $\widehat{\exp} f^* D$ on \mathbf{C} contains an ample information of \widehat{A} , D and f ; we have the following unicity theorem of the type of H. Cartan–P. Erdős–K. Yamanoi (cf. Yamanoi [28], Corvaja-Noguchi [4]).

Theorem 1.11 *Let A_j ($j = 1, 2$) be two semi-abelian varieties and let D_j ($j = 1, 2$) be effective reduced A_j -big divisors on \widehat{A}_j with $\widehat{\text{St}}(D_j) := \{x \in \widehat{A}_j : x + D_j = D_j\} = \{0\}$. Let $\widehat{\exp} f_j : \mathbf{C} \rightarrow \widehat{A}_j$ be entire curves induced from A_j -nondegenerate entire curves $f_j : \mathbf{C} \rightarrow \text{Lie}(A_j)$.*

Assume that $\text{Supp}(\widehat{\exp} f_1)^ D_1 = \text{Supp}(\widehat{\exp} f_2)^* D_2$ (as supports of divisors). Then there is an isomorphism $\alpha : A_1 \rightarrow A_2$ with the naturally induced isomorphism $\hat{\alpha} : \widehat{A}_1 \rightarrow \widehat{A}_2$ such that $\hat{\alpha}^* D_2 = D_1$ and $\widehat{\exp} f_2 = \hat{\alpha} \circ \widehat{\exp} f_1$, up to translations of \widehat{A}_j .*

The present paper is organized so that in §2 we prepare the notation from Nevanlinna theory and prove a key estimate lemma (see Lemma 2.5). We give a proof of Theorem 1.3 in §3 and those of Theorems 1.6 and 1.10 in §4. In §5 we give the proofs of Theorems 1.10 and 1.11. In §6 we discuss some examples. In §7 we discuss the generalization of the domain \mathbf{C} to other cases, to say, a punctured disk (big Picard type), affine algebraic curves, and others.

Remark 1.12 We would like to refer to some other results related to the present subject and remarks.

- (i) J. Tsimerman [25] gave another proof to the above Ax-Schanuel Theorem [1] for $A = (\mathbf{C}^*)^n$ by means of the “o-minimal structure” theory, and similarly Y. Peterzil and S. Starchenko [22] for semi-abelian varieties.
- (ii) In Noguchi [16], there is a direct application of the Log Bloch-Ochiai Theorem to the proof of Raynaud’s Theorem (Manin-Mumford Conjecture) through the aid of the “o-minimal structure” theory.
- (iii) In the analytic theory we can think of the “values”; it might be an advantage of the analytic theory, compared with the formally analytic theory.

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2. Order functions and a key lemma

2.1. Order functions

In general for order functions, cf. Hayman [8] Chap’s. 1, 2, Noguchi-Ochiai [17] Chap. V, and Noguchi-Winkelmann [18] Chap. 2.

Let X be a compact complex space with reduced structure sheaf \mathcal{O}_X , and let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf. Let $f : \mathbf{C} \rightarrow X$ be an entire curve. We are going to define an order function $T_f(r, \omega_{\mathcal{I}})$ of f with respect to \mathcal{I} (see [18] §2.4 for details: Note that the projective algebraic condition on X is unnecessary because of 1-dimensionality of the domain \mathbf{C}).

The pull-back $f^*\mathcal{I}$ is an effective divisor on \mathbf{C} , unless $f(\mathbf{C}) \subset \text{Supp } \mathcal{I} (= \{x \in X : \mathcal{I}_x \neq \mathcal{O}(X)_x\})$, which we assume. Denoting by $\text{ord}_z f^*\mathcal{I}$ the order of $f^*\mathcal{I}$ at $z \in \mathbf{C}$, we define the counting functions of $f^*\mathcal{I}$ truncated at level $l \in \mathbf{N} \cup \{\infty\}$ by

$$n_l(t, f^*\mathcal{I}) = \sum_{|z| < t} \min\{\text{ord}_z f^*\mathcal{I}, l\}, \quad N_l(t, f^*\mathcal{I}) = \int_1^t \frac{n_l(t, f^*\mathcal{I})}{t} dt, \quad r > 1.$$

Let $\phi_{\mathcal{I}}(x)$ ($x \in X$) be the proximity potential of \mathcal{I} (see [18] §2.4). Then we have the pull-back $f^*\phi_{\mathcal{I}}(z)$ such that it is of C^∞ -class outside the singular set $f^{-1}(\text{Supp } \mathcal{I})$ and at a singular point $a \in f^{-1}(\text{Supp } \mathcal{I})$ it is written locally as

$$f^*\phi_{\mathcal{I}}(z) = (\text{ord}_a f^*\mathcal{I}) \log |z - a| + C^\infty\text{-term.}$$

We set the proximity function of f for \mathcal{I} by

$$m_f(r, \mathcal{I}) = \frac{1}{2\pi} \int_{|z|=r} -f^*\phi_{\mathcal{I}}(re^{i\theta}) d\theta.$$

With $\omega_{\mathcal{I}} := \frac{1}{\pi i} \partial \bar{\partial} \phi_{\mathcal{I}}$ we define the order function of f with respect to $\omega_{\mathcal{I}}$ by

$$T_f(r, \omega_{\mathcal{I}}) = \int_1^r \frac{dt}{t} \int_{|z| < t} f^*\omega_{\mathcal{I}} \quad (r > 1).$$

We have a so-called First Main Theorem for f and \mathcal{I} (see [18] Theorem 2.4.9):

$$(2.1) \quad T_f(r, \omega_{\mathcal{I}}) = N_\infty(r, f^*\mathcal{I}) + m_f(r, \mathcal{I}) - m_f(1, \mathcal{I}).$$

In the case where $\mathcal{I} = \mathcal{I}\langle Y \rangle$ is the ideal sheaf of an analytic subset Y possibly non-reduced of X , we write

$$f^* \mathcal{I}\langle Y \rangle = f^* Y, \quad \omega_Y = \omega_{\mathcal{I}\langle Y \rangle}, \quad T_f(r, \omega_Y) = T_f(r, \omega_{\mathcal{I}\langle Y \rangle}).$$

If X is smooth and Y is a divisor D on X . Then we have the line bundle $L(D) \rightarrow X$ associated with D , and the first Chern class $c_1(D)$. Then, $\omega_D \in c_1(D)$, and it is common to write

$$T_f(r, \omega_D) = T_f(r, L(D)) = T_f(r, c_1(D))$$

for the order functions.

Assume that X is projective algebraic. Let D and D' be big divisors on X such that $f(\mathbf{C}) \not\subset \text{Supp } D \cup \text{Supp } D'$. Then we have

$$T_f(r, L(D)) = O(T_f(r, L(D'))), \quad T_f(r, L(D')) = O(T_f(r, L(D))).$$

Therefore in an estimate such as $O(T_f(r, L(D)))$ the choice of D or $L(D)$ does not matter; in such a case we simply write $T_f(r)$ for $T_f(r, L(D))$ with respect to some ample or big line bundle $L(D)$ over X as far as $f(\mathbf{C}) \not\subset \text{Supp } D$; however, once it is chosen, it is fixed.

Remark 2.2 The followings are equivalent for $f : \mathbf{C} \rightarrow X$:

- (i) f is rational (not transcendental);
- (ii) $\varliminf_{r \rightarrow \infty} \frac{T_f(r)}{\log r} < \infty$.
- (iii) $\overline{\lim}_{r \rightarrow \infty} \frac{T_f(r)}{\log r} < \infty$; i.e., $T_f(r) = O(\log r)$.

