The supersingular locus of the Shimura variety of $\mathrm{GU}(2, n-2)$

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Abstract

We study the supersingular locus of a reduction at an inert prime of the Shimura variety attached to $\mathrm{GU}(2, n-2)$. More concretely, we realize irreducible components of the supersingular locus as closed subschemes of flag schemes over Deligne–Lusztig varieties defined by explicit conditions after taking perfections. Moreover we study the intersections of the irreducible components. Stratifications of Deligne–Lusztig varieties defined using powers of Frobenius action appear in the description of the intersections.

1 Introduction

Shimura varieties play an important role in the study of number theory. One way to approach the arithmetic of Shimura varieties is to construct integral models and study their reductions. The geometry of the supersingular locus of the reduction of a Shimura variety is especially useful information. One of the striking results in this direction is the study of the supersingular locus of the reduction of the Shimura variety of $\mathrm{GU}(1, n-1)$ at an inert prime by Vollaard–Wedhorn in [VW11], where they give a description of the supersingular locus and their intersections in terms of Deligne–Lusztig varieties. This result is crucially used in [KR11].

A long standing problem since [VW11] is to extend such a result to unitary groups of other signatures. The only result in this line is the work [HP14] of Howard–Pappas on the $\mathrm{GU}(2, 2)$-case, which relies on an exceptional isomorphism. A source of difficulty is that the Shimura variety of $\mathrm{GU}(2, n-2)$ is not fully Hodge–Newton decomposable in the sense of [GHN19, Definition 3.1] if $n \geq 5$. In such a case, we can not expect that the supersingular locus is a union of Deligne–Lusztig varieties by [GHN19, Theorem B].

On the other hand, the study of the perfection of the supersingular locus is essentially reduced to a study of an affine Deligne–Lusztig variety via the Rapoport–Zink uniformization. Further, a construction of irreducible components of an affine Deligne–Lusztig variety under some unramified condition is given by Xiao–Zhu in [XZ17]. In their construction, we can rephrase the source of difficulty in the following way: Even though the affine Deligne–Lusztig variety related to the Shimura variety of $\mathrm{GU}(2, n-2)$ is defined using a minuscule cocharacter, non-minuscule cocharacters appear in the construction of its irreducible components if $n \geq 5$.

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The objective of this paper is to find an explicit description of the irreducible components of the affine Deligne–Lusztig variety related to the Shimura variety of $\mathrm{GU}(2, n-2)$ in terms of Deligne–Lusztig varieties.

Let F be a non-archimedean local field. We write L for the completion of the maximal unramified extension of F. Let G be the unramified general unitary group of degree n over F. Let μ be the cocharacter of G corresponding to $z \mapsto (\mathrm{diag}(z, z, 1, \ldots, 1), z)$ under an isomorphism $G_L \simeq \mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}$. Let $X_{\mu^*}(\varpi^{-1})$ denote the affine Deligne–Lusztig variety for the dual μ^* of μ and $\varpi^{-1} \in G(L)$, where ϖ is a uniformizer of F and we regard ϖ^{-1} as an element of $G(L)$ by embedding it into the $\mathbb{G}_{\mathrm{m}}$-component. We put $r = [n/2]$. Then $X_{\mu^*}(\varpi^{-1})$ has r isomorphism classes of irreducible components, whose representatives are given by $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ for $1 \leq i \leq r$ as explained in §8. If $i = 1$ or $i = n/2$, then $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ is isomorphic to the perfection of a Deligne–Lusztig variety as shown in Proposition 8.2 and Proposition 8.3.

Assume that $2 \leq i \leq [(n-1)/2]$. Then the action of a hyperspecial subgroup $J_{\tau_i}(\mathcal{O}) \subset G(F)$ on $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ does not factor through the finite reductive quotient $J_{\tau_i}(\mathcal{O}/\varpi)$ unlike the cases for $i = 1, n/2$. We construct a kind of Demazure resolution X_i of $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$. We write $\mathring{X}_i$ and $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ for the inverse images in X_i and $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ of the Schubert cell $\mathring{\mathrm{Gr}}_{\nu_i^*}$ of an affine Grassmannian $\mathrm{Gr}_{\nu_i^*}$ under natural morphisms $X_i \to X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*) \to \mathrm{Gr}_{\nu_i^*}$. Explicitly, we construct a vector bundle $\mathscr{V}_i$ of rank $2i-1$ over the perfection Y_i of a Deligne–Lusztig variety. We have a natural morphism

$$\phi_1 \colon \mathscr{V}_i \to F(\mathscr{V}_i^{\vee})$$

by a Hermitian pairing related to the unitary group G, where $F(\mathscr{V}_i^{\vee})$ is some Frobenius twist of $\mathscr{V}_i^{\vee}$ (*cf.* (8.4)). Let $\mathrm{Par}_{t_i}(\mathscr{G}_{Y_i})$ denote the flag scheme parametrizing subvector bundles $\mathscr{W} \subset \mathscr{V}_i$ of rank $i-1$.

Theorem 1.1 (Theorem 8.6, Proposition 8.7)**.** *The scheme X_i is isomorphic to the closed subscheme of $\mathrm{Par}_{t_i}(\mathscr{G}_{Y_i})$ defined by the condition $\phi_1(\mathscr{W}) \subset F(\mathscr{W}^{\perp})$ on $\mathscr{W}$. Further $\mathring{X}_i$ is isomorphic to $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$.*

Let us summarize the situation in the following diagram:

$$\begin{array}{ccccccc}
\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*) & \xrightarrow{\sim} & \mathring{X}_i & \hookrightarrow & X_i & \hookrightarrow & \mathrm{Par}_{t_i}(\mathscr{G}_{Y_i}) \\
 & & \downarrow & \square & \downarrow & & \downarrow \\
 & & \mathring{\mathrm{Gr}}_{\nu_i^*} & \hookrightarrow & \mathrm{Gr}_{\nu_i^*} & & Y_i.
\end{array}$$

By Theorem 1.1 and the above diagram, $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ is cut out in $\mathrm{Par}_{t_i}(\mathscr{G}_{Y_i})$ by two explicit conditions: one is a closed condition in Theorem 1.1 and another is an open condition given by $\mathring{\mathrm{Gr}}_{\nu_i^*} \subset \mathrm{Gr}_{\nu_i^*}$.

It is important to describe X_i, not only $\mathring{X}_i$, in order to study the intersections of irreducible components of $X_{\mu^*}(\varpi^{-1})$, because we need to understand a closure of $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$. We give a description of the intersections of the irreducible components in most cases in §9. Here we state one of the results, which exhibits an interesting new phenomenon.

Proposition 1.2 (Proposition 9.4)**.** *The intersection $X_{\mu^*}^{\mathbf{b}_1,x_0}(\tau_1^*) \cap X_{\mu^*}^{\mathbf{b}_2,x_0}(\tau_2^*)$ is isomorphic to the perfect closed subscheme of $(\mathbb{P}^{n-1})^{\mathrm{pf}}$ defined by two equations*

$$\sum_{i=1}^{n} x_i^{q+1} = 0, \quad \sum_{i=1}^{n} x_i^{q^3+1} = 0.$$

The perfect closed subscheme of $(\mathbb{P}^{n-1})^{\mathrm{pf}}$ in Proposition 1.2 is the perfection of a stratification of a Deligne–Lusztig variety with respect to relative positions of parabolic subgroups and their twists by the third power of the Frobenius action. Such an intersection did not appear in the preceding research in fully Hodge–Newton decomposable cases. Our study does not cover all the intersections in general because of some technical difficulty which involves the study of vanishing of a ring with explicit generators and relations, but it does cover all the cases if $n \leq 6$.

In the construction of irreducible components of an affine Deligne–Lusztig variety by Xiao–Zhu, they actually first construct $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$, and then construct $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ as a closure of $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$. In the study of the unitary case in this paper, we clarify that this step in the construction is really necessary, i.e. we can not construct $X_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ directly by a fiber product that is similar to the one used to construct $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$. This gives a negative answer to a question of Xiao–Zhu (*cf.* Remark 7.4).

The method in this paper should work for unitary groups of other signatures since the results in [XZ17] and equidimensionality of Satake cycle in §5 are available also for other signatures. On the other hand, they will be more complicated for general signatures since the number of isomorphism classes of irreducible components of the affine Deligne–Lusztig varieties become larger. In this paper, we study the perfection of the supersingular locus via affine Deligne–Lusztig varieties. However, once the geometry of the corresponding affine Deligne–Lusztig varieties is understood using Demazure resolutions, we should be able to write a similar moduli problem using p-divisible groups and study them without taking perfections. That is the subject of future work.

We explain the contents of each section. In §2, we recall a terminology on relative positions in flag schemes. We also give some gluing constructions of reductive group schemes. In §3, we recall Deligne–Lusztig varieties and their Bruhat stratifications. We give also a new stratification using twists by a power of Frobenius map. We study the irreducibility of the stratification in some unitary case. In §4, we recall affine Grassmannian and Satake cycles. In §5, we recall and generalize results on equidimensionality of Satake cycles in [Hai06]. In §6, we recall a construction of irreducible components of affine Deligne–Lusztig varieties in [XZ17]. In §7, we explain the setting of a unitary group and apply the result in §5 to the unitary case. In §8, we give an explicit description of irreducible components. In §9, we study the intersection of irreducible components. In §10, we explain the results in the $n = 6$ case as an example. In §11, we explain a relation between the affine Deligne–Lusztig varieties and the supersingular loci of reductions of Shimura varieties in our case.

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2 Flag scheme

2.1 Relative position

Let $\mathscr{G}$ be a reductive group scheme over a scheme $\mathscr{S}$. Let $\mathrm{Par}(\mathscr{G})$ be the $\mathscr{S}$-scheme of parabolic subgroups of $\mathscr{G}$. Let $\mathrm{Dyn}(\mathscr{G})$ be the $\mathscr{S}$-scheme of Dynkin for $\mathscr{G}$ constructed in [SGA3-3, XXIV, 3.3].

Remark 2.1. *If $(\mathscr{T}, M, R)$ is a splitting of $\mathscr{G}$ in the sense of [SGA3-3, XXII, Définition 1.13] and Δ is a set of simple roots, then we have a canonical isomorphism*

$$\mathrm{Dyn}(\mathscr{G}) \simeq \Delta_{\mathscr{S}}. \tag{2.1}$$

This is stated in [SGA3-3, XXIV, 3.4 (iii)] choosing a pinning, but the isomorphism actually depends only on $(\mathscr{T}, M, R)$ and Δ.

Let $\mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))$ be the $\mathscr{S}$-scheme of sets of open and closed subschemes of $\mathrm{Dyn}(\mathscr{G})$ (*cf.* [SGA3-3, XXVI, 3.1]). We have a projective smooth morphism

$$\mathbf{t}\colon \mathrm{Par}(\mathscr{G}) \to \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))$$

of schemes as [SGA3-3, XXVI, Théorème 3.3]. For $t, t' \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))(\mathscr{S})$, we put

$$\mathrm{Par}_t(\mathscr{G}) = \mathbf{t}^{-1}(t) \subset \mathrm{Par}(\mathscr{G}), \quad \mathrm{Par}_{t,t'}(\mathscr{G}) = (\mathbf{t} \times \mathbf{t})^{-1}(t, t') \subset \mathrm{Par}(\mathscr{G}) \times_{\mathscr{S}} \mathrm{Par}(\mathscr{G}).$$

We recall results from [SGA3-3, XXVI. 4.5.3, 4.5.4]. Let $\mathrm{Stand}(\mathscr{G})$ be the $\mathscr{S}$-scheme of pairs of parabolic subgroups of $\mathscr{G}$ in mutually standard positions. Let $\mathrm{TypeStand}(\mathscr{G})$ be the $\mathscr{S}$-scheme of types of mutually standard positions in $\mathscr{G}$. The natural morphism

$$\mathbf{t}_2\colon \mathrm{Stand}(\mathscr{G}) \to \mathrm{TypeStand}(\mathscr{G}),$$

which is the quotient morphism under the conjugacy action of $\mathscr{G}$, is smooth. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Stand}(\mathscr{G}) & \xrightarrow{\quad \mathbf{t}_2 \quad} & \mathrm{TypeStand}(\mathscr{G}) \\ \downarrow & & \downarrow{\scriptstyle q_{\mathscr{G}}} \\ \mathrm{Par}(\mathscr{G}) \times_{\mathscr{S}} \mathrm{Par}(\mathscr{G}) & \xrightarrow{\mathbf{t} \times \mathbf{t}} & \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G})) \times_{\mathscr{S}} \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G})). \end{array}$$

Let $\mathscr{P}$ be a parabolic subgroup scheme of $\mathscr{G}$. Let $\mathrm{Par}(\mathscr{G}; \mathscr{P})$ be the $\mathscr{S}$-scheme of parabolic subgroups of $\mathscr{G}$ in standard positions relative to $\mathscr{P}$. Let $t \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))(\mathscr{S})$. We put

$$\mathrm{Par}_t(\mathscr{G}; \mathscr{P}) = \mathrm{Par}(\mathscr{G}; \mathscr{P}) \cap \mathrm{Par}_t(\mathscr{G}).$$

Then we have a morphism

$$\mathbf{t}_{\mathscr{P}}\colon \mathrm{Par}_t(\mathscr{G}; \mathscr{P}) \to q_{\mathscr{G}}^{-1}(\mathbf{t}(\mathscr{P}), t)$$

induced by $\mathbf{t}_2$. For an $\mathscr{S}$-scheme $\mathscr{S}'$ and $r \in (q_{\mathscr{G}}^{-1}(\mathbf{t}(\mathscr{P}), t))(\mathscr{S}')$, we define $\mathrm{Par}_t(\mathscr{G}; \mathscr{P})_r$ by the fiber product

$$\begin{array}{ccc} \mathrm{Par}_t(\mathscr{G}; \mathscr{P})_r & \longrightarrow & \mathscr{S}' \\ \downarrow & & \downarrow{\scriptstyle r} \\ \mathrm{Par}_t(\mathscr{G}; \mathscr{P}) & \xrightarrow{\mathbf{t}_{\mathscr{P}}} & q_{\mathscr{G}}^{-1}(\mathbf{t}(\mathscr{P}), t). \end{array}$$

Remark 2.2. *Let $\mathscr{Q}$ be a parabolic subgroup scheme of $\mathscr{G}$. Let $\mathscr{S}'$ be an $\mathscr{S}$-scheme. we write $\mathscr{G}'$, $\mathscr{P}'$, $\mathscr{Q}'$ for the base change of $\mathscr{G}$, $\mathscr{P}$, $\mathscr{Q}$ to $\mathscr{S}'$. Assume that a maximal torus $\mathscr{T}'$ of $\mathscr{G}'$ is contained in $\mathscr{P}' \cap \mathscr{Q}'$. Then we have a natural isomorphism*

$$W_{\mathscr{P}'}(\mathscr{T}')\backslash W_{\mathscr{G}'}(\mathscr{T}')/W_{\mathscr{Q}'}(\mathscr{T}') \simeq q^{-1}(\mathbf{t}(\mathscr{P}), \mathbf{t}(\mathscr{Q})) \times_{\mathscr{S}} \mathscr{S}' \tag{2.2}$$

over $\mathscr{S}'$ as in [SGA3-3, XXVI. 4.5.3].

Notation 2.3. *Assume that $\mathscr{G}$ is split and $\mathscr{S}$ is connected. Let $(\mathscr{T}, M, R)$ be a splitting of $\mathscr{G}$ and Δ be a set of simple roots. Let (W, S) be the Coxeter system of (M, R, Δ). For $I \subset S$, let W_I be the subgroup of W generated by I, and let $t(I)$ be the element of $\mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))(\mathscr{S})$ corresponding to I under (2.1). Conversely, let $I(t)$ be the subset of S corresponding to t under (2.1) for $t \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))(\mathscr{S})$. We simply write W_t for $W_{I(t)}$.*

2.2 Inner gluing

Definition 2.4. *Let $\mathscr{G}_0$ be a reductive group scheme over a scheme $\mathscr{S}_0$. Let $\mathscr{S}$ be a scheme over $\mathscr{S}_0$. An inner gluing over $\mathscr{S}$ of $\mathscr{G}_0$ is a pair $(\mathscr{G}, \varphi)$, where $\mathscr{G}$ is a reductive group scheme over $\mathscr{S}$ and φ is a global section of the Zariski sheaf*

$$\underline{\mathrm{Isom}}_{\mathscr{S}}(\mathscr{G}_0 \times_{\mathscr{S}_0} \mathscr{S}, \mathscr{G})/\underline{\mathrm{Inn}}_{\mathscr{S}}(\mathscr{G}_0 \times_{\mathscr{S}_0} \mathscr{S})$$

on $\mathscr{S}$.