The order ρ_f of $f : \mathbf{C} \rightarrow X$ is defined by

$$0 \leq \rho_f = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} \leq \infty.$$

If $\rho_f < \infty$, f is said to be of finite order and

$$T_f(r) = O(r^{\rho_f + \varepsilon}), \quad \forall \varepsilon > 0.$$

2.2. Key lemma

Let A be a semi-abelian variety. Here we fix an isomorphism $\text{Lie}(A) \cong \mathbf{C}^n$ with coordinates (z_1, \dots, z_n) . Let $f = (f_1, \dots, f_n) : z \in \mathbf{C} \rightarrow f(z) \in \mathbf{C}^n (\cong \text{Lie}(A))$ be an entire curve. In the present paper we assume that f is non-constant. We denote by $T(r, f_j)$ Nevanlinna's order function (see [8], [18] Chap. 1), and set

$$T_f(r) = \max_{1 \leq j \leq n} T(r, f_j).$$

Let \bar{A} be an equivariant smooth projective compactification of A with boundary divisor $\partial A = \bar{A} \setminus A$ of simple normal crossing type (cf. [18] Chap. 5), and let $L \rightarrow \bar{A}$ be an ample line bundle. We set the order function of $\exp f : \mathbf{C} \rightarrow A$ by

$$T_{\exp f}(r) = T_{\exp f}(r, L).$$

For $\widehat{\exp} f = (\exp f, f) : \mathbf{C} \rightarrow A \times \mathbf{C}^n (\cong \widehat{A})$ we set

$$T_{\widehat{\exp} f}(r) := T_{\exp f}(r) + T_f(r).$$

By [18] Theorem 6.1.9 and Remark 2.2 we see

Proposition 2.3 *Let the notation be as above. The followings are equivalent:*

- (i) $f : \mathbf{C} \rightarrow \mathbf{C}^n$ is rational (i.e., f_j are polynomials).
- (ii) $\exp f$ is of finite order.
- (iii) $\widehat{\exp} f$ is of finite order.

As usual in Nevanlinna theory we use the symbol “ $S_{\exp f}(r)$ ” for a small term such that

$$(2.4) \quad |S_{\exp f}(r)| = O(\log^+ T_{\exp f}(r)) + O(\log r) + O(1) \quad ||,$$

where “||” stands for the validity of the estimate except for r in exceptional intervals with finite total length, and for $\exp f$ of finite order there are no such exceptional intervals. We use the notation through the paper.

The following is the key lemma for the estimates in the arguments henceforth.

Lemma 2.5 *With the notation above, we have:*

- (i) $T_f(r) = S_{\exp f}(r)$.
- (ii) $T_{\widehat{\exp} f}(r) = T_{\exp f}(r) + S_{\exp f}(r)$.

Proof. It suffices to prove (i). When the order of $\exp f$ is finite, then f is rational, and so $T_f(r) = O(\log r)$ without exceptional intervals.

In general, we take a representation of the semi-abelian variety A

$$0 \rightarrow (\mathbf{C}^*)^p \rightarrow A \rightarrow A_0 \rightarrow 0,$$

where A_0 is an abelian variety. Then A has a structure of locally flat $(\mathbf{C}^*)^p$ -principal bundle with transition transformation by $(S^1)^p := \{\zeta \in \mathbf{C} : |\zeta| = 1\}^p$ -multiplication (cf. [18] §6.1). After a change of indices of the coordinates (z_j) and a linear transform of (z_j) we have the following expression of the order function $T_{\exp f}(r)$ (cf. *ibid.*):

$$(2.6) \quad \begin{aligned} f(z) &= (f_1(z), \dots, f_p(z), f_{p+1}, \dots, f_n(z)), \\ T_1(r) &:= \sum_{j=1}^p T(r, e^{f_j}), \\ T_2(r) &:= \frac{1}{4\pi} \int_{|z|=r} \sum_{j=p+1}^n |f_j(z)|^2 d\theta - \frac{1}{4\pi} \int_{|z|=1} \sum_{j=p+1}^n |f_j(z)|^2 d\theta, \\ T_{\exp f}(r) &= T_1(r) + T_2(r). \end{aligned}$$

For f_j ($p+1 \leq j \leq n$) we have

$$\begin{aligned} T(r, f_j) &= \frac{1}{2\pi} \int_{|z|=r} \log^+ |f_j| d\theta \leq \frac{1}{4\pi} \int_{|z|=r} \log(1 + |f_j|^2) d\theta \\ &= \frac{1}{2} \log \left(1 + \frac{1}{2\pi} \int_{|z|=r} |f_j|^2 d\theta \right) \leq \frac{1}{2} \log^+ T_2(r) + O(1) \\ &= O(\log^+ T_{\exp f}(r)) + O(1). \end{aligned}$$

For f_j ($1 \leq j \leq p$) we have

$$T(r, e^{f_j}) = \frac{1}{2\pi} \int_{|z|=r} \log^+ |e^{f_j(z)}| d\theta = \frac{1}{2\pi} \int_{|z|=r} \Re^+ f_j(z) d\theta,$$

where $\Re^+ f_j := \max\{\Re f_j, 0\}$ with the real part $\Re f_j$. With the imaginary part $\Im f_j(0)$ and the complex Poisson kernel we write

$$f_j(z) = \frac{1}{2\pi} \int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z} \Re f_j(\zeta) d\theta + \Im f_j(0).$$

For $|z| = r < R$ we get

$$|f_j(z)| \leq \frac{R+r}{R-r} \cdot \frac{1}{2\pi} \int_{|\zeta|=R} \Re^+ f_j(\zeta) d\theta + |\Im f_j(0)|.$$

Then we have

$$\begin{aligned} T(r, f_j) &\leq \frac{1}{2\pi} \int_{|\zeta|=r} \log(1 + |f_j(z)|) d\theta \leq \log \left(1 + \frac{1}{2\pi} \int_{|\zeta|=r} |f_j(z)| d\theta \right) \\ &\leq \log \left(1 + \frac{R+r}{R-r} \cdot \frac{1}{2\pi} \int_{|\zeta|=R} \Re^+ f_j(\zeta) d\theta + O(1) \right) \\ &\leq \log \left(1 + \frac{R+r}{R-r} \cdot T_1(R) + O(1) \right) \\ &\leq \log^+ \left(\frac{R+r}{R-r} \cdot T_1(R) \right) + O(1) \end{aligned}$$

Now we take $R = r + 1/T_1(r)$, so that

$$T(r, f_j) \leq \log^+ \left((2r+1)T_1(r) \cdot T_1 \left(r + \frac{1}{T_1(r)} \right) \right) + O(1)$$

Borel's Lemma (cf. Hayman [8] Lemma 2.4) implies

$$T_1 \left(r + \frac{1}{T_1(r)} \right) \leq 2T_1(r) \quad ||.$$

Therefore it follows that

$$\begin{aligned} T(r, f_j) &= O(\log^+ T_1(r)) + O(\log r) + O(1) \\ &= O(\log^+ T_{\exp f}(r)) + O(\log r) + O(1) \quad ||. \end{aligned}$$

The proof is completed. □

3. Proof of Theorem 1.3

By [13] we see that the Zariski closure $\overline{\exp f(\mathbf{C})}^{\text{Zar}}$ in A is a translation of a connected algebraic subgroup (a semi-abelian subvariety) of A . It follows from A -nondegeneracy (cf. subgroup condition 1.2) that $\overline{\exp f(\mathbf{C})}^{\text{Zar}} = A$, so that

$$(3.1) \quad \text{tr. deg}_{\mathbf{C}} \exp f = n.$$

Let $\mathbf{C}(A)$ (resp. $\mathbf{C}(f)$) be the rational function field of A (resp. the field generated by f_j ($1 \leq j \leq n$) over \mathbf{C}), and let $\mathbf{C}(f, (\exp f)^*\mathbf{C}(A))$ be the field generated by elements of $\mathbf{C}(f)$ and the pull-backed field $(\exp f)^*\mathbf{C}(A)$ over \mathbf{C} . We denote the transcendence degree of the field $\mathbf{C}(f, (\exp f)^*\mathbf{C}(A))$ over $\mathbf{C}(f)$ by $\text{tr. deg}_{\mathbf{C}(f)} \mathbf{C}(f, (\exp f)^*\mathbf{C}(A))$. We prove:

Lemma 3.2 *With the notation above we have*

$$(3.3) \quad \text{tr. deg}_{\mathbf{C}(f)} \mathbf{C}(f, (\exp f)^* \mathbf{C}(A)) \geq 1.$$

Proof. We take a transcendence basis $\{\phi_j\}_{j=1}^n$ of $\mathbf{C}(A)$ over \mathbf{C} such that $\tilde{\phi}_j := \phi_j \circ \exp f$ are defined as non-constant meromorphic functions, and

$$\tilde{\phi} := (\tilde{\phi}_1, \dots, \tilde{\phi}_n).$$

Assume contrarily that (3.3) is false; i.e.,

$$(3.4) \quad \text{tr. deg}_{\mathbf{C}(f)} \tilde{\phi} = 0 \quad (\text{regarded as } \tilde{\phi} = \{\tilde{\phi}_j\}_{j=1}^n).$$

Then all $\tilde{\phi}_j$ are algebraic over $\mathbf{C}(f)$. There are non-zero polynomials $P_j(t)$ in one variable with coefficients in $\mathbf{C}(f)$ such that

$$(3.5) \quad P_j(\tilde{\phi}_j) = 0, \quad 1 \leq j \leq n.$$

By [18] Lemma 2.5.15 we have

$$T(r, \tilde{\phi}_j) = O(T_f(r)) + O(1).$$

With setting $\tilde{T}(r) := \max_{1 \leq j \leq n} T(r, \tilde{\phi}_j)$ we thus obtain

$$\tilde{T}(r) = O(T_f(r)) + O(1).$$

On the other hand, it follows from [18] Theorem 2.5.18 that

$$T_{\exp f}(r) = O(\tilde{T}(r)) + O(1).$$

Therefore we see that

$$T_{\exp f}(r) = O(T_f(r)) + O(1).$$

But this contradicts Lemma 2.5. □

Continuation of the proof of Theorem 1.3: By (3.1), $\text{tr. deg}_{\mathbf{C}} \{f, \tilde{\phi}\} \geq n$. For proof by contradiction we assume that

$$\text{tr. deg}_{\mathbf{C}} \{f, \tilde{\phi}\} = n.$$

Then all f_j are algebraic over $\mathbf{C}(\tilde{\phi})$, so that there are non-trivial algebraic relations,

$$(3.6) \quad P_j(f_j, \tilde{\phi}) = P_j(f_j, \tilde{\phi}_1, \dots, \tilde{\phi}_n) = 0, \quad 1 \leq j \leq n.$$

If $\text{tr. deg}_{\mathbf{C}} \{f_j\}_{j=1}^n = n$, the assumption implies $\text{tr. deg}_{\mathbf{C}(f)} \{f, \tilde{\phi}\} = 0$; this does not take place by Lemma 3.2. Therefore $\text{tr. deg}_{\mathbf{C}} \{f_j\}_{j=1}^n < n$, and hence there is a non-trivial algebraic relation over \mathbf{C} :

$$(3.7) \quad Q(f_1, \dots, f_n) = 0.$$

If, to say, f_1 is contained in (3.7), we take the resultant of (3.7) and (3.6) ($j = 1$) with respect to f_1 , which yields a non-trivial algebraic relation

$$Q_1(f_2, \dots, f_n, \tilde{\phi}) = 0.$$

After repeating this process at most n -times we eliminate f_1, \dots, f_n in (3.7) to obtain a non-trivial algebraic relation

$$\tilde{Q}(\tilde{\phi}_1, \dots, \tilde{\phi}_n) = 0;$$

this again contradicts the algebraic independence of $\tilde{\phi}_1, \dots, \tilde{\phi}_n$.

The proof of Theorem 1.3 is completed. \square

Remark 3.8 (i) It is noticed that the logarithmic function $\log(1+t) \in \mathbf{C}[[t]]$ can be dealt with in the Ax-Schanuel Theorem as a formal power series, but cannot in our Theorem 1.3. To deal with the case of a finite unit disk as a domain instead of \mathbf{C} we need some growth condition for $T_{\exp f}(r)$ (cf. §7.3).

- (ii) Let $\exp : \mathbf{C}^n = \text{Lie}(A) \rightarrow A$ be as above. We have a semi-lattice $\Lambda = \text{Ker } \exp \subset \mathbf{C}^n$ (the periods of A). Then an entire curve $f : \mathbf{C} \rightarrow \mathbf{C}^n$ is A -(resp. non)degenerate if and only if there is a (resp. no) complex vector subspace $E \subsetneq \mathbf{C}^n$ such that $E \supset (f(\mathbf{C}) - f(0))$ and $E/(E \cap \Lambda)$ is a semi-abelian variety. Therefore if Λ is concerned, it would be better to say f being Λ -(non)degenerate.

4. Nevanlinna theory of entire curves $\widehat{\exp} f$

4.1. Back ground

In the same monograph [10] (p. 32) (1966) as Schanuel's Conjecture 1.4 was mentioned, S. Lang raised an interesting question:

- 4.1 (i) Let $\varphi : \mathbf{C} \rightarrow A$ be a 1-parameter subgroup of an abelian variety A (say Zariski dense), and let D be a hyperplane section of A . Then, is $\varphi(\mathbf{C}) \cap D \neq \emptyset$?
- (ii) And unless φ is algebraic, is the cardinality $|\varphi(\mathbf{C}) \cap D| = \infty$?

It has developed roughly as the following list (far from completeness):

- 4.2 (1) J. Ax [2] (1972) gave an affirmative answer to question 4.1 (i) above.
- (2) P.A. Griffiths [7] (Problem F, 1972) generalized question 4.1 (i) for entire curves into A (so-called Lang's Conjecture).
- (3) At the Taniguchi Symposium "Geometric Function Theory in Katata 1978 organized by S. Murakami (chair), the author formulated a 2nd main theorem for entire curves $f : \mathbf{C} \rightarrow A$ and a divisor D on A as a conjecture, which implies (2) above and question 4.1 (i) as well; see Noguchi-Ochiai [17] (p. 248, 1984/'90).
- (4) Siu and Yeung [24] (1996) solved Lang's Conjecture (2) above for entire curves into abelian varieties, and Noguchi [15] (1998) generalized it for entire curves into semi-abelian varieties with another proof, which unifies the result for abelian varieties and the classical E. Borel's results for $(\mathbf{C}^*)^n$.
- (5) Noguchi, Winkelmann and Yamanoi [19] (2000), [20] (2002) proved (3) above, the 2nd Main Theorem for entire curves into semi-abelian varieties, and finally in [21] proved it with counting functions truncated at level one.
- (6) P. Corvaja and J. Noguchi [4] (2012) solved affirmatively question 4.1 (ii) for entire curves into semi-abelian varieties, $f : \mathbf{C} \rightarrow A$ by making use of the 2nd Main Theorem of (5) above. It is noticed that question 4.1 (ii) had been open even for 1-parameter subgroups of abelian varieties.

It is natural and interesting to ask questions similar to the above for

$$\widehat{\exp} f : z \in \mathbf{C} \rightarrow (\exp f(z), f(z)) \in \widehat{A} (= A \times \text{Lie}(A))$$

in view of the analytic Ax-Schanuel Theorem 1.3 and Schanuel's Conjecture 1.4.

4.2. 2nd Main Theorem

We denote by $S_{\varepsilon, \exp f}(r)$ (≥ 0) a small term such that for every small $\varepsilon > 0$ (cf. (2.4))

$$|S_{\varepsilon, \exp f}(r)| \leq \varepsilon T_{\exp f}(r) + O(\log r) \quad ||_{\varepsilon},$$

where “ $||_{\varepsilon}$ ” stands for the exceptional intervals being dependent on $\varepsilon > 0$.