Remark 2.5. *Let $\mathscr{V}$ be a vector bundle of rank n on $\mathscr{S}$. We put $\mathscr{G} = \mathrm{Aut}_{\mathscr{S}}(\mathscr{V})$. By taking Zariski local trivializations of $\mathscr{V}$, we obtain an inner gluing $(\mathscr{G}, \varphi_{\mathscr{V}})$ over $\mathscr{S}$ of $\mathrm{GL}_{n,\mathbb{Z}}$. This is independent of the choice of trivializations, because a difference of trivializations induces an inner automorphism of GL_n.*

Lemma 2.6. *Let $\pi\colon \mathscr{S} \to \mathscr{S}_0$ be a morphism of schemes. Let $\mathscr{G}_0$ be a reductive group scheme over $\mathscr{S}_0$. Let $(\mathscr{G}, \varphi)$ an inner gluing over $\mathscr{S}$ of $\mathscr{G}_0$.*

(1) The section φ induces isomorphisms

$$\mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}_0)) \times_{\mathscr{S}_0} \mathscr{S} \xrightarrow{\sim} \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G})),$$
$$\mathrm{TypeStand}(\mathscr{G}_0) \times_{\mathscr{S}_0} \mathscr{S} \xrightarrow{\sim} \mathrm{TypeStand}(\mathscr{G})$$

which are compatible with $q_{\mathscr{G}_0}$ and $q_{\mathscr{G}}$.

(2) Assume that $\mathscr{G}_0$ is split and $\mathscr{S}_0$ is connected. Let $(\mathscr{T}_0, M, R)$ be a splitting of $\mathscr{G}_0$ and Δ be a set of simple roots. Let (W, S) be the Coxeter system of (M, R, Δ). Let $t_0, t_0' \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}_0))(\mathscr{S}_0)$. Let $t, t' \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}))(\mathscr{S})$ denote the pullbacks to $\mathscr{S}$ of t_0, t_0'. Then φ induces an isomorphism

$$(W_{t_0}\backslash W/W_{t_0'})_{\mathscr{S}} \xrightarrow{\sim} q_{\mathscr{G}}^{-1}(t, t').$$

Proof. There is a Zariski covering $\{\mathscr{U}_\lambda\}_{\lambda\in\Lambda}$ of $\mathscr{S}$ and a family of isomorphisms $\varphi_\lambda\colon \mathscr{G}_0 \times_{\mathscr{S}_0} \mathscr{U}_\lambda \xrightarrow{\sim} \mathscr{G} \times_{\mathscr{S}} \mathscr{U}_\lambda$ such that φ_λ is compatible with $\varphi|_{\mathscr{U}_\lambda}$. Then the family of isomorphisms φ_λ induce isomorphisms

$$\mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}_0)) \times_{\mathscr{S}_0} \mathscr{U}_\lambda \xrightarrow{\sim} \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G} \times_{\mathscr{S}} \mathscr{U}_\lambda)).$$

These isomorphisms glue together to give the first isomorphism in the claim (1) by [SGA3-3, XXIV, 3.4 (iv)].

The family of isomorphisms φ_λ induce also isomorphisms

$$\mathrm{Stand}(\mathscr{G}_0) \times_{\mathscr{S}_0} \mathscr{U}_\lambda \xrightarrow{\sim} \mathrm{Stand}(\mathscr{G} \times_{\mathscr{S}} \mathscr{U}_\lambda).$$

By taking the quotients by the conjugacy actions of $\mathscr{G}_0 \times_{\mathscr{S}_0} \mathscr{U}_\lambda \simeq \mathscr{G} \times_{\mathscr{S}} \mathscr{U}_\lambda$, we obtain isomorphisms

$$\mathrm{TypeStand}(\mathscr{G}_0) \times_{\mathscr{S}_0} \mathscr{U}_\lambda \xrightarrow{\sim} \mathrm{TypeStand}(\mathscr{G} \times_{\mathscr{S}} \mathscr{U}_\lambda).$$

These isomorphisms glue together to give the second isomorphism in the claim (1) because we take quotients by conjugacy actions. By the constructions, two isomorphisms in the claim (1) are compatible with $q_{\mathscr{G}_0}$ and $q_{\mathscr{G}}$.

By (1), we have an isomorphism

$$q_{\mathscr{G}_0}^{-1}(t_0, t_0') \times_{\mathscr{S}_0} \mathscr{S} \xrightarrow{\sim} q_{\mathscr{G}}^{-1}(t, t') \tag{2.3}$$

induced by φ. The claim (2) follows from [SGA3-3, XXII, Proposition 3.4] and (2.3). □

3 Stratification of Deligne–Lusztig variety

3.1 Deligne–Lusztig variety

Let $\mathbf{G}_0$ be a connected reductive group over $\mathbb{F}_q$. We take a maximal torus and a Borel subgroup $\mathbf{T}_0 \subset \mathbf{B}_0 \subset \mathbf{G}_0$ over $\mathbb{F}_q$. We write $\mathbf{G}$, $\mathbf{B}$ and $\mathbf{T}$ for the base changes to $\overline{\mathbb{F}}_q$ of $\mathbf{G}_0$, $\mathbf{B}_0$ and $\mathbf{T}_0$. Let (W, S) be the Coxeter system of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{B}$. For $I, J \subset S$, we write $\mathrm{Par}_I(\mathbf{G})$ and $\mathrm{Par}_{I,J}(\mathbf{G})$ for $\mathrm{Par}_{t(I)}(\mathbf{G})$ and $\mathrm{Par}_{t(I),t(J)}(\mathbf{G})$.

For $I, J \subset S$ and $w \in W$, we put

$$\mathrm{Par}_{I,J}(\mathbf{G})_{[w]} = \mathbf{t}_2^{-1}(r_w),$$

where $r_w \in (q_{\mathbf{G}}^{-1}(t(I), t(J)))(\overline{\mathbb{F}}_q)$ corresponds to $[w] \in W_I \backslash W / W_J$ by Lemma 2.6 (2). Let $\mathrm{Par}_{I,J}(\mathbf{G})_{\leq [w]}$ be the closed reduced subscheme of $\mathrm{Par}_{I,J}(\mathbf{G})$ determined by

$$\bigcup_{[w'] \leq [w]} \mathrm{Par}_{I,J}(\mathbf{G})_{[w']}.$$

Let F be the q-th power Frobenius endomorphism of $\mathbf{G}$ obtained from $\mathbf{G}_0$. Let $I \subset S$ and $w \in W$. For $* \in \{[w], \leq [w]\}$ with $[w] \in W_I \backslash W / W_{\mathrm{F}(I)}$, let $X_I^{\mathrm{F}}(*)$ be the locally closed subscheme of $\mathrm{Par}_I(\mathbf{G})$ defined by the fiber product

$$\begin{array}{ccc} X_I^{\mathrm{F}}(*) & \longrightarrow & \mathrm{Par}_{I,\mathrm{F}(I)}(\mathbf{G})_* \\ \downarrow & & \downarrow \\ \mathrm{Par}_I(\mathbf{G}) & \xrightarrow{(\mathrm{id},\mathrm{F})} & \mathrm{Par}_I(\mathbf{G}) \times \mathrm{Par}_{\mathrm{F}(I)}(\mathbf{G}). \end{array}$$

If there is no confusion, we simply write $X_I(*)$ for $X_I^{\mathrm{F}}(*)$.

For $I \subset J \subset S$, we have a natural morphism

$$\pi_{I,J} \colon X_I([w]) \to X_J([w])$$

which sends a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ of type I to a unique parabolic subgroup $\mathbf{P}'$ of $\mathbf{G}$ of type J containing $\mathbf{P}$.

3.2 Bruhat stratification

Let $I, J \subset S$ and $w \in W$. Let $\mathbf{P}_J$ be a parabolic subgroup of $\mathbf{G}$ of type J. For $* \in \{[w'], \leq[w']\}$ with $[w'] \in W_I \backslash W / W_J$, we let $X_I([w])_{\mathbf{P}_J,*}$ be the locally closed subscheme of $X_I([w])$ defined by the fiber product

$$\begin{array}{ccc} X_I([w])_{\mathbf{P}_J,*} & \longrightarrow & \mathrm{Par}_{I,J}(\mathbf{G})_* \\ \downarrow & & \downarrow \\ X_I([w]) & \xrightarrow{(\mathrm{id},\mathbf{P}_J)} & \mathrm{Par}_I(\mathbf{G}) \times \mathrm{Par}_J(\mathbf{G}). \end{array}$$

3.3 Stratification relative to Frobenius twists

For $1 \leq i \leq m$, let F_i be an Frobenius endomorphism of $\mathbf{G}$ which descends it to an algebraic group over a finite field. Let $w_1, \ldots, w_m \in W$. For $*_i \in \{[w_i], \leq[w_i]\}$ with $[w_i] \in W_I \backslash W / W_{\mathrm{F}_i(I)}$ and $1 \leq i \leq m$, let $X_I^{\mathrm{F}_1,\ldots,\mathrm{F}_m}(*_1, \ldots, *_m)$ be the locally closed subscheme of Par_I defined by the fiber product

$$\begin{array}{ccc} X_I^{\mathrm{F}_1,\ldots,\mathrm{F}_m}(*_1, \ldots, *_m) & \longrightarrow & \prod_{1\leq i\leq m} \mathrm{Par}_{I,\mathrm{F}_i(I)}(\mathbf{G})_{*_i} \\ \downarrow & & \downarrow \\ \mathrm{Par}_I(\mathbf{G}) & \xrightarrow{\prod_{1\leq i\leq m}(\mathrm{id},F_i)} & \prod_{1\leq i\leq m} \left(\mathrm{Par}_I(\mathbf{G}) \times \mathrm{Par}_{\mathrm{F}_i(I)}(\mathbf{G})\right). \end{array}$$

Then $X_I^{\mathrm{F}_1,\ldots,\mathrm{F}_m}([w_1], \ldots, [w_m])$ for $[w_i] \in W_I \backslash W / W_{\mathrm{F}_i(I)}$ and $2 \leq i \leq m$ give a stratification of $X_I^{\mathrm{F}_1}([w_1])$. We note that $X_I^{\mathrm{F}_1,\ldots,\mathrm{F}_m}([w_1], \ldots, [w_m]) = \bigcap_{1\leq i\leq m} X_I^{\mathrm{F}_i}([w_i])$ by the definition.

3.4 Unitary case

We put $\mathbf{V}_0 = \mathbb{F}_{q^2}^d$ equipped with the hermitian form

$$\mathbb{F}_{q^2}^d \times \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2};\ ((a_i)_{1\leq i\leq d}, (a'_i)_{1\leq i\leq d}) \mapsto \sum_{i=1}^{d} a_i^q a'_{d+1-i}.$$

We put $\mathbf{G}_0 = \mathrm{GU}(\mathbf{V}_0)$. By taking the first factor of the decomposition

$$\mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2} \simeq \mathbb{F}_{q^2} \times \mathbb{F}_{q^2};\ a \otimes b \mapsto (ab, ab^q),$$

we have an isomorphism

$$\mathbf{G} \simeq \mathrm{GL}_d \times \mathbb{G}_{\mathrm{m}}. \tag{3.1}$$

Let $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ be the maximal torus and the Borel subgroup determined by the diagonal torus T_d and the upper triangular subgroup B_d of GL_d under (3.1). Let $(W_{\mathbf{G}}, \{s_1, \ldots, s_{d-1}\})$ be the Coxeter system of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{B}$. where s_i corresponds to the simple root

$$T_d \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}};\ (\mathrm{diag}(x_1, \ldots, x_d), z) \mapsto x_i x_{i+1}^{-1}$$

of $\mathrm{GL}_d \times \mathbb{G}_{\mathrm{m}}$ under (3.1). For $1 \leq i_1 < \cdots < i_l \leq d-1$, we put

$$I_d^{i_1,\ldots,i_l} = \{s_i\}_{i\in\{1,\ldots,d-1\}\setminus\{i_1,\ldots,i_l\}}.$$

Lemma 3.1. *Assume that* $2 \leq i \leq d/2$. *The schemes* $X^{\mathrm{F},\mathrm{F}^2,\mathrm{F}^3}_{I_d^{i-1}}([1], \leq[s_{i-1}], [1])$ *and* $X^{\mathrm{F},\mathrm{F}^2}_{I_d^{d-i}}([1], \leq[s_{d-i}])$ *are irreducible.*

Proof. The scheme $X_{I_d^{i-1,d-i}}([1])$ is irreducible by [BR06, Theorem 1]. Hence, it suffices to show the following claims:

(1) The image of
$$\pi_{I_d^{i-1,d-i}, I_d^{i-1}} \colon X_{I_d^{i-1,d-i}}([1]) \to X_{I_d^{i-1}}([1])$$
on $\overline{\mathbb{F}}_q$-valued points is equal to $X^{\mathrm{F},\mathrm{F}^2,\mathrm{F}^3}_{I_d^{i-1}}([1], \leq[s_{i-1}], [1])(\overline{\mathbb{F}}_q)$.

(2) The image of
$$\pi_{I_d^{i-1,d-i}, I_d^{d-i}} \colon X_{I_d^{i-1,d-i}}([1]) \to X_{I_d^{d-i}}([1])$$
on $\overline{\mathbb{F}}_q$-valued points is equal to $X^{\mathrm{F},\mathrm{F}^2}_{I_d^{d-i}}([1], \leq[s_{d-i}])(\overline{\mathbb{F}}_q)$.

We show the claim (1). We equip $\overline{\mathbb{F}}_q^d$ with the pairing
$$\overline{\mathbb{F}}_q^d \times \overline{\mathbb{F}}_q^d \to \overline{\mathbb{F}}_q;\ ((x_i)_{1\leq i\leq d}, (y_i)_{1\leq i\leq d}) \mapsto \sum_{i=1}^{d} x_i y_{d+1-i}. \tag{3.2}$$

For an $\overline{\mathbb{F}}_q$-vector subspace $V \subset \overline{\mathbb{F}}_q^d$, let $V^\perp$ denote the orthogonal complement of V with respect to the pairing (3.2). The q-th power Frobenius element F acts on $\overline{\mathbb{F}}_q^d$. A point of $X_{I_d^{i-1,d-i}}([1])(\overline{\mathbb{F}}_q)$ corresponds to a filtration $0 \subset V_1 \subset V_2 \subset \overline{\mathbb{F}}_q^d$ such that $\dim V_1 = i-1$, $\dim V_2 = d-i$ and
$$V_1 \subset \mathrm{F}(V_2^\perp) \subset V_2 \subset \mathrm{F}(V_1^\perp). \tag{3.3}$$
The condition (3.3) implies
$$V_1 + \mathrm{F}^2(V_1) \subset \mathrm{F}(V_2^\perp). \tag{3.4}$$
Therefore we have
$$\mathrm{F}^3(V_1) \subset \mathrm{F}(V_1 + \mathrm{F}^2(V_1)) \subset \mathrm{F}^2(V_2^\perp) \subset \mathrm{F}(V_2) \subset V_1^\perp \cap \mathrm{F}^2(V_1^\perp) \subset V_1^\perp. \tag{3.5}$$
The conditions (3.3), (3.4) and (3.5) imply that V_1 defines a point of
$$X^{\mathrm{F},\mathrm{F}^2,\mathrm{F}^3}_{I_d^{i-1}}([1], \leq[s_{i-1}], [1])(\overline{\mathbb{F}}_q),$$
because $\dim \mathrm{F}(V_2^\perp) = i$. To show the claim (1), it suffices to show that the image of $\pi_{I_d^{i-1,d-i}, I_d^{i-1}}$ on $\overline{\mathbb{F}}_q$-valued points contains
$$X^{\mathrm{F},\mathrm{F}^2,\mathrm{F}^3}_{I_d^{i-1}}([1], [s_{i-1}], [1])(\overline{\mathbb{F}}_q), \tag{3.6}$$
because $X_{I_d^{i-1,d-i}}([1])$ is proper. A point of (3.6) gives an $\overline{\mathbb{F}}_q$-vector subspace $V_1 \subset \overline{\mathbb{F}}_q^d$ of dimension $i-1$ such that
$$V_1 \subset \mathrm{F}(V_1^\perp), \quad \dim(V_1 + \mathrm{F}^2(V_1)) = i, \quad \mathrm{F}^3(V_1) \subset V_1^\perp. \tag{3.7}$$

The condition implies

$$\mathrm{F}(V_1 + \mathrm{F}^2(V_1)) \subset V_1^{\perp} \cap \mathrm{F}^2(V_1^{\perp})$$

and $\dim V_1^{\perp} \cap \mathrm{F}^2(V_1^{\perp}) = d - i$. We take $V_2 \subset \overline{\mathbb{F}}_q^d$ such that $\mathrm{F}(V_2) = V_1^{\perp} \cap \mathrm{F}^2(V_1^{\perp})$. Then (V_1, V_2) defines a point of $X_{I_d^{i-1,d-i}}([1])(\overline{\mathbb{F}}_q)$ whose image under $\pi_{I_d^{i-1,d-i}, I_d^{i-1}}$ is the point of (3.6) corresponding to V_1. Therefore we obtain the claim (1).

The claim (2) is proved similarly. □

4 Affine Grassmannian

Let F be a non-archimedean local field with residue field $k = \mathbb{F}_q$. Let $\mathcal{O}$ be the ring of integers of F. Let ϖ be a uniformizer of F. For a perfect k-algebra R, we put

$$W_{\mathcal{O}}(R) = \varprojlim_n W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n,$$

$D_R = \mathrm{Spec}(W_{\mathcal{O}}(R))$ and $D_R^* = \mathrm{Spec}(W_{\mathcal{O}}(R)[\frac{1}{\varpi}])$. For an affine group scheme H of finite type over $\mathcal{O}$, we define the jet group L^+H and the loop group $\mathrm{L}H$ by

$$\mathrm{L}^+H(R) = H(W_{\mathcal{O}}(R)), \quad \mathrm{L}H(R) = H(W_{\mathcal{O}}(R)[\frac{1}{\varpi}]).$$

We put $L = W_{\mathcal{O}}(\overline{k})[\frac{1}{\varpi}]$. We note that $\mathrm{L}H(\overline{k}) = H(L)$.

Let G be a reductive group scheme over $\mathcal{O}$. Let T be the abstract Cartan subgroup. Let $\Phi \subset \mathbb{X}^\bullet(T)$ denote the set of roots of G. We fix a Borel subgroup $B \subset G$, which determines the semi-group of dominant coweights $\mathbb{X}_\bullet(T)^+ \subset \mathbb{X}_\bullet(T)$. Let U be the unipotent radical of B. Let $\rho \in \mathbb{X}^\bullet(T)_{\mathbb{Q}}$ be the half sum of all positive roots.

Let Gr_G denote the affine Grassmannian over k of G defined by $\mathrm{Gr}_G = \mathrm{L}G/\mathrm{L}^+G$. For a finite etale extension $\mathcal{O}'$ of $\mathcal{O}$ with residue field k', we have a natural isomorphism

$$(\mathrm{Gr}_G)_{k'} \cong \mathrm{Gr}_{G_{\mathcal{O}'}} \tag{4.1}$$

by the construction. We simply write Gr for Gr_G if there is no confusion. Then Gr is an ind-perfectly projective scheme by [BS17, Corollary 9.6]. Let $\mathcal{E}^0$ denote the trivial G-torsor over $\mathcal{O}$. For a perfect k-algebra R, we have

$$\mathrm{Gr}(R) = \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } D_R, \\ \beta\colon \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ is a trivialization.} \end{array} \right\} \tag{4.2}$$

(*cf.* [Zhu17, Lemma 1.3]). We sometimes write $\beta\colon \mathcal{E} \dashrightarrow \mathcal{E}^0$ for $\beta\colon \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*}$ in (4.2), and call it a modification. Given a point $(\mathcal{E}, \beta)$, one can define a relative position invariant $\mathrm{inv}(\beta) \in \mathbb{X}_\bullet(T)^+$.

Let $\mu \in \mathbb{X}_\bullet(T)^+$. The Schubert variety Gr_μ is the closed subscheme of $\mathrm{Gr}_{\overline{k}}$ parametrizing pairs $(\mathcal{E}, \beta)$ such that $\mathrm{inv}(\beta) \preceq \mu$. The Schubert cell $\mathring{\mathrm{Gr}}_\mu$ is the open subscheme of Gr_μ parametrizing pairs $(\mathcal{E}, \beta)$ such that $\mathrm{inv}(\beta) = \mu$.

For a sequence $\mu_\bullet = (\mu_1, \ldots, \mu_n)$ of positive dominant coweights, let $\mathrm{Gr}_{\mu_\bullet}$ be the scheme over $\overline{k}$ parametrizing sequences of modifications $(\beta_i\colon \mathcal{E}_i \dashrightarrow \mathcal{E}_{i-1})_{1 \le i \le n}$ with $\mathcal{E}_0 = \mathcal{E}^0$ such that $\mathrm{inv}(\beta_i) \preceq \mu_i$ for each i. The open subscheme $\mathring{\mathrm{Gr}}_{\mu_\bullet} \subset \mathrm{Gr}_{\mu_\bullet}$ is defined by the condition that $\mathrm{inv}(\beta_i) = \mu_i$ for each i. The convolution map $m_{\mu_\bullet}\colon \mathrm{Gr}_{\mu_\bullet} \to \mathrm{Gr}_{\overline{k}}$ sends a sequence of modifications to the composition $(\mathcal{E}_n, \beta_1 \circ \cdots \circ \beta_n)$.

Let $\lambda_\bullet = (\lambda_1, \dots, \lambda_l)$ and $\mu_\bullet = (\mu_1, \dots, \mu_n)$ be two sequences. We put

$$\mathrm{Gr}^0_{\lambda_\bullet|\mu_\bullet} = \mathrm{Gr}_{\lambda_\bullet} \times_{\mathrm{Gr}_{\overline{k}}} \mathrm{Gr}_{\mu_\bullet}, \quad \mathring{\mathrm{Gr}}^0_{\lambda_\bullet|\mu_\bullet} = \mathring{\mathrm{Gr}}_{\lambda_\bullet} \times_{\mathrm{Gr}_{\overline{k}}} \mathring{\mathrm{Gr}}_{\mu_\bullet},$$

where the products are over the convolution maps $m_{\lambda_\bullet} \colon \mathrm{Gr}_{\lambda_\bullet} \to \mathrm{Gr}_{\overline{k}}$, $m_{\mu_\bullet} \colon \mathrm{Gr}_{\mu_\bullet} \to \mathrm{Gr}_{\overline{k}}$ and their restrictions respectively. We write

$$m_{\lambda_\bullet|\mu_\bullet} \colon \mathrm{Gr}^0_{\lambda_\bullet|\mu_\bullet} \to \mathrm{Gr}_{\overline{k}}$$

for the natural projection. We simply write m for $m_{\lambda_\bullet|\mu_\bullet}$ if there is no confusion. For $1 \le j \le l$, we define

$$\mathrm{pr}_j \colon \mathrm{Gr}^0_{\lambda_\bullet|\mu_\bullet} \to \mathrm{Gr}_{(\lambda_1,\dots,\lambda_j)}$$

by sending $((\alpha_i)_{1\le i\le l}, (\beta_i)_{1\le i\le n})$ to $(\alpha_i)_{1\le i\le j}$.

An irreducible component of $\mathrm{Gr}^0_{\lambda_\bullet|\mu_\bullet}$ of dimension $\langle \rho, |\lambda_\bullet| + |\mu_\bullet| \rangle$ is called a Satake cycle. Let $\mathbb{S}_{\lambda_\bullet|\mu_\bullet}$ be the set of Satake cycles in $\mathrm{Gr}^0_{\lambda_\bullet|\mu_\bullet}$. We sometimes write $\mathrm{Gr}^{0,\mathbf{a}}_{\lambda_\bullet|\mu_\bullet}$ instead of $\mathbf{a} \in \mathbb{S}_{\lambda_\bullet|\mu_\bullet}$ for the Satake cycle. We put

$$\mathring{\mathrm{Gr}}^{0,\mathbf{a}}_{\lambda_\bullet|\mu_\bullet} = \mathrm{Gr}^{0,\mathbf{a}}_{\lambda_\bullet|\mu_\bullet} \cap \mathring{\mathrm{Gr}}^0_{\lambda_\bullet|\mu_\bullet}.$$

Lemma 4.1. *For $\mathbf{a} \in \mathbb{S}_{\lambda_\bullet|\mu_\bullet}$, the scheme $\mathring{\mathrm{Gr}}^{0,\mathbf{a}}_{\lambda_\bullet|\mu_\bullet}$ is not empty.*

Proof. The dimension of $\mathrm{Gr}^0_{\lambda_\bullet|\mu_\bullet} \setminus \mathring{\mathrm{Gr}}^0_{\lambda_\bullet|\mu_\bullet}$ is less than $\langle \rho, |\lambda_\bullet| + |\mu_\bullet| \rangle$ by [XZ17, Proposition 3.1.10 (1)]. Hence we obtain the claim. □

We fix an embedding $T \subset B$. Let $\mu \in \mathbb{X}_\bullet(T)$. Let $\mathcal{O}'$ be $\mathcal{O}_L = W_{\mathcal{O}}(\overline{k})$ or a finite etale extension of $\mathcal{O}$ which splits G. For $\alpha \in \Phi$, let $U_{\alpha,\mathcal{O}'}$ denote the root subgroup of $G_{\mathcal{O}'}$ corresponding to α. Let $P_{\mu,\mathcal{O}'}$ denote the parabolic subgroup of $G_{\mathcal{O}'}$ generated by $T_{\mathcal{O}'}$ and $U_{\alpha,\mathcal{O}'}$ for $\alpha \in \Phi$ such that $\langle \alpha, \mu \rangle \ge 0$.

We write ϖ^μ for $\mu(\varpi) \in G(L) = \mathrm{L}G(\overline{k})$. Let $[\varpi^\mu]$ denote the point of $\mathrm{Gr}_{\overline{k}}$ determined by ϖ^μ. For $\mu \in \mathbb{X}_\bullet(T)^+$, the Schubert cell $\mathring{\mathrm{Gr}}_\mu$ is the L^+G-orbit of $[\varpi^\mu]$ by [Zhu17, Proposition 1.23 (1)].

Lemma 4.2. *For $\mathbf{a} \in \mathbb{S}_{\lambda_\bullet|\mu}$, the natural morphism $\mathrm{Gr}^{0,\mathbf{a}}_{\lambda_\bullet|\mu} \to \mathrm{Gr}_\mu$ is surjective.*

Proof. The natural morphism $\mathring{\mathrm{Gr}}^{0,\mathbf{a}}_{\lambda_\bullet|\mu} \to \mathring{\mathrm{Gr}}_\mu$ is surjective, because the action of L^+G on $\mathring{\mathrm{Gr}}_\mu$ is transitive and $\mathring{\mathrm{Gr}}^{0,\mathbf{a}}_{\lambda_\bullet|\mu}$ is a nonempty scheme stable under the action of L^+G by Lemma 4.1. Hence we obtain the claim because $\mathrm{Gr}^0_{\lambda_\bullet|\mu} \to \mathrm{Gr}_\mu$ is perfectly proper and $\mathring{\mathrm{Gr}}_\mu \subset \mathrm{Gr}_\mu$ is Zariski dense by [Zhu17, Proposition 1.23 (3)]. □

For $\lambda \in \mathbb{X}_\bullet(T)$, let S_λ be the $(\mathrm{L}U)_{\overline{k}}$-orbit of ϖ^λ in $\mathrm{Gr}_{\overline{k}}$. For $\lambda \in \mathbb{X}_\bullet(T)$ and $\mu \in \mathbb{X}_\bullet(T)^+$, an irreducible component of $S_\lambda \cap \mathrm{Gr}_\mu$ is called a Mirković–Vilonen cycle after [MV07]. Let $\mathbb{MV}_\mu(\lambda)$ be the set of the Mirković–Vilonen cycles in $S_\lambda \cap \mathrm{Gr}_\mu$. We sometimes write $(S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}}$ instead of $\mathbf{b} \in \mathbb{MV}_\mu(\lambda)$ for the Mirković–Vilonen cycle.

Let $(\widehat{G}, \widehat{B}, \widehat{T})$ be the Langlands dual over $\overline{\mathbb{Q}}_\ell$ of (G, B, T). For $\mu \in \mathbb{X}_\bullet(T)^+ = \mathbb{X}^\bullet(\widehat{T})^+$, let V_μ denote the irreducible algebraic representation of $\widehat{G}$ of highest weight μ. For an algebraic representation V of $\widehat{G}$ and $\lambda \in \mathbb{X}_\bullet(T) = \mathbb{X}^\bullet(\widehat{T})$, let $V(\lambda)$ denote the λ-weight space of V. Then we have

$$|\mathbb{MV}_\mu(\lambda)| = \dim V_\mu(\lambda) \tag{4.3}$$

by [GHKR06, Proposition 5.4.2] and [Zhu17, Corollary 2.8].

For $\nu, \mu \in \mathbb{X}_\bullet(T)^+$ and $\lambda \in \mathbb{X}_\bullet(T)$ such that $\nu + \lambda \in \mathbb{X}_\bullet(T)^+$, there is an injective map

$$i_\nu^{\mathbb{MV}} \colon \mathbb{S}_{(\nu,\mu)|\nu+\lambda} \to \mathbb{MV}_\mu(\lambda)$$

constructed by [XZ17, Lemma 3.2.7].

5 Equidimensionality of Satake cycles

Lemma 5.1. *The morphism $m_{\mu_\bullet} \colon \mathrm{Gr}_{\mu_\bullet} \to \mathrm{Gr}_{\overline{k}}$ is Zariski-locally trivial over $\mathring{\mathrm{Gr}}_\lambda$.*

Proof. Taking the base change to an unramified extension of $\mathcal{O}$, we may assume that G is split by (4.1). As in the proof of [Hai06, Lemma 2.1], it suffices to show that

$$\mathrm{L}^+G \to \mathrm{L}^+G/(\mathrm{L}^+G \cap \varpi^\lambda \mathrm{L}^+G \varpi^{-\lambda})$$

has a section Zariski-locally. Since $\mathrm{L}^+U/(\mathrm{L}^+U \cap \varpi^\lambda \mathrm{L}^+U \varpi^{-\lambda})$ is an open subscheme of $\mathrm{L}^+G/(\mathrm{L}^+G \cap \varpi^\lambda \mathrm{L}^+G \varpi^{-\lambda})$, it suffices to show that

$$\mathrm{L}^+U \to \mathrm{L}^+U/(\mathrm{L}^+U \cap \varpi^\lambda \mathrm{L}^+U \varpi^{-\lambda})$$

has a section. We fix an identification $\mathbb{G}_\mathrm{a} \simeq U_{\alpha,\mathcal{O}}$ for a positive root α. For a positive root α, let $\mathrm{L}^+_{<\langle\alpha,\lambda\rangle} U_{\alpha,\mathcal{O}}$ be the closed subscheme of $\mathrm{L}^+U_{\alpha,\mathcal{O}}$ defined by the condition $x_i = 0$ for $i \geq \langle \alpha, \lambda \rangle$ for a point $\sum_{i=0}^\infty \varpi^i [x_i]$ of $\mathrm{L}^+U_{\alpha,\mathcal{O}}$. Then the composition

$$\prod_\alpha \mathrm{L}^+_{<\langle\alpha,\lambda\rangle} U_{\alpha,\mathcal{O}} \to \mathrm{L}^+U \to \mathrm{L}^+U/(\mathrm{L}^+U \cap \varpi^\lambda \mathrm{L}^+U \varpi^{-\lambda})$$

is an isomorphism. Hence we have a section. □

Lemma 5.2. *Assume that μ is a dominant minuscule cocharacter and $w \in W$. We have an isomorphism*

$$S_{w\mu} \cap \mathrm{Gr}_\mu \simeq \mathrm{L}^+U_{\overline{k}}/((\mathrm{L}^+U)_{\overline{k}} \cap \varpi^{w\mu} (\mathrm{L}^+U)_{\overline{k}} \varpi^{-w\mu}).$$

In particular, $S_{w\mu} \cap \mathrm{Gr}_\mu$ is the perfection of an affine space of dimension $\langle \rho, \mu + w\mu \rangle$.