The main aim is to prove Theorem 1.6 for $\widehat{\exp} f : \mathbf{C} \rightarrow \widehat{A}$, which is analogous to [18] Theorem 6.5.1 (cf. [21]) for $\exp f : \mathbf{C} \rightarrow A$. We prove Theorem 1.6 applying the arguments in [18] Chap. 6 (cf. [20], [21]) by making use of the key Lemmata 2.5, 4.9. Henceforward we will describe the key points.

The way to obtain the compactifications of \bar{A} and $X_k(\widehat{\exp} f)$ in Theorem 1.6 is not written precisely in [18] Theorem 6.5.1, but it follows from the arguments of the proof there.

4.3. Reduction

Let $f : \mathbf{C} \rightarrow \text{Lie}(A)$ be an entire curve. By the Log Bloch–Ochiai Theorem ([18] Theorem 6.2.1, [12], [13]) $\overline{\exp f(\mathbf{C})}^{\text{Zar}}$ is a translation of a subgroup B of A . By a translation we may assume that $\overline{\exp f(\mathbf{C})}^{\text{Zar}} = B$. Then $f(\mathbf{C}) \subset \text{Lie}(B) (\subset \text{Lie}(A))$, and

$$\widehat{\exp} f : \mathbf{C} \longrightarrow B \times \text{Lie}(B) = \widehat{B} \subset \widehat{A}.$$

Now, $f : \mathbf{C} \rightarrow \text{Lie}(B)$ is B -nondegenerate. Therefore without loss of generality we may assume that $B = A$, i.e. f is A -nondegenerate.

4.4. Jet bundles

We keep the same notation as in the previous subsections. Let $f : \mathbf{C} \rightarrow \mathbf{C}^n$ be an A -nondegenerate entire curve. We would like to study the value distribution of $\widehat{\exp} f : \mathbf{C} \rightarrow A \times \mathbf{C}^n$.

Let $J_k(A) \rightarrow A$ (resp. $J_k(\text{Lie}(A)) \rightarrow \text{Lie}(A)$) be the k -th jet bundle over A (resp. $\text{Lie}(A)$). Because of the flat structure of the logarithmic tangent (and cotangent, as well) bundle over A (cf. [18] §4.6.3), we have the trivializations:

$$(4.3) \quad \begin{aligned} J_k(A) &\cong A \times J_{k,A}, & J_{k,A} &\cong \mathbf{C}^{nk}, \\ J_k(\text{Lie}(A)) &\cong \text{Lie}(A) \times J_{k,\text{Lie}(A)}, & J_{k,\text{Lie}(A)} &\cong \mathbf{C}^{nk}, \end{aligned}$$

where $J_{k,A}$ and $J_{k,\text{Lie}(A)}$ are the so-called jet-parts of A and $\text{Lie}(A)$, respectively. Through the exponential map $\exp : \text{Lie}(A) \rightarrow A$ we have the natural isomorphism $J_{k,A} \cong J_{k,\text{Lie}(A)}$, which are identified. Therefore we have

$$(4.4) \quad J_k(\widehat{A}) \cong A \times \text{Lie}(A) \times J_{k,A} \times J_{k,A}.$$

Let $\Delta_k \subset J_{k,A} \times J_{k,A}$ be the diagonal, and let $J_k(\widehat{\exp} f) : \mathbf{C} \rightarrow J_k(\widehat{A})$ be the k -jet lift of $\widehat{\exp} f$. Then we see that

$$(4.5) \quad J_k(\widehat{\exp} f) : z \in \mathbf{C} \rightarrow (\exp f(z), f(z), J_{k,f}(z), J_{k,f}(z)) \in A \times \text{Lie}(A) \times \Delta_k,$$

where $J_{k,f}$ is the jet part of $J_k(f) = (f, J_{k,f}) \in \text{Lie}(A) \times J_{k,A}$. For the sake of simplicity we identify $\Delta_k = J_{k,A}$ and write

$$(4.6) \quad J_k(\widehat{\exp} f) : z \in \mathbf{C} \rightarrow (\exp f(z), f(z), J_{k,f}(z)) \in A \times \text{Lie}(A) \times J_{k,A} \subset J_k(A \times \text{Lie}(A)).$$

We put

$$(4.7) \quad \begin{aligned} \widehat{J}_{k,A} &= \text{Lie}(A) \times J_{k,A} \cong \mathbf{C}^n \times \mathbf{C}^{nk} \quad (\text{extended jet part}), \\ X_k(\widehat{\exp} f) &= \overline{J_k(\widehat{\exp} f)(\mathbf{C})}^{\text{Zar}} \subset A \times \widehat{J}_{k,A}. \end{aligned}$$

We define the *extended jet projection* by

$$(4.8) \quad \widehat{I}_k : X_k(\widehat{\exp} f) \subset A \times \widehat{J}_{k,A} \longrightarrow \widehat{J}_{k,A},$$

which will play the role of the jet projection I_k (cf. [18] p.151) for entire curves into semi-abelian varieties in [18] Chap's. 4–6.

Let $L_{\bar{A}} \rightarrow \bar{A}$ be a big line bundle over a projective compactification \bar{A} of A . We take a compactification $\widehat{\widehat{J}}_{k,A}$ of $\widehat{J}_{k,A}$, e.g.,

$$\widehat{\widehat{J}}_{k,A} = \overline{\mathbf{C}^n \times \mathbf{C}^{nk}} = \mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^{nk}(\mathbf{C}),$$

and the ample line bundle $H = \mathcal{O}_{\mathbf{P}^n(\mathbf{C})}(1) \otimes \mathcal{O}_{\mathbf{P}^{nk}(\mathbf{C})}(1) \rightarrow \widehat{\widehat{J}}_{k,A}$, with which we define

$$T_{\widehat{I}_k \circ J_k(\widehat{\exp} f)}(r) = T_{\widehat{I}_k \circ J_k(\widehat{\exp} f)}(r, H).$$

Lemma 4.9 *For \widehat{I}_k we have*

$$(4.10) \quad T_{\widehat{I}_k \circ J_k(\widehat{\exp} f)}(r) = S_{\exp f}(r).$$

Proof. Since

$$\widehat{I}_k \circ J_k(\widehat{\exp} f) : z \in \mathbf{C} \rightarrow (f(z), J_{k,\exp f}(z)) \in \text{Lie}(A) \times J_{k,A},$$

it follows from Lemma on logarithmic derivative for $\exp f$ ([12], [18] §4.7) that

$$T_{J_{k,\exp f}}(r) = S_{\exp f}(r).$$

This combined with Lemma 2.5 implies (4.10). □

4.5. A -action

We consider an A -action on $A \times \text{Lie}(A) \times J_{k,A} \subset J_k(A \times \text{Lie}(A))$ by

$$(a, (x, v, w)) \in A \times (A \times \text{Lie}(A) \times J_{k,A}) \rightarrow (a + x, v, w) \in A \times \text{Lie}(A) \times J_{k,A}.$$

We denote the stabilizer subgroup of $X_k(\widehat{\exp} f)$ by

$$\text{St}(X_k(\widehat{\exp} f)) = \text{St}_A(X_k(\widehat{\exp} f)) = \{a \in A : a + X_k(\widehat{\exp} f) = X_k(\widehat{\exp} f)\},$$

and by $\text{St}(X_k(\widehat{\exp} f))^0$ the identity component.