Proof. The first claim follows from [XZ17, (3.2.3)]. The second claim follows from the first one as in the proof of [Hai06, Lemma 3.2]. □

Theorem 5.3. *Assume that μ_i are minuscule. For a point y of $\mathring{\mathrm{Gr}}_\lambda$, the fiber of $m_{\mu_\bullet} \colon \mathrm{Gr}_{\mu_\bullet} \to \mathrm{Gr}_{\overline{k}}$ at y is equidimensional of dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$.*

Proof. This is proved in the same way as [Hai06, Theorem 3.1] using Lemma 5.1 and Lemma 5.2 instead of [Hai06, Lemma 2.1 and Lemma 3.2] respectively. □

Proposition 5.4. *Assume that each μ_i is a sum of minuscule cocharacters. Then, for a point y of $\mathring{\mathrm{Gr}}_\lambda$, any irreducible component of the fiber $m_{\mu_\bullet}^{-1}(y)$ whose generic point belongs to $\mathring{\mathrm{Gr}}_{\mu_\bullet}$ has dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$.*

Proof. This follows from Theorem 5.3 in the same way as [Hai06, Proposition 4.1]. □

6 Affine Deligne–Lusztig variety

Recall that $L = W_{\mathcal{O}}(\overline{k})[\frac{1}{\varpi}]$. Let $b \in G(L)$ and $\mu \in \mathbb{X}_\bullet(T)$. Let σ denote the q-th power Frobenius element. We define the affine Deligne–Lusztig variety $X_\mu(b)$ by

$$X_\mu(b) = \{g(\mathrm{L}^+G)_{\overline{k}} \in \mathrm{Gr}_{\overline{k}} \mid g^{-1}b\sigma(g) \in \overline{(\mathrm{L}^+G)_{\overline{k}}\varpi^\mu(\mathrm{L}^+G)_{\overline{k}}}\}.$$

Let $B(G)$ be the set of σ-conjugacy classes of $G(L)$. We define $B(G,\mu) \subset B(G)$ as in [Kot97, 6.2]. Then $X_\mu(b)$ is non-empty if and only if $[b] \in B(G,\mu)$ by [Gas10, Theorem 5.1].

An element of $B(G)$ is called unramified if it is contained in the image of the natural map $B(T) \to B(G)$. Let $B(G)_{\mathrm{ur}}$ denote the set of unramified elements of $B(G)$.

For $\chi \in \mathbb{X}^\bullet(T)$, we put

$$\overline{\chi} = \frac{1}{|\langle\sigma\rangle\chi|}\sum_{\chi' \in \langle\sigma\rangle\chi}\chi' \in \mathbb{X}^\bullet(T)^\sigma_{\mathbb{Q}}.$$

The natural pairing $\mathbb{X}_\bullet(T) \times \mathbb{X}^\bullet(T) \to \mathbb{Z}$ induces a pairing $\langle\ ,\ \rangle \colon \mathbb{X}_\bullet(T)_\sigma \times \mathbb{X}^\bullet(T)^\sigma_{\mathbb{Q}} \to \mathbb{Q}$. We put

$$\mathbb{X}_\bullet(T)^+_\sigma = \{[\lambda] \in \mathbb{X}_\bullet(T)_\sigma \mid \langle[\lambda],\overline{\alpha}\rangle \geq 0 \text{ for every } \alpha \in \Delta\}.$$

Then we have the bijection

$$\mathbb{X}_\bullet(T)^+_\sigma \cong B(G)_{\mathrm{ur}};\ [\lambda] \mapsto [\varpi^\lambda]$$

as in [XZ17, Lemma 4.2.3].

For $\tau \in \mathbb{X}_\bullet(T)$, we write $X_\mu(\tau)$ for $X_\mu(\varpi^\tau)$. We assume that $b = \varpi^\tau$ for $\tau \in \mathbb{X}_\bullet(T)$ such that $[\tau] \in \mathbb{X}_\bullet(T)^+_\sigma$. We can define the twisted centralizer J_τ over $\mathcal{O}$ for ϖ^τ as in [XZ17, 4.2.13]. We note that $J_\tau = G$ if $[b] \in B(G)_{\mathrm{ur}}$ is basic.

We assume that G satisfies [XZ17, Hypothesis 4.4.1]. Further, we assume that Z_G is connected.

Let $\lambda \in \mathbb{X}_\bullet(T)$ such that $[\lambda] = [\tau] \in \mathbb{X}_\bullet(T)^+_\sigma$. We take $\delta_\lambda \in \mathbb{X}_\bullet(T)$ such that $\lambda = \tau + \delta_\lambda - \sigma(\delta_\lambda)$. Let $\mathbf{b} \in \mathbb{MV}_\mu(\lambda)$. We take $\nu_{\mathbf{b}} \in \mathbb{X}_\bullet(T)$ as in [XZ17, Lemma 4.4.3]. We put $\tau_{\mathbf{b}} = \lambda + \nu_{\mathbf{b}} - \sigma(\nu_{\mathbf{b}})$. Then we have the isomorphism

$$J_\tau(F) \cong J_{\tau_{\mathbf{b}}}(F);\ g \mapsto \varpi^{\delta_\lambda+\nu_{\mathbf{b}}}g\varpi^{-\delta_\lambda-\nu_{\mathbf{b}}}.$$

We consider the isomorphism

$$X_\mu(b) = X_\mu(\tau) \simeq X_\mu(\tau_{\mathbf{b}});\ g\mathrm{L}^+G \mapsto \varpi^{\delta_\lambda+\nu_{\mathbf{b}}}g\mathrm{L}^+G. \tag{6.1}$$

Let $\mathbf{a} \in \mathbb{S}_{(\nu_{\mathbf{b}},\nu)|\lambda+\nu_{\mathbf{b}}}$ be the unique element such that $\mathbf{b} = i^{\mathbb{MV}}_{\nu_{\mathbf{b}}}(\mathbf{a})$.

We define $X_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}})$ by the fiber product

$$\begin{array}{ccc} X_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}}) & \longrightarrow & \mathrm{Gr}^0_{(\nu_{\mathbf{b}},\mu)|\tau_{\mathbf{b}}+\sigma(\nu_{\mathbf{b}})} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1 \times m} \\ \mathrm{Gr}_{\nu_{\mathbf{b}}} & \xrightarrow{1\times\varpi^{\tau_{\mathbf{b}}}\sigma} & \mathrm{Gr}_{\nu_{\mathbf{b}}} \times \mathrm{Gr}_{\tau_{\mathbf{b}}+\sigma(\nu_{\mathbf{b}})}\ . \end{array}$$

Further, we define $X^{\mathbf{a}}_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}})$ by the fiber product

$$\begin{array}{ccc} X^{\mathbf{a}}_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}}) & \longrightarrow & \mathrm{Gr}^{0,\mathbf{a}}_{(\nu_{\mathbf{b}},\mu)|\tau_{\mathbf{b}}+\sigma(\nu_{\mathbf{b}})} \\ \downarrow & & \downarrow \\ X_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}}) & \longrightarrow & \mathrm{Gr}^{0}_{(\nu_{\mathbf{b}},\mu)|\tau_{\mathbf{b}}+\sigma(\nu_{\mathbf{b}})} . \end{array}$$

Let x_0 denote $[1] \in J_\tau(F)/J_\tau(\mathcal{O})$. We put

$$\mathring{X}^{\mathbf{b},x_0}_{\mu}(\tau_{\mathbf{b}}) = X^{\mathbf{a}}_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}}) \cap \mathring{\mathrm{Gr}}_{\nu_{\mathbf{b}}}.$$

Let $X^{\mathbf{b},x_0}_{\mu}(\tau_{\mathbf{b}})$ denote the closure of $\mathring{X}^{\mathbf{b},x_0}_{\mu}(\tau_{\mathbf{b}})$ in $X^{\mathbf{a}}_{\mu,\nu_{\mathbf{b}}}(\tau_{\mathbf{b}})$. The scheme $X^{\mathbf{b},x_0}_{\mu}(\tau_{\mathbf{b}})$ is irreducible of dimension $\langle \rho, \mu - \tau_{\mathbf{b}} \rangle$ by [XZ17, Theorem 4.4.5].

By [XZ17, Theorem 4.4.14], there is a bijection between the set

$$\coprod_{\lambda \in \mathbb{X}_\bullet(T),\ [\lambda]=[\tau]\in \mathbb{X}_\bullet(T)^+_\sigma} \mathbb{MV}_\mu(\lambda) \times J_\tau(F)/J_\tau(\mathcal{O})$$

and the set of irreducible components of $X_\mu(b)$ given by

$$(\mathbf{b}, [g]) \mapsto X^{\mathbf{b},[g]}_{\mu}(\tau_{\mathbf{b}}) := g X^{\mathbf{b},x_0}_{\mu}(\tau_{\mathbf{b}}),$$

where we regard $X^{\mathbf{b},[g]}_{\mu}(\tau_{\mathbf{b}})$ as a subscheme of $X_\mu(b)$ by (6.1).

7 Unitary group

7.1 Setting

Let F_2 be the quadratic unramified extension of F. Let $\mathcal{O}_2$ denote the ring of integers of F_2. Let ϖ be a uniformizer of F. We put $\Lambda = \mathcal{O}_2^n$ equipped with the hermitian form

$$\mathcal{O}_2^n \times \mathcal{O}_2^n \to \mathcal{O}_2;\ ((a_i)_{1\le i\le n}, (a'_i)_{1\le i \le n}) \mapsto \sum_{i=1}^{n} \sigma(a_i) a'_{n+1-i}.$$

We put $G = \mathrm{GU}(\Lambda)$. By taking the first factor of the decomposition

$$\mathcal{O}_2 \otimes_{\mathcal{O}} \mathcal{O}_2 \simeq \mathcal{O}_2 \times \mathcal{O}_2;\ a \otimes b \mapsto (ab, a\sigma(b)),$$

we have an isomorphism

$$G_{\mathcal{O}_2} \simeq \mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}. \tag{7.1}$$

We put $V = \Lambda \otimes_{\mathcal{O}} F$. Let $\widehat{G} = \mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}$ denote the dual group over $\overline{\mathbb{Q}}_\ell$ with a maximal torus $\widehat{T}$ and a Borel subgroup $\widehat{B}$, which are the diagonal torus and the upper triangular subgroup on the GL_n-component. For $\mu \in \mathbb{X}_\bullet(T)^+$, let $\mu^* \in \mathbb{X}_\bullet(T)^+$ be the element such that $V_{\mu^*} = V_\mu^*$.

For an index $i \in \{1, \ldots, n\}$, we will use the notation $i^\vee = n+1-i$. The group $\mathbb{X}^\bullet(\widehat{T})$ has a basis $\{\varepsilon_i\}_{i=0}^n$, where ε_0 is the projection to the $\mathbb{G}_{\mathrm{m}}$-component and ε_i is the character of $\widehat{T}$ given by evaluating the (i,i) entry for $i \geq 1$. In the following, all cocharacters of T (equivalently, characters of $\widehat{T}$) will be written according to this basis. We have $\sigma(\varepsilon_0) = \varepsilon_0$ and $\sigma(\varepsilon_i) = -\varepsilon_{i^\vee}$ for $1 \le i \le n$. For $\mu = \sum_{i=0}^n m_i \varepsilon_i \in \mathbb{X}_\bullet(T)^+$, we have

$$\mu^* = -m_0 \varepsilon - \sum_{i=1}^{n} m_{n+1-i} \varepsilon_i \in \mathbb{X}_\bullet(T)^+.$$

7.2 Satake cycle

Let $\mu = \varepsilon_0 + \varepsilon_1 + \varepsilon_2 \in \mathbb{X}_\bullet(T)$. We put $r = [n/2]$. We put

$$\nu_i = \varepsilon_1 + \cdots + \varepsilon_{i-1} - \varepsilon_{i^\vee} - \cdots - \varepsilon_{1^\vee}, \quad \tau_i = \varepsilon_0$$

for $1 \leq i \leq [(n-1)/2]$, and

$$\nu_r = \varepsilon_1 + \cdots + \varepsilon_{r-1}, \quad \tau_r = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_n$$

if n is even. We put $\lambda_i = -\varepsilon_0 - \varepsilon_i - \varepsilon_{i^\vee}$ for $1 \leq i \leq r$.

Lemma 7.1. *For $\lambda \in -\varepsilon_0 + (1-\sigma)\mathbb{X}_\bullet(T)$, we have $\mathbb{MV}_{\mu^*}(\lambda) \neq \emptyset$ if and only if $\lambda \in \{\lambda_1, \ldots, \lambda_r\}$. Further, $\mathbb{MV}_{\mu^*}(\lambda_i)$ is a singleton for $1 \leq i \leq r$.*

Proof. For $\lambda \in \mathbb{X}_\bullet(T)$, we have $\dim V_{\mu^*}(\lambda) \leq 1$, and $V_{\mu^*}(\lambda)$ is nonzero if and only if $\lambda = -\varepsilon_0 - \varepsilon_i - \varepsilon_j$ for some $1 \leq i < j \leq n$. If $-\varepsilon_0 - \varepsilon_i - \varepsilon_j \in -\varepsilon_0 + (1-\sigma)\mathbb{X}_\bullet(T)$ for some $1 \leq i < j \leq n$, we must have $j = i^\vee$. Hence the claim follows from (4.3). □

Let $1 \leq i \leq r$. Note that $\tau_i^* + \sigma(\nu_i^*) = \nu_i^* + \lambda_i = \nu_i + \tau_i^*$. Let $\mathbf{b}_i$ be the unique element of $\mathbb{MV}_{\mu^*}(\lambda_i)$. There is $\mathbf{a}_i \in \mathbb{S}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}$ such that $i_{\nu_i^*}^{\mathbb{MV}}(\mathbf{a}_i) = \mathbf{b}_i$. Since $i_{\nu_i^*}^{\mathbb{MV}}$ is injective by [XZ17, Lemma 3.2.7], the set $\mathbb{S}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}$ is also a singleton.

We study the Satake cycle $\mathrm{Gr}^{0,\mathbf{a}_i}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}$.

Lemma 7.2. *(1) The scheme $\mathring{\mathrm{Gr}}^{0}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}$ is irreducible.*

(2) We have

$$\mathrm{Gr}^{0,\mathbf{a}_i}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*} = \overline{\mathring{\mathrm{Gr}}^{0}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}}.$$

Proof. By the definition, $\mathrm{Gr}^{0}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}$ is equal to the inverse image of $\mathrm{Gr}_{\nu_i+\tau_i^*}$ under the convolution morphism

$$m_{(\nu_i^*,\mu^*)} \colon \mathrm{Gr}_{(\nu_i^*,\mu^*)} \to \mathrm{Gr}.$$

By Lemma 5.1 and Proposition 5.4, we obtain the claim (1), since $\mathbb{S}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*}$ is a singleton. The claim (2) follows from (1). □

We do not use the following lemma in the sequel, but it shows that a study of intersections of irreducible components of affine Deligne–Lusztig varieties is more subtle than intersections of Satake cycles.

Lemma 7.3. *Assume that $n \geq 5$.*

(1) The actions of L^+G on $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ and $\mathrm{Gr}^{0}_{(\nu_1^*,\mu^*)|\nu_1+\tau_1^*}$ are transitive.

(2) The Satake cycle $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ contains $\mathrm{Gr}^{0,\mathbf{a}_1}_{(\nu_1^*,\mu^*)|\nu_1+\tau_1^*}$.

Proof. We show (1). It suffices to show that the number of the orbits under the action of L^+G on $\mathrm{Gr}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ is 2. Let $(\mathrm{L}^+G)_{\nu_1+\tau_1^*}$ be the stabilizer of $[\varpi^{\nu_1+\tau_1^*}] \in \mathrm{Gr}_{\nu_1+\tau_1^*}$ in L^+G. Since the action of L^+G on $\mathrm{Gr}_{\nu_1+\tau_1^*}$ is transitive, it suffices to show that the number of the orbits in $m^{-1}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}([\varpi^{\nu_1+\tau_1^*}])$ under the action of $(\mathrm{L}^+G)_{\nu_1+\tau_1^*}$ is 2. These orbits are in a bijection with $(P_{\nu_1+\tau_1^*,\mathcal{O}_L})_{\overline{k}} \backslash G_{\overline{k}} / (P_{\mu,\mathcal{O}_L})_{\overline{k}}$. Hence the number of the orbits is 2.

We show (2). By Lemma 4.2, the natural morphism $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*} \to \mathrm{Gr}_{\nu_2+\tau_2^*}$ is surjective. Hence the intersection of $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ and $\mathrm{Gr}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ is not empty.

If the intersection of $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ and $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ is not empty, then $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ contains $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ because L^+G acts transitively on $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ and $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ is stable under the action of L^+G. Then $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ contains $\mathrm{Gr}^{0}_{(\nu_1^*,\mu^*)|\nu_1+\tau_1^*}$, since $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ is dense in $\mathrm{Gr}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$.