Lemma 4.11 *With the notation above, $\text{St}(X_k(\widehat{\text{exp}} f))^0 \neq \{0\}$.*

Proof. If otherwise, $\text{St}(X_k(\widehat{\text{exp}} f))^0 = \{0\}$. We consider the l -jet space $J_l(X_k(\widehat{\text{exp}} f))$ of $X_k(\widehat{\text{exp}} f)$ (“jet of jet”) with induced projection

$$d^l \widehat{I}_k : J_l(X_k(\widehat{\text{exp}} f)) (\subset J_l(A \times \text{Lie}(A) \times J_{k,A})) \rightarrow J_l(\text{Lie}(A) \times J_{k,A}).$$

By [18] Lemma 6.2.4, there is a large number $l \in \mathbf{N}$ such that the differential $d(d^l \widehat{I}_k)$ is non-degenerate at general points of $J_l(X_k(\widehat{\text{exp}} f))$. Therefore we have

$$T_{\text{exp} f}(r) \leq T_{\widehat{\text{exp}} f}(r) = O(T_{J_l(J_k(\widehat{\text{exp}}_A f))}(r)) = O(T_{d^l \widehat{I}_k(J_k(\widehat{\text{exp}} f))}(r)).$$

On the other hand, $T_{d^l \widehat{I}_k(J_k(\widehat{\text{exp}} f))}(r) = S_{\varepsilon, \text{exp} f}(r)$ by Lemma 4.9; it is a contradiction. \square

Proposition 4.12 (cf. [18] Theorem 6.2.6) *Let $B = \text{St}(X_k(\widehat{\text{exp}} f))^0$ and set the quotient map*

$$(4.13) \quad q_B : X_k(\widehat{\text{exp}} f) \rightarrow X_k(\widehat{\text{exp}} f)/B (\subset (A/B) \times \text{Lie}(A) \times J_{k,A}).$$

Then $T_{q_B \circ J_k(\widehat{\text{exp}} f)}(r) = S_{\text{exp} f}(r)$.

Proof. The semi-abelian variety A/B acts on $(A/B) \times \text{Lie}(A) \times J_{k,A}$ by the translations of the first factor and the identity for the other factors. Then $\text{St}_{A/B}(X_k(\widehat{\text{exp}} f)/B)^0 = \{0\}$. As in the proof of Lemma 4.11, with a large l the projection

$$\rho_l : J_l(X_k(\widehat{\text{exp}} f)/B) \rightarrow J_l(\text{Lie}(A) \times J_{k,A})$$

has a non-degenerate differential $d\rho_l$ at general points (see [18] Lemma 6.2.4). Therefore we have

$$T_{d^l q_B \circ J_k(\widehat{\text{exp}} f)}(r) = O(T_{\rho_l \circ d^l q_B \circ J_k(\widehat{\text{exp}} f)}(r)),$$

where $d^l q_B : J_l(X_k(\widehat{\text{exp}} f)) \rightarrow J_l(X_k(\widehat{\text{exp}} f)/B)$ is the induced morphism from q_B . It follows from $\rho_l \circ d^l q_B = d^l \widehat{I}_k$ and Lemma 4.9 that

$$T_{d^l q_B \circ J_k(\widehat{\text{exp}} f)}(r) = S_{\text{exp} f}(r).$$

On the other hand we have by Lemma 2.5

$$T_{d^l q_B \circ J_k(\widehat{\text{exp}} f)}(r) = T_{q_B \circ J_k(\widehat{\text{exp}} f)}(r) + S_{\text{exp} f}(r).$$

Thus we deduce that $T_{q_B \circ J_k(\widehat{\text{exp}} f)}(r) = S_{\text{exp} f}(r)$. \square

4.6. Proof of Theorem 1.6

Let the notation be as in Theorem 1.6. Through the arguments of the proof of [18] Theorem 6.5.6 with replacing the jet projection I_k there by the extended jet projection \widehat{I}_k (cf. (4.8)) and q_k^B by q_B (cf. (4.13)), we deduce that there are a number $l_0 \in \mathbf{N}$ and a compactification \bar{A} of A such that

$$(4.14) \quad T_{J_k(\widehat{\text{exp}} f)}(r, \omega_{\bar{Z}}) = N_{l_0}(r, J_k(\widehat{\text{exp}} f)^* Z) + S_{\text{exp} f}(r),$$

where $\bar{X}_k(\widehat{\exp} f)$ (resp. \bar{Z}) is the closure of $X_k(\widehat{\exp} f)$ (resp. Z) in $\bar{A} \times \bar{J}_{k,A}$.

Next, we show (ii). It follows from (4.14) that

$$T_{J_k(\widehat{\exp} f)}(r, \omega_{\bar{Z}}) \leq l_0 N_1(r, J_k(\widehat{\exp} f)^* Z) + S_{\exp f}(r).$$

Now, making use of the assumption $\text{codim}_{X_k(\widehat{\exp} f)} Z \geq 2$ together with Lemmata 2.5 and 4.9, we deduce from the arguments of [18] §6.5.3, adapted for $\widehat{\exp} f : \mathbf{C} \rightarrow A \times \text{Lie}(A)$ that

$$(4.15) \quad N_1(r, J_k(\widehat{\exp} f)^* Z) = S_{\varepsilon, \exp f}(r).$$

Thus, (1.8) is deduced, and (ii) is finished.

Now, we go back to the proof of (i). It follows from the First Main Theorem 2.1 that

$$(4.16) \quad N_1(r, J_k(\widehat{\exp} f)^* Z) \leq N_\infty(r, J_k(\widehat{\exp} f)^* Z) \leq T_{J_k(\widehat{\exp} f)}(r, \bar{Z}).$$

Note that (i) is finished in the case of $\text{codim}_{X_k(\widehat{\exp} f)} Z \geq 2$. Thus we consider the case where Z is an effective reduced divisor D on $X_k(\widehat{\exp} f)$. Let $D = \sum D_i$ be the irreducible decomposition. We deduce from (4.14) that

$$\begin{aligned} T_{J_k(\widehat{\exp} f)}(r, \omega_{\bar{D}}) &\leq N_{l_0}(r, J_k(\widehat{\exp} f)^* D) + S_{\exp f}(r) \\ &\leq N_1(r, J_k(\widehat{\exp} f)^* D) + l_0 \sum_{i < j} N_1(r, J_k(\widehat{\exp} f)^*(D_i \cap D_j)) \\ &\quad + l_0 \sum_i N_1(r, J_{k+1}(\widehat{\exp} f)^* J_1(D_i)) + S_{\exp f}(r) \end{aligned}$$

(cf. [18] (6.5.51)). Since $\text{codim}_{X_k(\widehat{\exp} f)} D_i \cap D_j \geq 2$, (4.15) implies

$$N_1(r, J_k(\widehat{\exp} f)^*(D_i \cap D_j)) = S_{\varepsilon, \exp f}(r).$$

We have $J_{k+1}(\widehat{\exp} f) : \mathbf{C} \rightarrow X_{k+1}(\widehat{\exp} f) \subset J_{k+1}(A \times \text{Lie}(A))$ and $B = \text{St}(X_{k+1}(\widehat{\exp} f))^0$. For each D_i we have two cases: (1) $B \subset \text{St}(D_i)^0$ and (2) $B \not\subset \text{St}(D_i)^0$. In the first case (1) we have by using q_B that

$$N_1(r, J_{k+1}(\widehat{\exp} f)^* J_1(D)) = S_{\exp f}(r)$$

(see [18] §6.5.4 (b)). In the second case (2), we have by [18] Lemma 6.5.50

$$\text{codim}_{X_{k+1}(\widehat{\exp} f)}(X_{k+1}(\widehat{\exp} f) \cap J_1(D_i)) \geq 2.$$

Then it is deduced from (4.15) with $k+1$ that

$$N_1(r, J_{k+1}(\widehat{\exp} f)^* J_1(D)) = S_{\varepsilon, \exp f}(r).$$

Thus, (1.7) follows.

(iii) The case of $k=0$ is a special case of (i); the proof of Theorem 1.6 is completed. \square

We consider the fundamental case where $k=0$ and Z is a reduced divisor D on $X_0(f)$. Let $p_1 : X_0(f) (\subset A \times \text{Lie}(A)) \rightarrow A$ be the projection to the first factor A .

Definition 4.17 We say that D is A -big if for a big divisor E on A (i.e., the closure \bar{E} in a compactification \bar{A} of A is big) the complete linear system $|mD - p_1^*E|$ with large $m \in \mathbf{N}$ contains an effective divisor on $X_0(f)$.

If D is A -big, then

$$(4.18) \quad \begin{aligned} T_{\exp f}(r) &= O(T_{\widehat{\exp f}}(r, \omega_{\bar{D}})), \\ T_{\widehat{\exp f}}(r) &= O(T_{\widehat{\exp f}}(r, \omega_{\bar{D}})) \quad ||. \end{aligned}$$

Corollary 4.19 *Let $f : \mathbf{C} \rightarrow \text{Lie}(A)$ be an A -nondegenerate entire curve and let D be a reduced divisor on $X_0(f)$. If $\text{ord}_z(\widehat{\exp f})^*D \geq 2$ for all $z \in \text{Supp}(\widehat{\exp f})^*D$ except for finitely points of $\text{Supp}(\widehat{\exp f})^*D$, then D is not A -big.*

Proof. If D is A -big, it follows from the 2nd Main Theorem 1.6 and (4.18) that

$$\begin{aligned} T_{\widehat{\exp f}}(r, \omega_{\bar{D}}) &= N_1(r, (\widehat{\exp f})^*D) + S_{\varepsilon, \widehat{\exp f}} \\ &\leq \frac{1}{2}N_{\infty}(r, (\widehat{\exp f})^*D) + S_{\varepsilon, \widehat{\exp f}} \\ &\leq \frac{1}{2}T_{\widehat{\exp f}}(r, \omega_{\bar{D}}) + \varepsilon T_{\widehat{\exp f}}(r, \omega_{\bar{D}}) + O(\log r) \quad ||_{\varepsilon}. \end{aligned}$$

This implies a contradiction ' $1 \leq \frac{1}{2}$ '. □

Remark 4.20 This corollary is motivated through a discussion with Corvaja and Zannier on their related or analogous results in rational function fields of Corvaja and Zannier [6] (see [14] too).

5. Applications

Here we give the proofs of two theorems stated in §1 as applications of Theorem 1.6.

5.1. Proof of Theorem 1.10

With the notation of the theorem we set

$$Z = \overline{\widehat{\exp f}(\mathbf{C})} \cap D^{\text{Zar}} \subset D.$$

If $\text{codim}_{X_0(\widehat{\exp f})}Z \geq 2$, Theorem 1.6 (ii) would imply

$$N_1(r, (\widehat{\exp f})^*D) = N_1(r, (\widehat{\exp f})^*Z) = S_{\varepsilon, \exp f}(r).$$

It follows from Theorem 1.6 (i) and (4.18) that for every $\varepsilon > 0$

$$T_{\widehat{\exp f}}(r, \omega_{\bar{D}}) \leq \varepsilon T_{\widehat{\exp f}}(r, \omega_{\bar{D}}) + O(\log r) \quad ||_{\varepsilon} :$$

This is a contradiction.

Therefore, Z has an irreducible component of codimension one in $X_0(\widehat{\exp f})$, which is an irreducible component of D . □

5.2. Proof of Theorem 1.11

By the assumption there is a point $z_0 \in \mathbf{C}$ such that $f_j(z_0) \in D_j$ ($j = 1, 2$); by parallel translations we may assume that $f_j(0) = 0 \in D_j$ ($j = 1, 2$). We first consider

$$f_3 : z \in \mathbf{C} \rightarrow (f_1(z), f_2(z)) \in \text{Lie}(A_1) \times \text{Lie}(A_2) \cong \text{Lie}(A_1 \times A_2).$$

By the Log Bloch-Ochiai Theorem, $A_3 := \overline{\exp f_3(\mathbf{C})}$ is a semi-abelian subvariety of $A_1 \times A_2$, so that $f_3 : \mathbf{C} \rightarrow \text{Lie}(A_3) \subset \text{Lie}(A_1 \times A_2)$. It follows that the projections

$$\pi_j : \widehat{A}_3 \longrightarrow \widehat{A}_j \quad (j = 1, 2)$$

are surjective, and that the entire curve $f_3 : \mathbf{C} \rightarrow \text{Lie}(A_3)$ is A_3 -nondegenerate. We set $\widehat{B}_j = \ker \pi_j \subset \widehat{A}_3$, $D'_j = \pi_j^{-1}D_j$ ($j = 1, 2$), and $D_3 = (D'_1 + D'_2) \cap \widehat{A}_3$. It follows from the assumption that

$$\mu(r) := N_1(r, (\widehat{\exp} f_3)^* D_3) = N_1(r, (\widehat{\exp} f_j)^* D_j) = N_1(r, (\widehat{\exp} f_3)^* D'_j), \quad j = 1, 2.$$

We then have by Theorem 1.6

$$T_{\widehat{\exp} f_j}(r, \omega_{\widehat{D}_j}) = \mu(r) + S_\varepsilon(r), \quad j = 1, 2, 3 :$$

Here $S_\varepsilon(r)$ denotes a small term such as

$$|S_\varepsilon(r)| \leq \varepsilon \mu(r) + O(\log r) \|\varepsilon$$

for an arbitrarily small $\varepsilon > 0$.

Let E_1 be any irreducible component of D'_1 . Then there are finitely many elements $x_h \in \widehat{A}_3$ ($1 \leq h \leq l$) such that $D'_1 = \bigcup_{h=1}^l (x_h + E_1)$. It follows that

$$N_1(r, (\widehat{\exp} f_3)^* E_1) = \frac{1}{l} \mu(r) + S_\varepsilon(r).$$

Suppose that $E_1 \not\subset D'_2$. Then $\text{codim}_{A_3} E_1 \cap D'_2 \geq 2$. We deduce from Theorem 1.6 (ii) that

$$N(r, (\widehat{\exp} f_3)^*(E_1 \cap D'_2)) = S_\varepsilon(r).$$

It follows from the assumption that $\text{Supp}((\widehat{\exp} f_3)^*(E_1 \cap D'_2)) = \text{Supp}((\widehat{\exp} f_3)^* E_1)$. Therefore, we have

$$\mu(r) \leq \varepsilon \mu(r) + O(\log r) \|\varepsilon, \quad \forall \varepsilon > 0.$$

This is a contradiction. Thus $E_1 \subset D'_2$, and hence $D'_1 \subset D'_2$. Similarly, $D'_2 \subset D'_1$, so that $D' := D'_1 = D'_2$. Note that $\widehat{\text{St}}(D'_j) := \{x \in \widehat{A}_j : x + D'_j = D'_j\} = \widehat{B}_j$ ($j = 1, 2$). Hence $\widehat{B} := \widehat{B}_1 = \widehat{B}_2$.

We see that $\widehat{A}_1 \cong \widehat{A}_3 / \widehat{B} \cong \widehat{A}_2$, $\pi_j(D') = D_j$, and $\widehat{\exp} f_j = \pi_j \circ \widehat{\exp} f_3$ ($j = 1, 2$). \square

6. Examples

To discuss examples it is convenient to write $\text{Lie}(A) \times A$ for \widehat{A} , so that in this section we use the notation

$$\widehat{\exp} f = (f, \exp f) : \mathbf{C} \longrightarrow \text{Lie}(A) \times A;$$

there will be no confusion.