If the intersection of $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ and $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_1+\tau_1^*}$ is empty, the intersection of $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ and $\mathrm{Gr}^{0}_{(\nu_1^*,\mu^*)|\nu_1+\tau_1^*}$ is not empty. Then $\mathrm{Gr}^{0,\mathbf{a}_2}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ contains $\mathrm{Gr}^{0}_{(\nu_1^*,\mu^*)|\nu_1+\tau_1^*}$ because L^+G acts transitively on $\mathrm{Gr}^{0}_{(\nu_1^*,\mu^*)|\nu_1+\tau_1^*}$ and $\mathring{\mathrm{Gr}}^{0}_{(\nu_2^*,\mu^*)|\nu_2+\tau_2^*}$ is stable under the action of L^+G. □

Remark 7.4. *Lemma 7.3 (2) shows that $X^{\mathbf{a}_2}_{\mu^*,\nu_2^*}(\tau_2)$ is not irreducible. This answers a question in [XZ17, Remark 4.4.6 (3)].*

8 Irreducible Components

We note that $[\varpi^{-\varepsilon_0}] \in B(G,\mu^*)$ is the basic class. Let $X^{\mathbf{b}_i,x_0}_{\mu^*}(\tau_i^*)$ be the closure in $X_{\mu^*}(\tau_i^*)$ of $\mathring{X}^{\mathbf{b}_i,x_0}_{\mu^*}(\tau_i^*)$ fitting in the cartesian diagram

$$\begin{array}{ccc} \mathring{X}^{\mathbf{b}_i,x_0}_{\mu^*}(\tau_i^*) & \longrightarrow & \mathring{\mathrm{Gr}}^{0,\mathbf{a}_i}_{(\nu_i^*,\mu^*)|\nu_i+\tau_i^*} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1\times m} \\ \mathring{\mathrm{Gr}}_{\nu_i^*} & \xrightarrow{1\times\varpi^{\tau_i^*}\sigma} & \mathring{\mathrm{Gr}}_{\nu_i^*}\times\mathring{\mathrm{Gr}}_{\nu_i+\tau_i^*}. \end{array}$$

By results in §6 and Lemma 7.1, we obtain the following proposition:

Proposition 8.1. *The number of the $G(F)$-orbits of the irreducible components of $X_{\mu^*}(\varepsilon_0^*)$ is r. Representatives of r orbits are given by $X^{\mathbf{b}_i,x_0}_{\mu^*}(\tau_i^*)$ for $1 \leq i \leq r$. The $G(F)$-orbit of $X^{\mathbf{b}_i,x_0}_{\mu^*}(\tau_i^*)$ is parametrized by $G(F)/G(\mathcal{O})$. The dimension of $X^{\mathbf{b}_i,x_0}_{\mu^*}(\tau_i^*)$ is $\langle \rho, \mu^*-\tau_i^*\rangle = n-2$.*

If $i=1$, the above construction defines a Deligne–Lusztig variety. If $i=2$ and $n \geq 5$, this defines a variety that is not a Deligne–Lusztig variety.

By (4.1) and (7.1), we have an isomorphism

$$\mathrm{Gr}_G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2} \simeq \mathrm{Gr}_{\mathrm{GL}_n\times\mathbb{G}_{\mathrm{m}}}. \tag{8.1}$$

We put $\mathcal{E}^0 = \mathcal{O}_2^n$ and $\mathcal{L}^0 = \mathcal{O}_2$ and view them as trivial vector bundles on $D_{\mathbb{F}_{q^2}}$. For any perfect $\mathbb{F}_{q^2}$-algebra R,

$$\mathrm{Gr}_{\mathrm{GL}_n\times\mathbb{G}_{\mathrm{m}}}(R) = \left\{ (\mathcal{E},\mathcal{L},\beta,\beta') \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a vector bundle on } D_R \text{ of rank } n, \\ \mathcal{L} \text{ is a line bundle on } D_R, \\ \beta\colon \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ and} \\ \beta'\colon \mathcal{L}|_{D_R^*} \simeq \mathcal{L}^0|_{D_R^*} \text{ are trivializations.} \end{array} \right\} \tag{8.2}$$

by (4.2). Under the identification by (8.1), the Frobenius endomorphism of $\mathrm{Gr}_G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ sends $(\mathcal{E}, \mathcal{L}, \beta, \beta')$ in (8.2) to

$$(\mathrm{F}(\mathcal{E}^\vee) \otimes \mathcal{L}, \mathcal{L}, \mathrm{F}(\beta^\vee)^{-1} \otimes \beta', \beta')$$

where

$$\mathrm{F}(\beta^\vee)^{-1} \otimes \beta' \colon (\mathrm{F}(\mathcal{E}^\vee) \otimes \mathcal{L})|_{D_R^*} \simeq (\mathrm{F}((\mathcal{E}^0)^\vee) \otimes \mathcal{L}^0)|_{D_R^*} = \mathcal{E}^0|_{D_R^*}.$$

We regard $\mathrm{Gr}_{\mathrm{GL}_n}$ as an open and closed sub-ind-scheme of $\mathrm{Gr}_{\mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}}$ by

$$(\mathcal{E}, \beta) \mapsto (\mathcal{E}, \mathcal{L}^0, \beta, \mathrm{id}).$$

If $\lambda \in \mathbb{X}_\bullet(T)$ is trivial on $\mathbb{G}_{\mathrm{m}}$-component under the identification (7.1), then we view $\mathrm{Gr}_{G,\lambda}$ as a subscheme of $\mathrm{Gr}_{\mathrm{GL}_n} \subset \mathrm{Gr}_{\mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}}$ under the identification (8.1). Under the identification by (8.1), the Frobenius endomorphism of $\mathrm{Gr}_G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ becomes

$$(\mathcal{E}, \beta) \mapsto (\mathrm{F}(\mathcal{E}^\vee), \mathrm{F}(\beta^\vee)^{-1})$$

in $\mathrm{Gr}_{\mathrm{GL}_n}$. We put $\mu_{\mathrm{GL}} = \varepsilon_1 + \varepsilon_2$.

8.1 Component for ν_1

The following proposition describes an analogue of a component studied in [VW11].

Proposition 8.2. *The irreducible component $X_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}^0$ bounded by ν_1^* such that $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}$. In particular, it is isomorphic to $X_{I_n^1}([1])^{\mathrm{pf}}$.*

Proof. We have

$$\mathrm{Gr}^{0,\mathbf{a}_1}_{(\nu_1^*, \mu^*)|\nu_1 + \tau_1^*} = \mathrm{Gr}^{0}_{(\nu_1^*, \mu^*)|\nu_1 + \tau_1^*} = \mathring{\mathrm{Gr}}^{0}_{(\nu_1^*, \mu^*)|\nu_1 + \tau_1^*}$$

since ν_1 is minuscule. Hence we have $X_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*) = \mathring{X}_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*) = X_{\mu^*, \nu_1^*}(\tau_1^*)$. By the definition, $X_{\mu^*, \nu_1^*}(\tau_1^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}^0$ bounded by ν_1^* such that $F(\mathcal{E}^\vee) \dashrightarrow \mathcal{E}$ is bounded by μ_{GL}^*. The condition that $F(\mathcal{E}^\vee) \dashrightarrow \mathcal{E}$ is bounded by μ_{GL}^* is equivalent to $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}$.

The last isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}^\vee / \mathcal{E}^0 \subset \frac{1}{\varpi} \mathcal{E}^0 / \mathcal{E}^0$. □

By Proposition 8.2, $X_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*)$ is isomorphic to the perfection of the Fermat hypersurface defined by

$$\sum_{i=1}^{n} x_i x_{n+1-i}^q = 0$$

in $\mathbb{P}^{n-1}$.

8.2 Components for ν_r when n is even.

The following proposition describes a generalization of a component studied in [HP14].

Proposition 8.3. *The irreducible component $X_{\mu^*}^{\mathbf{b}_r, x_0}(\tau_r^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}^0$ bounded by ν_r^* such that $\varpi \mathcal{E} \subset F(\mathcal{E}^\vee)$. In particular, it is isomorphic to $X_{I_n^{r-1}}([1])^{\mathrm{pf}}$.*

Proof. We have

$$\mathrm{Gr}^{0,\mathbf{a}_r}_{(\nu_r^*,\mu^*)|\nu_r+\tau_r^*} = \mathrm{Gr}^{0}_{(\nu_r^*,\mu^*)|\nu_r+\tau_r^*} = \mathring{\mathrm{Gr}}^{0}_{(\nu_r^*,\mu^*)|\nu_r+\tau_r^*}$$

since ν_r^* and $\nu_r+\tau_r^*$ are minuscule. Hence we have $X_{\mu^*}^{\mathbf{b}_r,x_0}(\tau_r^*) = \mathring{X}_{\mu^*}^{\mathbf{b}_r,x_0}(\tau_r^*) = X_{\mu^*,\nu_r^*}(\tau_r^*)$. It is parametrized by $\mathcal{E}_+ \dashrightarrow \mathcal{E}^0$ bounded by ν_r^* such that $\varpi\mathcal{E}_+ \subset F(\mathcal{E}_+^\vee)$.

By the definition, $X_{\mu^*,\nu_r^*}(\tau_r^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}^0$ bounded by ν_r^* such that $\frac{1}{\varpi}F(\mathcal{E}^\vee) \dashrightarrow \mathcal{E}$ is bounded by μ^*_{GL}. The condition that $\frac{1}{\varpi}F(\mathcal{E}^\vee) \dashrightarrow \mathcal{E}$ is bounded by μ^*_{GL} is equivalent to $\varpi\mathcal{E} \subset F(\mathcal{E}^\vee)$.

The last isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}/\mathcal{E}^0 \subset \frac{1}{\varpi}\mathcal{E}^0/\mathcal{E}^0$. □

Example 8.4. *Assume that $n=4$. Then $X_{\mu^*}^{\mathbf{b}_r,x_0}(\tau_r^*)$ is isomorphic to the perfection of the Fermat hypersurface defined by*

$$x_1x_4^q + x_2x_3^q + x_3x_2^q + x_4x_1^q = 0$$

in $\mathbb{P}^3$. This is a component which appears in [HP14, p.1689].

8.3 Non-minuscule case

Let $2 \leq i \leq [(n-1)/2]$. We put

$$\nu_{i,+} = \varepsilon_1 + \cdots + \varepsilon_{i-1}, \quad \nu_{i,-} = -\varepsilon_{i^\vee} - \cdots - \varepsilon_{1^\vee}.$$

We put $\xi_i = \varepsilon_1 + \cdots + \varepsilon_{2i-1}$. Let $(\mathrm{Gr}_{\nu_{i,+}^*} \times \mathrm{Gr}_{\nu_{i,-}^*})_{\xi_i}$ be the subspace of $\mathrm{Gr}_{\nu_{i,+}^*} \times \mathrm{Gr}_{\nu_{i,-}^*}$ defined by the condition that

$$\mathcal{E}_- \overset{\beta_-}{\dashrightarrow} \mathcal{E}^0 \overset{\beta_+^{-1}}{\dashrightarrow} \mathcal{E}_+$$

is bounded by ξ_i for a point $(\mathcal{E}_+ \overset{\beta_+}{\dashrightarrow} \mathcal{E}^0, \mathcal{E}_- \overset{\beta_-}{\dashrightarrow} \mathcal{E}^0)$ of $\mathrm{Gr}_{\nu_{i,+}^*} \times \mathrm{Gr}_{\nu_{i,-}^*}$. Let

$$(\mathrm{Gr}_{(\nu_{i,+}^*,\nu_{i,-}^*)} \times_{\mathrm{Gr}_{\nu_i^*}} \mathrm{Gr}_{(\nu_{i,-}^*,\nu_{i,+}^*)})_{\xi_i}$$

be the subspace of $\mathrm{Gr}_{(\nu_{i,+}^*,\nu_{i,-}^*)} \times_{\mathrm{Gr}_{\nu_i^*}} \mathrm{Gr}_{(\nu_{i,-}^*,\nu_{i,+}^*)}$ defined by the condition that

$$\mathcal{E}_- \overset{\beta_-}{\dashrightarrow} \mathcal{E}^0 \overset{\beta_+^{-1}}{\dashrightarrow} \mathcal{E}_+$$

is bounded by ξ_i for a point

$$(\mathcal{E} \overset{\beta'_+}{\dashrightarrow} \mathcal{E}_+ \overset{\beta_+}{\dashrightarrow} \mathcal{E}^0, \mathcal{E} \overset{\beta'_-}{\dashrightarrow} \mathcal{E}_- \overset{\beta_-}{\dashrightarrow} \mathcal{E}^0)$$

of $\mathrm{Gr}_{(\nu_{i,+}^*,\nu_{i,-}^*)} \times_{\mathrm{Gr}_{\nu_i^*}} \mathrm{Gr}_{(\nu_{i,-}^*,\nu_{i,+}^*)}$. We have a natural morphism

$$\pi_1 \colon (\mathrm{Gr}_{(\nu_{i,+}^*,\nu_{i,-}^*)} \times_{\mathrm{Gr}_{\nu_i^*}} \mathrm{Gr}_{(\nu_{i,-}^*,\nu_{i,+}^*)})_{\xi_i} \to (\mathrm{Gr}_{\nu_{i,+}^*} \times \mathrm{Gr}_{\nu_{i,-}^*})_{\xi_i}.$$

Let $\mathscr{V}_i$ be the vector bundle of rank $2i-1$ over $(\mathrm{Gr}_{\nu_{i,+}^*} \times \mathrm{Gr}_{\nu_{i,-}^*})_{\xi_i}$ defined by $\mathcal{E}_+/\mathcal{E}_-$, where $(\mathcal{E}_+,\mathcal{E}_-)$ is a point of $(\mathrm{Gr}_{\nu_{i,+}^*} \times \mathrm{Gr}_{\nu_{i,-}^*})_{\xi_i}$. We put $\mathscr{G}_i = \mathrm{Aut}(\mathscr{V}_i)$. Let $t_{i,\mathbb{Z}} \in \mathrm{Oc}(\mathrm{Dyn}(\mathrm{GL}_{2i-1,\mathbb{Z}}))(\mathbb{Z})$ be the image under

$$\mathbf{t}(\mathbb{Z}) \colon \mathrm{Par}(\mathrm{GL}_{2i-1,\mathbb{Z}})(\mathbb{Z}) \to \mathrm{Oc}(\mathrm{Dyn}(\mathrm{GL}_{2i-1,\mathbb{Z}}))(\mathbb{Z})$$

of the parabolic subgroup of $\mathrm{GL}_{2i-1,\mathbb{Z}}$ defined as the stabilizer of $\mathbb{Z}^{i-1} \subset \mathbb{Z}^{i-1} \oplus \mathbb{Z}^{i} = \mathbb{Z}^{2i-1}$. Let

$$t_i \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}_i))((\mathrm{Gr}_{\nu^*_{i,+}} \times \mathrm{Gr}_{\nu^*_{i,-}})_{\xi_i}) \tag{8.3}$$

be the element determined from $t_{i,\mathbb{Z}}$ by Remark 2.5 and Lemma 2.6 (2).

We define a morphism

$$\Psi\colon (\mathrm{Gr}_{(\nu^*_{i,+},\nu^*_{i,-})} \times_{\mathrm{Gr}_{\nu^*_i}} \mathrm{Gr}_{(\nu^*_{i,-},\nu^*_{i,+})})_{\xi_i} \to \mathrm{Par}_{t_i}(\mathscr{G}_i)$$

by sending

$$(\mathcal{E} \overset{\beta'_+}{\dashrightarrow} \mathcal{E}_+ \overset{\beta_+}{\dashrightarrow} \mathcal{E}^0, \mathcal{E} \overset{\beta'_-}{\dashrightarrow} \mathcal{E}_- \overset{\beta_-}{\dashrightarrow} \mathcal{E}^0)$$

to the stabilizer of $\mathcal{E}/\mathcal{E}_- \subset \mathcal{E}_+/\mathcal{E}_-$. Then Ψ is an isomorphism. Note that a natural morphism

$$\pi_0\colon (\mathrm{Gr}_{(\nu^*_{i,+},\nu^*_{i,-})} \times_{\mathrm{Gr}_{\nu^*_i}} \mathrm{Gr}_{(\nu^*_{i,-},\nu^*_{i,+})})_{\xi_i} \to \mathrm{Gr}_{\nu^*_i}$$

is an isomorphism over $\mathring{\mathrm{Gr}}_{\nu^*_i}$.

Recall that $X^{\mathbf{a}_i}_{\mu^*,\nu^*_i}(\tau^*_i)$ and $X_{\mu^*,\nu^*_i}(\tau^*_i)$ are closed subspaces of $\mathrm{Gr}_{\nu^*_i}$. The condition for the subspace

$$\pi_0^{-1}(X_{\mu^*,\nu^*_i}(\tau^*_i)) \subset (\mathrm{Gr}_{(\nu^*_{i,+},\nu^*_{i,-})} \times_{\mathrm{Gr}_{\nu^*_i}} \mathrm{Gr}_{(\nu^*_{i,-},\nu^*_{i,+})})_{\xi_i}$$

is that $\mathcal{E} \subset F(\mathcal{E}^\vee) \subset \frac{1}{\varpi}\mathcal{E}$.

For a point $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ of $(\mathrm{Gr}_{(\nu^*_{i,+},\nu^*_{i,-})} \times_{\mathrm{Gr}_{\nu^*_i}} \mathrm{Gr}_{(\nu^*_{i,-},\nu^*_{i,+})})_{\xi_i}$, we put $\mathscr{W} = \mathcal{E}/\mathcal{E}_- \subset \mathcal{E}_+/\mathcal{E}_-$, which is a subvector bundle of rank $i-1$. Let $\mathscr{W}^\perp \subset \mathcal{E}_-^\vee/\mathcal{E}_+^\vee$ be the annihilator of $\mathscr{W}$. Then we have $\mathscr{W}^\perp = \mathcal{E}^\vee/\mathcal{E}_+^\vee$.

Let Y_i be the closed subscheme of $(\mathrm{Gr}_{\nu^*_{i,+}} \times \mathrm{Gr}_{\nu^*_{i,-}})_{\xi_i}$ defined by the conditions

(1) $\mathcal{E}_+ \subset F(\mathcal{E}_-^\vee)$,

(2) $\mathcal{E}_- \subset F(\mathcal{E}_+^\vee)$,

(3) $\varpi F(\mathcal{E}_-^\vee) \subset \mathcal{E}_-$.

Then we have $Y_i = X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}$ under the identification given by sending $(\mathcal{E}_+, \mathcal{E}_-)$ to

$$0 \subset \varpi\mathcal{E}_+/\varpi\mathcal{E}^0 \subset \mathcal{E}_-/\varpi\mathcal{E}^0 \subset \mathcal{E}^0/\varpi\mathcal{E}^0.$$

Assume that $(\mathcal{E}_+, \mathcal{E}_-)$ is a point of Y_i. The condition $\mathcal{E} \subset F(\mathcal{E}^\vee)$ is equivalent to that the image of $\mathscr{W}$ under the natural morphism

$$\phi_1\colon \mathcal{E}_+/\mathcal{E}_- \to F(\mathcal{E}_-^\vee)/F(\mathcal{E}_+^\vee) = F(\mathcal{E}_-^\vee/\mathcal{E}_+^\vee) \tag{8.4}$$

is contained in $F(\mathscr{W}^\perp)$. The condition $F(\mathcal{E}^\vee) \subset \frac{1}{\varpi}\mathcal{E}$ is equivalent to that the image of $F(\varpi\mathscr{W}^\perp)$ under the natural morphism

$$\phi_2\colon \mathcal{E}^0/\varpi\mathcal{E}^0 \to \mathcal{E}_+/\mathcal{E}_-$$

is contained in $\mathscr{W}$. We put

$$X_i = \pi_0^{-1}(X_{\mu^*,\nu^*_i}(\tau^*_i)) \cap \pi_1^{-1}(Y_i).$$

Then X_i is the subscheme of $\pi_1^{-1}(Y_i)$ cut out by the conditions

$$\phi_1(\mathscr{W}) \subset F(\mathscr{W}^{\perp}), \tag{8.5}$$
$$\phi_2(F(\varpi\mathscr{W}^{\perp})) \subset \mathscr{W}. \tag{8.6}$$

Let π_0' and π_1' be the restrictions of π_0 and π_1 to X_i respectively. We have

$$\begin{array}{ccccc} \pi_0^{-1}(X_{\mu^*,\nu_i^*}(\tau_i^*)) & \longleftarrow & X_i & \longrightarrow & \pi_1^{-1}(Y_i) \\ \downarrow{\scriptstyle \pi_0} & \swarrow{\scriptstyle \pi_0'} & & \searrow{\scriptstyle \pi_1'} & \downarrow{\scriptstyle \pi_1} \\ X_{\mu^*,\nu_i^*}(\tau_i^*) & & & & Y_i. \end{array}$$

We note that π_0 and π_0' are isomorphisms over $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$.

Lemma 8.5. *The inverse image* $\pi_0^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*))$ *is contained in* X_i.

Proof. Let $(\mathcal{E},\mathcal{E}_+,\mathcal{E}_-)$ be a point of $\pi_0^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*))$. Then we have $\mathcal{E}_- = \mathcal{E}\cap\mathcal{E}^0$. By the condition $F(\mathcal{E}^{\vee}) \subset \frac{1}{\varpi}\mathcal{E}$, we have

$$\varpi F(\mathcal{E}_-^{\vee}) = \varpi F((\mathcal{E}\cap\mathcal{E}^0)^{\vee}) = \varpi(F(\mathcal{E}^{\vee})+\mathcal{E}^0) \subset \mathcal{E}.$$

Hence we have

$$\varpi F(\mathcal{E}_-^{\vee}) \subset \mathcal{E}\cap\mathcal{E}^0 = \mathcal{E}_-.$$

This means that $(\mathcal{E}_+,\mathcal{E}_-)$ is a point of Y_i. □

Let $\mathscr{G}_{Y_i}$ denote the restriction of $\mathscr{G}_i$ to Y_i. We have an isomorphism

$$\Psi_{Y_i}\colon \pi_1^{-1}(Y_i) \simeq \mathrm{Par}_{t_i}(\mathscr{G}_{Y_i}) \tag{8.7}$$

induced by Ψ.

Theorem 8.6. *The closed subscheme* $X_i \subset \pi_1^{-1}(Y_i) \simeq \mathrm{Par}_{t_i}(\mathscr{G}_{Y_i})$ *is defined by the condition* $\phi_1(\mathscr{W}) \subset F(\mathscr{W}^{\perp})$.

Proof. It suffices to show that the condition (8.6) is automatic. The condition (8.6) is equivalent to $\varpi F(\mathcal{E}^{\vee}) \subset \mathcal{E}$ under (8.7). Let $(\mathcal{E},\mathcal{E}_+,\mathcal{E}_-)$ be a point of $\pi_1^{-1}(Y_i)$. Then we have

$$\varpi F^{-1}(\mathcal{E}_-^{\vee}) \subset \mathcal{E}_- \subset \mathcal{E}.$$

By taking the dual, we have $\varpi F(\mathcal{E}^{\vee}) \subset \mathcal{E}_-$. Hence the condition $\varpi F(\mathcal{E}^{\vee}) \subset \mathcal{E}$ is satisfied. □

Proposition 8.7. *The scheme* $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ *is isomorphic to the subscheme of* $\pi_1^{-1}(Y_i)$ *defined by the condition* $\mathcal{E} \subset F(\mathcal{E}^{\vee})$ *and* $\mathcal{E}\cap\mathcal{E}^0 = \mathcal{E}_-$.

Proof. Let $\mathring{X}_i$ and $\mathring{X}_{\mu^*,\nu_i^*}(\tau_i^*)$ be the inverse images of $\mathring{\mathrm{Gr}}_{\nu_i^*}$ in X_i and $X_{\mu^*,\nu_i^*}(\tau_i^*)$. By Theorem 8.6, $\mathring{X}_i$ is equal to the subscheme of $\pi_1^{-1}(Y_i)$ defined by the condition $\mathcal{E} \subset F(\mathcal{E}^{\vee})$ and $\mathcal{E}\cap\mathcal{E}^0 = \mathcal{E}_-$. The natural morphism $\pi_0'^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)) \to \mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ is an isomorphism. Hence it suffices to show that $\pi_0'^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)) = \mathring{X}_i$.

By Lemma 8.5, we have $\pi_0'^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)) \subset \mathring{X}_i$. On the other hand, $\mathring{X}_i$ is contained in $\pi_0^{-1}(\mathring{X}_{\mu^*}^{b_i,x_0}(\tau_i^*))$, since $\mathring{X}_{\mu^*,\nu_i^*}(\tau_i^*) = \mathring{X}_{\mu^*}^{b_i,x_0}(\tau_i^*)$ by Lemma 7.2. □

9 Intersections

Let $x, x' \in J_\tau(F)/J_\tau(\mathcal{O})$. Let Λ_x and $\Lambda_{x'}$ be the lattices of V determined by x and x'. We put

$$l_{x,x'} = \mathrm{length}_{\mathcal{O}}\left(\Lambda_x/(\Lambda_x \cap \Lambda_{x'})\right).$$

Let $\mathcal{E}_x$ and $\mathcal{E}_{x'}$ be the modifications of $\mathcal{E}^0$ corresponding to Λ_x and $\Lambda_{x'}$. Let $\mathbf{P}_{x,x'}$ be the parabolic subgroup of $\mathbf{G}$ that is the stabilizer of the filtration

$$\varpi\Lambda_x \subset \varpi^2\Lambda_{x'} + \varpi\Lambda_x \subset (\Lambda_x \cap \varpi\Lambda_{x'}) + \varpi\Lambda_x \subset (\Lambda_x \cap \Lambda_{x'}) + \varpi\Lambda_x \subset \Lambda_x.$$

We note that $\varpi\Lambda_x \subset \Lambda_{x'}$ if and only if $\varpi\Lambda_{x'} \subset \Lambda_x$ by taking dual with respect to the hermitian pairing. We put

$$d_1 = \dim((\varpi^2\Lambda_{x'} + \varpi\Lambda_x)/\varpi\Lambda_x),$$
$$d_2 = \dim(((\Lambda_x \cap \varpi\Lambda_{x'}) + \varpi\Lambda_x)/\varpi\Lambda_x).$$

We note that $d_1 + d_2 = l_{x,x'}$. In the identification (6.1) for $\mathbf{b}_i$, we use $\delta_{\lambda_i} = -\nu_i^*$ if $1 \le i \le r-1$ and $\delta_{\lambda_r} = -\varepsilon_{r+1}$ if $i = r$.

9.1 Intersection of components for ν_i and $\nu_{i'}$, where $i, i' \neq r$ if n is even.

9.1.1 Different hyperspecial subgroups

We assume that $x \neq x'$. For a subscheme X of $X_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*)$, let $X_{\mathbf{P}_{x,x'},[w]}$ be the inverse image of $X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w]}$ under $X \hookrightarrow X_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*) \to X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}$.

We recall that

$$\mathring{X}_{\mu^*}^{\mathbf{b}_{i'},x'}(\tau_{i'}^*) = X_{\mu^*,\nu_{i'}^*}^{\mathbf{a}_{i'},x'}(\tau_{i'}^*) \setminus X_{\mu^*}^{\mathbf{b}_{i'-1},x'}(\tau_{i'-1}^*).$$

Assume that $i \le i'$. For $j_1, j_2 \in \mathbb{N}$ such that $i-1-d_2 \le j_1 \le i-1$ and $d_2 - i \le j_2 \le n-i-d_2-j_1$, we define $w_{j_1,j_2} \in S_n$ by

$$w_{j_1,j_2}(j) = \begin{cases} j + j_1 & \text{if } i - j_1 \le j \le d_2, \\ j + i - j_1 - d_2 - 1 & \text{if } d_2 + 1 \le j \le d_2 + j_1, \\ j + j_2 & \text{if } n - j_2 - i + 1 \le j \le n - d_2, \\ j + d_2 - i - j_2 & \text{if } n - d_2 + 1 \le j \le n - d_2 + j_2, \\ j & \text{otherwise.} \end{cases}$$

We put $\mathcal{E}_{x,x'}^+ = (\mathcal{E}_x + \mathcal{E}_{x'}) \cap \frac{1}{\varpi}\mathcal{E}_x$ and $\mathcal{E}_{x,x'}^- = (\mathcal{E}_x \cap \mathcal{E}_{x'}) + \varpi\mathcal{E}_x$. Let $\mathcal{E}_+$ and $\mathcal{E}_-$ be the universal vector bundles on $X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$. We note that

$$\mathrm{length}((\mathcal{E}_+ + \mathcal{E}_{x,x'}^+)/\mathcal{E}_{x,x'}^+) = j_1, \quad \mathrm{length}((\mathcal{E}_- + \mathcal{E}_{x,x'}^-)/\mathcal{E}_{x,x'}^-) = j_2.$$

We put $\mathcal{E}_{+,-} = \mathcal{E}_+ \cap (\mathcal{E}_- + \mathcal{E}_{x'})$ and $d_{j_1,j_2} = j_2 - j_1 + 2i - 1 - d_2$. We note that

$$\mathcal{E}_{+,-} = \mathcal{E}_- + \mathcal{E}_+ \cap \mathcal{E}_{x'} \subset F(\mathcal{E}_-^\vee) \cap (F(\mathcal{E}_+^\vee) + \mathcal{E}_{x'}) = F(\mathcal{E}_{+,-}^\vee)$$

using $\mathcal{E}_+ \subset F(\mathcal{E}_-^\vee)$ and $\mathcal{E}_- \subset F(\mathcal{E}_+^\vee)$.

Lemma 9.1. *We have* $\mathrm{length}(\mathcal{E}_{+,-}/\mathcal{E}_{-}) = d_{j_1,j_2}$. *Further* $\mathcal{E}_{+,-}/\mathcal{E}_{-}$ *is a vector bundle on* $X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$.

Proof. We have

$$\begin{aligned}\mathrm{length}(\mathcal{E}_{+}/\mathcal{E}_{+,-}) &= \mathrm{length}((\mathcal{E}_{+}+\mathcal{E}_{x'})/(\mathcal{E}_{-}+\mathcal{E}_{x'}))\\ &= \mathrm{length}((\mathcal{E}_{+}+\mathcal{E}_{x'})/(\varpi\mathcal{E}_{x}+\mathcal{E}_{x'})) - \mathrm{length}((\mathcal{E}_{-}+\mathcal{E}_{x'})/(\varpi\mathcal{E}_{x}+\mathcal{E}_{x'}))\\ &= \mathrm{length}((\mathcal{E}_{+}+\mathcal{E}_{x'})/(\mathcal{E}_{x}+\mathcal{E}_{x'})) + \mathrm{length}((\mathcal{E}_{x}+\mathcal{E}_{x'})/(\varpi\mathcal{E}_{x}+\mathcal{E}_{x'})) - j_2\\ &= j_1 + \mathrm{length}((\mathcal{E}_{x}\cap\varpi\mathcal{E}_{x'})/(\varpi\mathcal{E}_{x}\cap\varpi\mathcal{E}_{x'})) - j_2 = j_1 + d_2 - j_2.\end{aligned}$$

Hence we obtain the first claim. By the above equalities, $\mathrm{length}(\mathcal{E}_{+}/\mathcal{E}_{+,-})$ is constant on $X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$. Hence $\mathcal{E}_{+}/\mathcal{E}_{+,-}$ is a vector bundle on $X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$ by [BS17, Lemma 7.3]. Therefore $\mathcal{E}_{+,-}/\mathcal{E}_{-}$ is also a vector bundle. □

Let $\mathscr{G}_{j_1,j_2}$ be the restriction of $\mathscr{G}$ to $X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$. Let

$$t_{j_1,j_2} \in \mathrm{Oc}(\mathrm{Dyn}(\mathscr{G}_{j_1,j_2}))(X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]})$$

denote the restriction of t_i in (8.3). Let $\mathscr{P}_{j_1,j_2}$ be the parabolic subgroup of $\mathscr{G}_{j_1,j_2}$ determined by

$$\mathcal{E}_{-} \subset \mathcal{E}_{+} \cap (\mathcal{E}_{-}+\mathcal{E}_{x'}) \subset \mathcal{E}_{+}.$$

We put $l_{j_1,j_2} = i' - 1 - j_2 - d_1$. We define $s_{j_1,j_2} \in S_{2i-1}$ by

$$s_{j_1,j_2}(j) = \begin{cases} j + l_{j_1,j_2} & \text{if } i - l_{j_1,j_2} \le j \le d_{j_1,j_2},\\ j + i - 1 - d_{j_1,j_2} - l_{j_1,j_2} & \text{if } d_{j_1,j_2}+1 \le j \le d_{j_1,j_2} + l_{j_1,j_2},\\ j & \text{otherwise.}\end{cases}$$

Let r_{j_1,j_2} be the element of

$$(q^{-1}_{\mathscr{G}_{j_1,j_2}}(\mathbf{t}(\mathscr{P}_{j_1,j_2}), t_{j_1,j_2}))(X_{I_n^{i-1,n-i}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]})$$

corresponding to $[s_{j_1,j_2}]$ by Lemma 2.6 (2).