(a) The optimality of (3.1): Let $A = (\mathbf{C}^*)^n$ and let $\alpha_j, 1 \leq j \leq n$, be complex numbers, linearly independent over \mathbf{Q} . Then the entire curve $\phi(z) = (\alpha_1 z, \dots, \alpha_n z) (\in \mathbf{C}^n = \text{Lie}(A))$ is A -nondegenerate with the natural exponential map $\exp : \mathbf{C}^n \ni (z_j) \mapsto (e^{z_j}) \in A$, and so

$$\text{tr. deg}_{\mathbf{C}}\{z, e^{\alpha_1 z}, \dots, e^{\alpha_n z}\} = n + 1.$$

Let $\mathbf{P}^n(\mathbf{C}) \supset \mathbf{C}^n$ and $\bar{A} := \mathbf{P}^1(\mathbf{C})^n \supset A$ be the compactifications and let $T_{\exp \phi}(r)$ and $T_{\widehat{\exp \phi}}(r)$ denote the order functions with respect to the products of point bundles. We write $\widehat{\exp \phi} = (\phi, \exp \phi)$. Then,

$$T_{\widehat{\exp \phi}}(r) = T_{\exp \phi}(r) + O(\log r) = \frac{\sum_{j=1}^n |\alpha_j|}{\pi} r + O(\log r).$$

Let $P(z_1, \dots, z_n, w_1, \dots, w_n)$ be a polynomial of degree d_j in w_j ($1 \leq j \leq n$). We assume the condition:

- 6.1 (i) The divisor on $\mathbf{P}^n(\mathbf{C}) \times \bar{A}$ defined by the zero of P is reduced and equal to the closure \bar{D}_P of the divisor D_P defined by $\{P = 0\} \cap (\text{Lie}(A) \times A)$.
(ii) D_P is A -big (see Definition 4.17).

We need this condition; otherwise, to say, if $P = z_1$, then $D_P = \{z_1 = 0\}$ and D_P is not A -big; there is only one root $z = 0$ in $P(\widehat{\exp \phi}(z)) = 0$. We have

$$T_{\widehat{\exp \phi}}(r, L(\bar{D}_P)) = \frac{\sum_{j=1}^n d_j |\alpha_j|}{\pi} r + O(\log r).$$

It is yet, in general, hard to find a root of $P(\widehat{\exp \phi}(z)) = 0$, but by Theorem 1.6 (iii)

$$(6.2) \quad \begin{aligned} T_{\widehat{\exp \phi}}(r, L(\bar{D}_P)) &= N_{\infty}(r, (\widehat{\exp \phi})^* D_P) + O(\log r) \\ &= N_1(r, (\widehat{\exp \phi})^* D_P) + S_{\varepsilon, \exp \phi}(r). \end{aligned}$$

The above estimate of $N_{\infty}(r, (\widehat{\exp \phi})^* D_P)$ is classical due to Borel-Nevanlinna, but that of $N_1(r, (\widehat{\exp \phi})^* D_P)$ is new; moreover from Theorem 1.10 we obtain

$$(6.3) \quad \overline{\widehat{\exp \phi}(\mathbf{C}) \cap D_P}^{\text{Zar}} = D_P.$$

By Corollary 4.19 there is no entire function $g(z)$ such that

$$(6.4) \quad P(\widehat{\exp \phi}(z)) = (g(z))^m \quad (m \in \mathbf{N} \geq 2).$$

In view of the transcendence problem of π and e the above example in the case of $n = 2$ and complex vector $\varpi_0 = (1, 2\pi i) \in \mathbf{C}^2$ is of a special interest. We consider the induced 1-parameter subgroup

$$(6.5) \quad \begin{aligned} \phi_0(z) &= z\varpi_0 \in \mathbf{C}^2, \quad z \in \mathbf{C}, \\ \widehat{\exp \phi_0}(z) &= (z, 2\pi iz, e^z, e^{2\pi iz}) = (z_1, z_2, w_1, w_2) \in \mathbf{C}^2 \times (\mathbf{C}^*)^2. \end{aligned}$$

Let $P(z_1, z_2, w_1, w_2)$ be a polynomial with integral coefficients satisfying condition 6.1. If $P(\widehat{\exp \phi_0}(\zeta)) = 0$, then

$$\zeta, 2\pi i\zeta, e^{\zeta}, e^{2\pi i\zeta}$$

are algebraically dependent, and there are infinitely many such points, for which (6.3) and (6.2) hold.

(c) Let $\exp_{(j)}(z)$ ($j = 1, 2, \dots$) denote the j -times iteration of the exponential function e^z . We set

$$f_1(z) = z, \quad f_j(z) = \exp_{(j-1)}(z), \quad 2 \leq j \leq n.$$

Then we have

$$\text{tr. deg}_{\mathbf{C}}\{f_1, \dots, f_n, e^{f_1}, \dots, e^{f_n}\} = \text{tr. deg}_{\mathbf{C}}\{f_1, \dots, f_n, f_{n+1}\} = n + 1.$$

In this case, $\exp f$ is of infinite order and $T_{\exp f}(r)$ has a growth such that $T_{\exp f}(r) \sim \exp_{(n)}(r)$.

(d) (Cf. Brownawell-Kubota [3]) Let $\exp : \text{Lie}(A) \cong \mathbf{C}^n \rightarrow A$ be an exponential map of a semi-abelian variety A . In general if f_j ($1 \leq j \leq n$) are entire functions, linearly independent over \mathbf{C} , then $f = (f_j) : \mathbf{C} \rightarrow \mathbf{C}^n$ is A -nondegenerate. In particular, let $n = l + m$ and let $\wp_j(w)$ ($1 \leq j \leq m$) be Weierstrass' pe -functions. Then

$$\text{tr. deg}_{\mathbf{C}}\{f_1, \dots, f_{l+m}, e^{f_1}, \dots, e^{f_l}, \wp_1(f_{l+1}), \dots, \wp_m(f_{l+m})\} \geq l + m + 1.$$

(e) Let $f_1(z) = z, f_2(z) = z$. Then they are not linearly independent over \mathbf{C} . Let E_j ($j = 1, 2$) be elliptic curves which are not isogenous to each other. Let $A = E_1 \times E_2$ and $\exp : \mathbf{C}^2 \rightarrow A$ be an exponential map. Then $f = (f_1, f_2) : \mathbf{C} \rightarrow \mathbf{C}^2$ is A -nondegenerate, and so

$$\text{tr. deg}_{\mathbf{C}}\{z, \wp_1(z), \wp_2(z)\} = 3.$$

This should be known classically.

Let $\text{Lie}(E_1) \times \text{Lie}(E_2) \times A = \mathbf{P}^2(\mathbf{C}) \times E_1 \times E_2$ be the compactification with product line bundle L (resp. L_0) of the hyperplane bundle and the point-bundles over $\mathbf{P}^2(\mathbf{C}) \times E_1 \times E_2$ (resp. $E_1 \times E_2$). Then, L is ample and the order functions satisfy

$$T_{\widehat{\exp} f}(r, L) = T_{\exp f}(r, L_0) + 2 \log r + O(1) = \frac{\pi r^2}{2} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + o(1) \right),$$

where λ_j is the surface area of the fundamental domain of \wp_j ($j = 1, 2$). Let $P(z_1, z_2, w_1, w_2)$ be an irreducible polynomial, involving w_1 and w_2 , and of degree d_1 (resp. d_2) with respect to w_1 (resp. w_2). We consider $w_j = \wp_j$ ($j = 1, 2$) a rational function of E_j , and denote by D_P the divisor defined by the zeros of P on $\text{Lie}(A) \times A$. Let Ξ_P denote the zero divisor on \mathbf{C} defined by $P(z, z, \wp_1(z), \wp_2(z)) = 0$. Then we have

$$N_{\infty}(r, \Xi_P) = N_1(r, \Xi_P) + S_{\varepsilon, \exp f}(r) = \pi r^2 \left(\frac{d_1}{\lambda_1} + \frac{d_2}{\lambda_2} + o(1) \right) + S_{\exp f}(r).$$

It also follows from Theorem 1.10 that $\overline{\widehat{\exp} f(\mathbf{C}) \cap D_P}^{\text{Zar}}$ contains an irreducible component of D_P . By Corollary 4.19 we see that there is no meromorphic function $g(z)$ on \mathbf{C} satisfying

$$(6.6) \quad P(\widehat{\exp} f(z)) = g(z)^m \quad (m \in \mathbf{N}, \geq 2).$$

(f) Set $f_1 = z, f_2 = z^2, f_3 = z$. Then these are not linearly independent over \mathbf{C} . Let $\exp_E : \mathbf{C} \rightarrow E$ be an exponential map of an elliptic curve E with Weierstrass' $\wp(w)$. Set $A = (\mathbf{C}^*)^2 \times E$ with $\exp : \mathbf{C}^3 \rightarrow A$. Then $f = (f_j) : \mathbf{C} \rightarrow \mathbf{C}^3$ is A -nondegenerate, and so

$$\text{tr. deg}_{\mathbf{C}}\{z, e^z, e^{z^2}, \wp(z)\} = 4.$$

The order function of $\widehat{\exp} f$ has a growth, $T_{\widehat{\exp} f}(r) \sim r^2$.

7. Remarks to affine algebraic curves and other domains

7.1. Affine algebraic curves

In this section A denotes a semi-abelian variety.

Let R be a complex affine algebraic curve and let $f : R \rightarrow \text{Lie}(A)$ be a holomorphic curve. We may consider $\exp f : R \rightarrow A$ and $\widehat{\exp} f : R \rightarrow \widehat{A} = A \times \text{Lie}(A)$.

In general for a holomorphic curve $g : R \rightarrow A$ the arguments up to obtaining an estimate such as (4.14) work (cf. [18] Chap. 6), but further to advance to the estimates of the 2nd Main Theorem 1.6 with counting functions truncated at level one, we need to lift g to $\tilde{g} : R \rightarrow \text{Lie}(A)$, which does not exist in general, since R is not simply connected. But in the present case we begin with a holomorphic curve $f : R \rightarrow \text{Lie}(A)$, which is a lift of $\exp f : R \rightarrow A$. Therefore we can advance the arguments further there.

Let \bar{R} be the compactification of R by adding a finite number of points. To study $\widehat{\exp} f$ we may localize the problem about an infinite point $a \in \bar{R} \setminus R$; we take a disk neighborhood Δ of a in \bar{R} . Then $\Delta^* = \Delta \setminus \{a\}$ is a punctured disk, and the analysis of transcendental properties of $\widehat{\exp} f$ is reduced to that of the restriction $\widehat{\exp} f|_{\Delta^*}$ (see the next).

7.2. Punctured disk

Let Δ^* be a punctured disk. As mentioned above, although Δ^* is not simply connected, it does not cause a difficulty here, since we give in first a holomorphic curve

$$(7.1) \quad f : z \in \Delta^* \longrightarrow (f_1(z), f_2(z), \dots, f_n(z)) \in \mathbf{C}^n \cong \text{Lie}(A).$$

For a notational convenience we put the puncture at infinity and introduce a coordinate z such that

$$\Delta^* = \{|z| > 1\}, \quad \Delta = \Delta^* \cup \{\infty\}.$$

Let $F(z)$ be one of $f_j(z)$. Then $F(z)$ is expanded to a Laurent series

$$(7.2) \quad F(z) = \sum_{\nu > 0} c_\nu z^\nu + \sum_{\nu \leq 0} c_\nu z^\nu = F_m(z) + F_b(z).$$

Fix $r_0 > 1$. Then $F_b(z)$ and their derivatives are bounded in $\{|z| \geq r_0\}$, and $F_m(z)$ is the main part of the expansion:

$$(7.3) \quad \frac{d^k}{dz^k} F(z) = \frac{d^k}{dz^k} F_m(z) + O(1), \quad |z| \geq r_0, \quad k \geq 0.$$

Note that $F_m(z)$ is holomorphic in \mathbf{C} . Applying the key Lemma 2.5 for $F_m(z)$, we deduce the key Lemma 2.5 for f , $\exp f$ and $r > r_0$. We can then deduce the 2nd Main Theorem 1.6 for $\widehat{\exp} f : \Delta^* \rightarrow A \times \text{Lie}(A)$ and $J_k(\widehat{\exp} f)$ as well for $r > r_0$. Cf. Corvaja-Noguchi-Zannier [5] for related results.

7.3. Finite disk

We consider the case of a disk of \mathbf{C} with finite radius, to say, the unit disk Δ . Let $f : \Delta \rightarrow \text{Lie}(A)$ be a holomorphic curve. In this hyperbolic case, to make the proofs of key Lemma 2.5 and Lemma on logarithmic derivatives to work at least for a sequence

$r_\nu \nearrow 1$ ($\nu \rightarrow \infty$), we need a technical condition on the growth of the order function $T_{\exp f}(r)$ such that

$$(7.4) \quad \overline{\lim}_{r \rightarrow 1} \frac{T_{\exp f}(r)}{\log \frac{1}{1-r}} = \infty$$

(cf. Nevanlinna [11] Chap. VI, Hayman [8] §2.3). Under this condition the 2nd Main Theorem 1.6 for $\widehat{\exp} f : \Delta \rightarrow A \times \text{Lie}(A)$ is deduced.

7.4. Open Riemann surfaces

Let R be an open Riemann surface. The generalizations of Nevanlinna theory for meromorphic functions on R , holomorphic maps from R to another Riemann surface and holomorphic curves from R into $\mathbf{P}^n(\mathbf{C})$ are classical (cf., e.g., Sario-Noshiro [23], Wu [27]). There one uses a finite (hyperbolic case) or infinite (parabolic case) exhaustion function $\tau : R \rightarrow [0, r_0)$ with $r_0 \leq \infty$ such that τ is harmonic outside a compact subset of R .

Similarly it is formally possible to extend the 2nd Main Theorem 1.6 for holomorphic curves $f : R \rightarrow \text{Lie}(A)$ and $\widehat{\exp} f : R \rightarrow \widehat{A}$. There we use the differential $\partial\tau$, holomorphic where τ is harmonic. For a logarithmic 1-form ω on A we take the ratio $F = (\exp f)^*\omega/\partial\tau$, which may have poles at zeros of $\partial\tau$. The counting functions of those zeros is the counting function $E_R(r)$ of the Euler numbers of $\{\tau < r\}$, which appears in the estimates (4.14) and (1.7) under a growth assumption such as (7.4) with $r_0 = 1$ in hyperbolic case (i.e., $r_0 < \infty$):

$$(7.5) \quad T_{J_k(\widehat{\exp} f)}(r, \omega_{\bar{Z}}) = N_{l_0}(r, J_k(\widehat{\exp} f)^*Z) + C(k, l_0)E_R(r) \\ + O(\log^+ T_{\exp f}(r)) + O(\log r) \quad ||,$$

$$(7.6) \quad (1 - \varepsilon)T_{J_k(\widehat{\exp} f)}(r, \omega_{\bar{Z}, J_k(\widehat{\exp} f)}) \leq N_1(r, J_k(\widehat{\exp} f)^*Z) + C'(\varepsilon, k)E_R(r) \\ + O(\log r) \quad ||_\varepsilon.$$

Here $C(k, l_0)$ and $C'(\varepsilon, k)$ are positive constants such that $C(k, l_0), C'(\varepsilon, k) \nearrow \infty$ as $k, l_0 \nearrow \infty$ and $\varepsilon \searrow 0$; there are no estimates for $C(k, l_0), C'(\varepsilon, k)$. Therefore, in order to obtain a meaningful consequence we need a technical condition (besides (7.4) in hyperbolic case) such that

$$\overline{\lim}_{r \rightarrow r_0} \frac{E_R(r)}{T_{\exp f}(r)} = 0.$$

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