Proposition 9.2. *Assume that* $\mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*)$ *is not empty. Then we have* $1 \le l_{x,x'} \le i + i' - 1$.

The subscheme $\mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*) \subset \mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*)$ *is the locus defined by the condition that* $\mathcal{E}_x + \varpi\mathcal{E}_{x'} \subset \mathcal{E}_{+} \subset \frac{1}{\varpi}\mathcal{E}^{-}_{x,x'}$, $\varpi\mathcal{E}^{+}_{x,x'} \subset \mathcal{E}_{-} \subset \mathcal{E}_x \cap \frac{1}{\varpi}\mathcal{E}_{x'}$,

$$\mathrm{length}((\mathcal{E}_{-}+\mathcal{E}^{-}_{x,x'})/\mathcal{E}^{-}_{x,x'}) \le [(i'-i+d_2-d_1)/2]$$

and

$$\mathrm{length}((\mathcal{E}+\mathcal{E}_{+,-})/\mathcal{E}_{+,-}) + \mathrm{length}((\mathcal{E}_{-}+\mathcal{E}^{-}_{x,x'})/\mathcal{E}^{-}_{x,x'}) = i' - 1 - d_1.$$

In particular, $\mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*)$ *is the union of* $\left(\mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*)\right)_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$ *for* $j_1, j_2 \in \mathbb{N}$ *such that* $j_1 + d_2 - i \le j_2 \le j_1 + d_2 - i + 1$,

$$\begin{aligned}&i - 1 - d_2 \le j_1 \le i - 1 - d_1,\\ &i' - i - d_1 \le j_2 \le \min\{[(i'-i+d_2-d_1)/2], n - i - d_2 - j_1\}.\end{aligned}$$

Further we have

$$\left(\mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*)\right)_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]} = \mathring{X}^{\mathbf{b}_i,x}_{\mu^*}(\tau_i^*)_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]} \cap \mathrm{Par}_{t_{j_1,j_2}}(\mathscr{G}_{j_1,j_2};\mathscr{P}_{j_1,j_2})^{\mathrm{pf}}_{r_{j_1,j_2}}.$$

Proof. The intersection $\mathring{X}^{\mathbf{b}_{i},x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}_x$ which is equal to ν_i^* such that $\mathcal{E} \subset F(\mathcal{E}^\vee) \subset \frac{1}{\varpi}\mathcal{E}$ and $\mathcal{E} \dashrightarrow \mathcal{E}_{x'}$ is equal to $\nu_{i'}^*$. Let $\mathcal{E}$ be a point of $\mathring{X}^{\mathbf{b}_{i},x}_{\mu^*}(\tau_i^*) \cap \mathring{X}^{\mathbf{b}_{i'},x'}_{\mu^*}(\tau_{i'}^*)$. We put

$$\mathcal{E}_+ = \mathcal{E} + \mathcal{E}_x, \quad \mathcal{E}_- = \mathcal{E} \cap \mathcal{E}_x, \quad \mathcal{E}'_+ = \mathcal{E} + \mathcal{E}_{x'}, \quad \mathcal{E}'_- = \mathcal{E} \cap \mathcal{E}_{x'}.$$

Then we have

$$\begin{aligned} &\mathrm{length}(\mathcal{E}_x/\mathcal{E}_-) = i, \quad \mathrm{length}(\mathcal{E}/\mathcal{E}_-) = i-1, \\ &\mathrm{length}(\mathcal{E}_{x'}/\mathcal{E}'_-) = i', \quad \mathrm{length}(\mathcal{E}/\mathcal{E}'_-) = i'-1. \end{aligned}$$

Hence we have $\mathrm{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}'_-)) \leq i'-1$ and $\mathrm{length}(\mathcal{E}_x/(\mathcal{E}_- \cap \mathcal{E}'_-)) \leq i+i'-1$. Therefore the inclusion $\mathcal{E}_- \cap \mathcal{E}'_- \subset \mathcal{E}_x \cap \mathcal{E}_{x'}$ implies that

$$1 \leq l_{x,x'} \leq i+i'-1.$$

We have $\mathcal{E}_x + \varpi\mathcal{E}_{x'} \subset \mathcal{E}_+$ and $\varpi\mathcal{E}^+_{x,x'} = \varpi\mathcal{E}_x + (\mathcal{E}_x \cap \varpi\mathcal{E}_{x'}) \subset \mathcal{E}_-$, since $\varpi\mathcal{E}_{x'} \subset \mathcal{E}$. We have $\varpi\mathcal{E}_+ \subset \mathcal{E}_x \cap (\mathcal{E}_{x'} + \varpi\mathcal{E}_x) = \mathcal{E}^-_{x,x'}$ and $\mathcal{E}_- \subset \mathcal{E}_x \cap \frac{1}{\varpi}\mathcal{E}_{x'}$, since $\varpi\mathcal{E}_+ \subset \mathcal{E}_x$ and $\varpi\mathcal{E} \subset \mathcal{E}_{x'}$.

We put $j_1 = \mathrm{length}((\mathcal{E}_+ + \mathcal{E}^+_{x,x'})/\mathcal{E}^+_{x,x'})$ and $j_2 = \mathrm{length}((\mathcal{E}_- + \mathcal{E}^-_{x,x'})/\mathcal{E}^-_{x,x'})$. We have

$$\begin{aligned} \mathrm{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}'_-)) = \mathrm{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}_{x'})) &= j_2 + \mathrm{length}((\mathcal{E}_- \cap \mathcal{E}^-_{x,x'})/(\mathcal{E}_- \cap \mathcal{E}_{x'})) \\ &= j_2 + \mathrm{length}(\mathcal{E}^-_{x,x'}/(\mathcal{E}_x \cap \mathcal{E}_{x'})) = j_2 + d_1. \end{aligned}$$

We have

$$j_2 + i - d_2 = \mathrm{length}((\mathcal{E}_- + \mathcal{E}^-_{x,x'})/\mathcal{E}_-) \leq 1 + \mathrm{length}((F(\mathcal{E}_+^\vee) + \mathcal{E}^-_{x,x'})/F(\mathcal{E}_+^\vee))$$

since $\mathrm{length}(F(\mathcal{E}_+^\vee)/\mathcal{E}_-) = 1$. Further we have

$$\begin{aligned} \mathrm{length}((F(\mathcal{E}_+^\vee) + \mathcal{E}^-_{x,x'})/F(\mathcal{E}_+^\vee)) &= \mathrm{length}(\mathcal{E}_+/(\mathcal{E}_+ \cap \mathcal{E}^+_{x,x'})) \leq \mathrm{length}(\mathcal{E}_+/(\mathcal{E}'_- + \mathcal{E}_x)) \\ &\leq \mathrm{length}(\mathcal{E}/(\mathcal{E}_- + \mathcal{E}'_-)) = i'-1-\mathrm{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}'_-)) = i'-1-j_2-d_1. \end{aligned}$$

Therefore we obtain $j_2 \leq [(i'-i+d_2-d_1)/2]$.

Further, $j_1 + d_2 - i \leq j_2 \leq j_1 + d_2 - i + 1$ follows from $\mathrm{length}(F(\mathcal{E}_-^\vee)/\mathcal{E}_+) = 1$. This implies $i-1-d_2 \leq j_1$ and $d_2 - i \leq j_2$. We have $j_1 \leq i-1-d_1$ and $j_2 \leq n-i-d_2-j_1$ by the inclusions $\mathcal{E}_x + \varpi\mathcal{E}_{x'} \subset \mathcal{E}_+ \cap \mathcal{E}^+_{x,x'}$ and $\mathcal{E}_+ + \mathcal{E}^+_{x,x'} \subset \mathcal{E}_- \cap \mathcal{E}^-_{x,x'}$. The equality

$$\mathrm{length}((\mathcal{E} + \mathcal{E}_{+,-})/\mathcal{E}_{+,-}) + \mathrm{length}((\mathcal{E}_- + \mathcal{E}^-_{x,x'})/\mathcal{E}^-_{x,x'}) = i'-1-d_1.$$

and $\mathrm{length}((\mathcal{E} + \mathcal{E}_{+,-})/\mathcal{E}_{+,-}) \leq i-1$ imply $j_2 \geq i'-i-d_1$.

We have

$$\begin{aligned} \mathrm{length}((\mathcal{E} + \mathcal{E}_{+,-})/\mathcal{E}_{+,-}) &= \mathrm{length}(\mathcal{E}/(\mathcal{E} \cap \mathcal{E}_{+,-})) = \mathrm{length}(\mathcal{E}/(\mathcal{E} \cap (\mathcal{E}_- + \mathcal{E}_{x'}))) \\ &= \mathrm{length}((\mathcal{E} + \mathcal{E}_{x'})/(\mathcal{E}_- + \mathcal{E}_{x'})) = \mathrm{length}((\mathcal{E} + \mathcal{E}_{x'})/\mathcal{E}_{x'}) - \mathrm{length}((\mathcal{E}_- + \mathcal{E}_{x'})/\mathcal{E}_{x'}) \\ &= \mathrm{length}((\mathcal{E} + \mathcal{E}_{x'})/\mathcal{E}_{x'}) - \mathrm{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}_{x'})). \end{aligned}$$

Hence, $\mathrm{length}((\mathcal{E}+\mathcal{E}_{+,-})/\mathcal{E}_{+,-}) = i'-1-j_2-d_1$ if and only if $\mathrm{length}((\mathcal{E}+\mathcal{E}_{x'})/\mathcal{E}_{x'}) = i'-1$. This implies the last claim. □

9.1.2 Same hyperspecial subgroup

Assume that $x = x'$. It suffices to consider the case where $x = x' = x_0$, since all the hyperspecial subgroups are conjugate.

Let $2 \leq i \leq [(n-1)/2]$. Let $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ be a point of X_i. Let s be the rank of $(\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$. We put $\mathcal{V}_1 = \mathcal{E}/\mathcal{E}_-$ and take $\mathcal{V}_2 \subset \mathcal{E}^0/\mathcal{E}_-$ and $\mathcal{V}_3 \subset \mathcal{E}_+/\mathcal{E}_-$ such that projections induce isomorphisms $\mathcal{V}_2 \simeq (\mathcal{E} + \mathcal{E}^0)/\mathcal{E}$ and $\mathcal{V}_3 \simeq \mathcal{E}_+/(\mathcal{E} + \mathcal{E}^0)$. An open neighbourhood of $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ in $\mathrm{Gr}(i-1, \mathscr{V}_{Y_i})$ under (8.7) is given by $\mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2 \oplus \mathcal{V}_3)$ sending $f \in \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2 \oplus \mathcal{V}_3)$ to the inverse image $\mathcal{E}_f$ of

$$\{v + f(v) \mid v \in \mathcal{V}_1\} \subset \mathcal{E}_+/\mathcal{E}_-$$

in $\mathcal{E}_+$. By Theorem 8.6, the condition that $\mathcal{E}_f$ belongs to X_i is equivalent to

$$\langle v + f(v), F(v' + f(v')) \rangle = 0 \tag{9.1}$$

in $\varpi^{-1} W_{\mathcal{O}}(R)/W_{\mathcal{O}}(R)$ for $v, v' \in \mathcal{V}_1$. We write f as $f_2 + f_3$ for $f_2 \in \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ and $f_3 \in \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_3)$ For $v, v' \in (\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$, the condition (9.1) is equivalent to

$$\langle v + f_2(v), F(f_3(v')) \rangle + \langle f_3(v), F(v' + f_2(v') + f_3(v')) \rangle = 0 \tag{9.2}$$

in $\varpi^{-1} W_{\mathcal{O}}(R)/W_{\mathcal{O}}(R)$.

Take a basis $v_1, \ldots, v_{i-1}$ of $\mathcal{V}_1$ such that $v_1, \ldots, v_s$ form a basis of $(\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$. Take a basis $v_i, \ldots, v_{2i-s-1}$ of $\mathcal{V}_2$ and a basis $v_{2i-s}, \ldots, v_{2i-1}$ of $\mathcal{V}_3$. Write $f(v_l)$ as $x_{l,i} v_i + \cdots + x_{l,2i-1} v_{2i-1}$. Then the condition (9.2) is equivalent to

$$\langle v_l + \sum_{j=i}^{2i-s-1} x_{l,j} v_j, F(\sum_{k=2i-s}^{2i-1} x_{m,k} v_k) \rangle + \langle \sum_{k=2i-s}^{2i-1} x_{l,k} v_k, F(v_m + \sum_{j=i}^{2i-1} x_{m,j} v_j) \rangle = 0$$

for $1 \leq l, m \leq s$. We can write this as

$$(\langle v_l + \sum_{j=i}^{2i-s-1} x_{l,j} v_j, F(v_k) \rangle)_{l,k} (x_{m,k}^q)_{k,m} = -(x_{l,k})_{l,k} (\langle v_k, F(v_m + \sum_{j=i}^{2i-1} x_{m,j} v_j) \rangle)_{k,m}.$$

Taking the determinant, we obtain

$$\begin{aligned} \det(x_{l,k})_{l,k} \Big(\det(\langle v_l + \sum_{j=i}^{2i-s-1} x_{l,j} v_j, F(v_k) \rangle)_{l,k} (\det(x_{l,k})_{l,k})^{q-1} \\ - (-1)^s \det(\langle v_k, F(v_m + \sum_{j=i}^{2i-1} x_{m,j} v_j) \rangle)_{k,m} \Big) = 0. \end{aligned}$$

The condition $\mathcal{E}_f \cap \mathcal{E}^0 = \mathcal{E}_-$ is equivalent to $\det(x_{l,k})_{l,k} \neq 0$. Hence, if $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ belongs to the closure of $\pi_0'^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i, x_0}(\tau_i^*))$, then we have $\det(\langle v_k, F(v_m) \rangle)_{k,m} = 0$. This means $F^{-1}(\mathcal{E}_+^\vee) \subset \mathcal{E}$. Hence we have obtained the following proposition:

Proposition 9.3. *The intersection*

$$\pi_0'^{-1}(X_{\mu^*}^{\mathbf{b}_{i-s}, x_0}(\tau_{i-s}^*)) \cap \overline{\pi_0'^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_i, x_0}(\tau_i^*))}$$

is contained in the locus defined by the condition $F^{-1}(\mathcal{E}_+^\vee) \subset \mathcal{E}$.

Conversely, we assume that $F^{-1}(\mathcal{E}_+^\vee) \subset \mathcal{E}$. Then we may assume that v_1 is a basis of $F^{-1}(\mathcal{E}_+^\vee)/\mathcal{E}_-$, v_i is an element of $(F(\mathcal{E}^\vee) \cap \mathcal{E}^0)/\mathcal{E}_-$ lifting a basis of $(F(\mathcal{E}^\vee) \cap \mathcal{E}^0)/(\mathcal{E} \cap \mathcal{E}^0)$ such that $v_i \notin F^{-1}(\mathcal{E}_+^\vee)/\mathcal{E}_-$ and v_{2i-s} is an element of $F(\mathcal{E}^\vee)/\mathcal{E}_-$ lifting a basis of $(F(\mathcal{E}^\vee) + \mathcal{E}^0)/(\mathcal{E} + \mathcal{E}^0)$. Further, we may assume that $v_2, \ldots, v_{i-1}$ and $v_{2i-1}, \ldots, v_{2i-s+1}, v_{2i-s-1}, \ldots, v_{i+1}$ form dual base with respect to the pairing

$$\mathcal{E}/(F^{-1}(\mathcal{E}_+^\vee)) \times \mathcal{E}_+/(F(\mathcal{E}^\vee)); (v, v') \mapsto \langle F(v), v' \rangle.$$

and that $\langle v_j, F(v_k) \rangle = 0$ for $i+1 \leq j \leq 2i-s-1$ and $i \leq k \leq 2i-1$. Then the condition (9.1) is equivalent to

$$\langle \sum_{j=2i-s}^{2i-1} x_{l,j} v_j, F(\sum_{k'=i}^{2i-1} x_{m,k'} v_{k'}) \rangle + \begin{cases} \langle v_l + x_{l,i} v_i, F(\sum_{k=2i-s}^{2i-1} x_{m,k} v_k) \rangle & \text{if } 1 \leq l \leq r, \\ \langle v_l + x_{l,i} v_i, F(\sum_{k'=i}^{2i-1} x_{m,k'} v_{k'}) \rangle & \text{if } s+1 \leq l \leq i-1, \end{cases}$$
$$+ \begin{cases} 0 & \text{if } m = 1, \\ x_{l,2i+1-m} & \text{if } 2 \leq m \leq s, \\ x_{l,2i-m} & \text{if } s+1 \leq m \leq i-1, \end{cases} = 0 \tag{9.3}$$

for $1 \leq l, m \leq i-1$.

We put

$$y = \det(x_{l,j})_{1 \leq l \leq s,\, 2i-s \leq j \leq 2i-1}.$$

We want to show that the quotient of $k[[x_{l,j}]]_{1 \leq l \leq i-1,\, i \leq j \leq 2i-1}$ by the relation (9.3) is nonzero after inverting y.

Proposition 9.4. *(1) The intersection*

$$\pi_0'^{-1}(X_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*)) \cap \overline{\pi_0'^{-1}(\mathring{X}_{\mu^*}^{\mathbf{b}_2, x_0}(\tau_2^*))}$$

is equal to the locus defined by the condition $F^{-1}(\mathcal{E}_+^\vee) = \mathcal{E}$.

(2) We have an isomorphism $X_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*) \cap X_{\mu^*}^{\mathbf{b}_2, x_0}(\tau_2^*) \simeq X_{I_n^1}^{F, F^3}([1], [1])^{\mathrm{pf}}$ given by $\mathcal{E} \mapsto \mathcal{E}^\vee/\mathcal{E}^0$.

Proof. In this case, (9.3) becomes

$$\langle x_{1,3} v_3, F(x_{1,2} v_2 + x_{1,3} v_3) \rangle + \langle v_1 + x_{1,2} v_2, F(x_{1,3} v_3) \rangle = 0.$$

If the quotient of $k[[x_{1,2}, x_{1,3}]]$ by this relation is zero after inverting $x_{1,3}$, there is a positive integer N such that $x_{1,3}^N$ is divisible by

$$\langle x_{1,3} v_3, F(x_{1,2} v_2 + x_{1,3} v_3) \rangle + \langle v_1 + x_{1,2} v_2, F(x_{1,3} v_3) \rangle$$

in $k[[x_{1,2}, x_{1,3}]]$. This does not happen because $\langle v_3, F(v_2) \rangle \neq 0$, which follows from $v_2 \notin F^{-1}(\mathcal{E}_+^\vee)/\mathcal{E}_-$. Hence we have (1). The claim (2) follows from (1). □

By Proposition 9.4, $X_{\mu^*}^{\mathbf{b}_1, x_0}(\tau_1^*) \cap X_{\mu^*}^{\mathbf{b}_2, x_0}(\tau_2^*)$ is isomorphic to the perfect closed subscheme of $(\mathbb{P}^{n-1})^{\mathrm{pf}}$ defined by two equations

$$\sum_{i=1}^n x_i x_{n+1-i}^q = 0, \quad \sum_{i=1}^n x_i x_{n+1-i}^{q^3} = 0.$$

Since all non-degenerate hermitian forms on $\mathbb{F}_{q^2}^n$ are isomorphic, the above scheme is isomorphic to the perfect closed subscheme of $(\mathbb{P}^{n-1})^{\mathrm{pf}}$ defined by two equations

$$\sum_{i=1}^n x_i^{q+1} = 0, \quad \sum_{i=1}^n x_i^{q^3+1} = 0.$$

9.2 Intersection of components for ν_i and ν_r when n is even.

We put $g_r = \varpi^{\delta_{\lambda_r}+\nu_r^*} = \varpi^{-(\varepsilon_{r+1}+\cdots+\varepsilon_n)}$, $\Lambda_{x,r} = g_r\Lambda_x$ and $\mathcal{E}_{x,r} = g_r\mathcal{E}_x$.

Proposition 9.5. *Let $i \neq r$. Assume that $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ is non-empty. Then we have $\Lambda_{x'} \subset \Lambda_{x,r}$, $\varpi\Lambda_{x,r} \subset \varpi^{-1}\Lambda_{x'}$ and*

$$\mathrm{length}(\Lambda_{x,r}/\Lambda_{x,r}\cap\Lambda_{x'}) = \mathrm{length}(\Lambda_{x'}/\Lambda_{x,r}\cap\Lambda_{x'}) + r. \tag{9.4}$$

Let $\mathbf{P}_{x,r,x'}$ be the parabolic subgroup of $\mathbf{G}$ that is the stabilizer of the filtration

$$\mathcal{E}_{x'} \subset \varpi\mathcal{E}_{x,r} + \mathcal{E}_{x'} \subset \mathcal{E}_{x,r} \cap \varpi^{-1}\mathcal{E}_{x'} \subset \varpi^{-1}\mathcal{E}_{x'}.$$

We put $j_1 = \mathrm{length}((\varpi\Lambda_{x,r} + \Lambda_{x'})/\Lambda_{x'})$, $j_2 = \mathrm{length}((\Lambda_{x,r} \cap \varpi^{-1}\Lambda_{x'})/\Lambda_{x'})$ and define $w_r \in S_n$ by

$$w_r(j) = \begin{cases} j+i-1 & \text{if } j_1+1 \leq j \leq j_2 \\ j-j_2-i+r & \text{if } j_2+1 \leq j \leq j_2+i-1 \\ j & \text{otherwise.} \end{cases}$$

Then we have

$$\mathring{X}_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*) = X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)_{\mathbf{P}_{x,r,x'},[w_r]}.$$

Proof. By Proposition 8.3, $X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}_{x'}$ bounded by ν_r^* such that $\varpi\mathcal{E} \subset F(\mathcal{E}^\vee)$. By the identification (6.1), the subscheme $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*) \subset X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ is given by the conditions

$$\mathrm{length}((\mathcal{E}+\mathcal{E}_{x,r})/\mathcal{E}_{x,r}) = i-1, \quad \mathrm{length}(\mathcal{E}_{x,r}/\mathcal{E}\cap\mathcal{E}_{x,r}) = i \tag{9.5}$$

and $\varpi\mathcal{E}_{x,r} \subset \mathcal{E} \subset \varpi^{-1}\mathcal{E}_{x,r}$. Let $\mathcal{E} \in \mathring{X}_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$. Since $\varpi\mathcal{E}_{x,r} \subset \mathcal{E}$, we have $F(\mathcal{E}^\vee) \subset \mathcal{E}_{x,r}$. Hence $\mathcal{E}_{x'} \subset \mathcal{E} \subset F(\mathcal{E}^\vee) \subset \mathcal{E}_{x,r}$. We also have $\varpi\mathcal{E}_{x,r} \subset \mathcal{E} \subset \varpi^{-1}\mathcal{E}_{x'}$. By the equality

$$\begin{aligned} &\mathrm{length}((\mathcal{E}+\mathcal{E}_{x,r})/\mathcal{E}_{x,r}) + \mathrm{length}(\mathcal{E}_{x,r}/\mathcal{E}_{x,r}\cap\mathcal{E}_{x'}) \\ &\quad = \mathrm{length}((\mathcal{E}+\mathcal{E}_{x,r})/\mathcal{E}) + \mathrm{length}(\mathcal{E}/\mathcal{E}_{x'}) + \mathrm{length}(\mathcal{E}_{x'}/\mathcal{E}_{x,r}\cap\mathcal{E}_{x'}), \end{aligned}$$

$\mathrm{length}(\mathcal{E}/\mathcal{E}_{x'}) = r-1$ and (9.5), we have (9.4).

Since we have (9.4), by the above argument, for any $\mathcal{E}$ parametrizing $X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ the condition (9.5) holds if and only if $\mathrm{length}((\mathcal{E}+\mathcal{E}_{x,r})/\mathcal{E}_{x,r}) = i-1$, which is equivalent to $\mathrm{length}((\mathcal{E}+(\mathcal{E}_{x,r}\cap\varpi^{-1}\mathcal{E}_{x'}))/(\mathcal{E}_{x,r}\cap\varpi^{-1}\mathcal{E}_{x'})) = i-1$. Therefore the subscheme $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x}(\tau_i^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*) \subset X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ is given by the conditions $\mathrm{length}((\mathcal{E}+(\mathcal{E}_{x,r}\cap\varpi^{-1}\mathcal{E}_{x'}))/(\mathcal{E}_{x,r}\cap\varpi^{-1}\mathcal{E}_{x'})) = i-1$ and $\varpi\mathcal{E}_{x,r}+\mathcal{E}_{x'} \subset \mathcal{E}$. This implies the claim. □

Assume that $x \neq x'$

Proposition 9.6. *Assume that $X_{\mu^*}^{\mathbf{b}_r,x}(\tau_r^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ is not empty. Then we have $1 \leq l_{x,x'} \leq r-1$ and $\varpi\Lambda_x \subset \Lambda_{x'}$. The intersection $X_{\mu^*}^{\mathbf{b}_r,x}(\tau_r^*) \cap X_{\mu^*}^{\mathbf{b}_r,x'}(\tau_r^*)$ is parametrized by $\mathcal{E} \dashrightarrow \mathcal{E}_x$ bounded by ν_r^* such that $\varpi\mathcal{E} \subset F(\mathcal{E}^\vee)$ and $\mathcal{E} \dashrightarrow \mathcal{E}_{x'}$ is also bounded by ν_r^*. In particular, it is isomorphic to*

$$\left\{H \in \mathrm{Gr}^{\mathrm{pf}}(r-1-l_{x,x'}, \varpi^{-1}(\Lambda_x\cap\Lambda_{x'})/(\Lambda_x+\Lambda_{x'})) \mid H \subset \mathrm{Frob}(H^\perp)\right\}.$$

Proof. Assume that $\mathcal{E}$ is a point of $X^{\mathbf{b}_r,x}_{\mu^*}(\tau_r^*) \cap X^{\mathbf{b}_r,x'}_{\mu^*}(\tau_r^*)$. Since $\varpi\mathcal{E} \subset F(\mathcal{E}^\vee)$ and both $\mathcal{E} \dashrightarrow \mathcal{E}_x$ and $\mathcal{E} \dashrightarrow \mathcal{E}_{x'}$ are bounded by ν_r^*, we have the following chain conditions:

$$\mathcal{E}_x \subset \mathcal{E} \subset \varpi^{-1}F(\mathcal{E}^\vee) \subset \varpi^{-1}\mathcal{E}_x,$$
$$\mathcal{E}_{x'} \subset \mathcal{E} \subset \varpi^{-1}F(\mathcal{E}^\vee) \subset \varpi^{-1}\mathcal{E}_{x'}.$$

The inclusion follows from $\mathcal{E}_x \subset \mathcal{E} \subset \varpi^{-1}\mathcal{E}_{x'}$. Note that $\mathrm{length}(\varpi^{-1}F(\mathcal{E}^\vee)/\mathcal{E}) = 2$, while both $\mathrm{length}(\mathcal{E}/\mathcal{E}_x)$ and $\mathrm{length}(\mathcal{E}/\mathcal{E}_{x'})$ are $r-1$. Then $\mathcal{E}_x \cap \mathcal{E}_{x'}$ and $\mathcal{E}$ are related by

$$\mathcal{E}_x + \mathcal{E}_{x'} \subset \mathcal{E} \subset \varpi^{-1}F(\mathcal{E}) \subset \varpi^{-1}(\mathcal{E}_x \cap \mathcal{E}_{x'}).$$

Since $l_{x,x'} = \mathrm{length}((\mathcal{E}_x + \mathcal{E}_{x'})/\mathcal{E}_x)$, we have

$$l_{x,x'} = r - 1 - \mathrm{length}(\mathcal{E}/(\mathcal{E}_x + \mathcal{E}_{x'})).$$

Hence we have $1 \leq l_{x,x'} \leq r-1$.

The isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}/(\mathcal{E}_x + \mathcal{E}_{x'}) \subset \varpi^{-1}(\mathcal{E}_x \cap \mathcal{E}_{x'})/(\mathcal{E}_x + \mathcal{E}_{x'})$. □

10 Example

In this section, we study in details the case where $n = 6$. We identify the moduli parametrizing modification $\mathcal{E} \subset \mathcal{E}_x$ bounded by ν_1 with $(\mathbb{P}^5)^{\mathrm{pf}}$ by taking a basis of Λ_x such that the Hermitian pairing is the standard one. Let $\mathbb{P}_{x,x',+}$ be the projective subspace of $(\mathbb{P}^5)^{\mathrm{pf}}$ defined by the condition $\varpi\mathcal{E}^+_{x,x'} \subset \mathcal{E}$. Let $\mathbb{P}_{x,x',-}$ be the projective subspace of $(\mathbb{P}^5)^{\mathrm{pf}}$ defined by the condition $\varpi\mathcal{E}^-_{x,x'} \subset \mathcal{E}$. We note that $\mathbb{P}_{x,x',+}$ and $\mathbb{P}_{x,x',-}$ are isomorphic to $(\mathbb{P}^{5-d_2})^{\mathrm{pf}}$ and $(\mathbb{P}^{d_2-1})^{\mathrm{pf}}$ respectively.

10.1 Intersection of components for ν_1

We may assume that $x \neq x'$. The intersection is not empty only if $l_{x,x'} = 1$. In this case, $d_1 = 0$, $d_2 = 1$, $j_1 = j_2 = 0$. The intersection is $\mathbb{P}_{x,x',-}$, which is a point given by $\mathcal{E}_x \cap \mathcal{E}_{x'}$.

10.2 Intersection of components for ν_1 and ν_2

If $x = x'$, then the intersection is isomorphic to the perfect closed subscheme of $(\mathbb{P}^5)^{\mathrm{pf}}$ defined by two equations

$$\sum_{i=1}^{6} x_i x_{7-i}^q = 0, \quad \sum_{i=1}^{6} x_i x_{7-i}^{q^3} = 0.$$

We assume that $x \neq x'$.

10.2.1 $d_1 = 0$, $d_2 = 1$

In this case, $j_1 = 0$ and $j_2 = 1$. The intersection is equal to the perfect closed subscheme of $\mathbb{P}_{x,x',+}$ defined by equation

$$\sum_{i=1}^{5} x_i x_{6-i}^q = 0.$$

10.2.2 $d_1 = 0$, $d_2 = 2$

In this case, $j_1 = 0$ and $j_2 = 1$. The intersection is $\mathbb{P}_{x,x',-}$, which is isomorphic to $(\mathbb{P}^1)^{\mathrm{pf}}$.

Remark 10.1. *If $d_1 = d_2 = 1$, then there is no $j_2 \in \mathbb{N}$ satisfying the condition in Proposition 9.2.*

10.3 Intersection of components for ν_2

Let $(\mathcal{E}_+, \mathcal{E}_-)$ be a point of $X_{I_6^{1,4}}([1])^{\mathrm{pf}}$. The hermitian pairing on V induces a pairing on $\mathcal{E}_+/\mathcal{E}_-$ since we have $\mathcal{E}_+ \subset F(\mathcal{E}_-^\vee)$ and $\mathcal{E}_- \subset F(\mathcal{E}_+^\vee)$. We take a basis v_1, v_2, v_3 of $\mathcal{E}_+/\mathcal{E}_-$ such that $v_1 \in F^{-1}(\mathcal{E}_+^\vee)/\mathcal{E}_-$, $v_2 \in \mathcal{E}_x/\mathcal{E}_-$. Let $\mathcal{E}$ be a point of $\mathring{X}_{\mu^*}^{\mathbf{b}_2,x}(\tau_2^*)$ in the fiber of $(\mathcal{E}_+, \mathcal{E}_-)$ under

$$\pi\colon \mathring{X}_{\mu^*}^{\mathbf{b}_2,x}(\tau_2^*) \to X_{I_6^{1,4}}([1])^{\mathrm{pf}}.$$

We can take a generator $v = x_1 v_1 + x_2 v_2 + v_3$ of $\mathcal{E}/\mathcal{E}_-$ for $x_1, x_2 \in k$, since $\mathcal{E} \not\subset \mathcal{E}_x$. Then we have

$$\langle v, F(v)\rangle = x_1\langle v_1, F(v_3)\rangle + x_2\langle v_2, F(v_3)\rangle + x_2^q\langle v_3, F(v_2)\rangle + \langle v_3, F(v_3)\rangle$$

because $\langle w, F(w')\rangle = 0$ for $w, w' \in \mathcal{E}_x/\mathcal{E}_-$ and $\langle v_3, F(v_1)\rangle = 0$. Hence the fiber of $(\mathcal{E}_+, \mathcal{E}_-)$ under π is defined by

$$x_1\langle v_1, F(v_3)\rangle + x_2\langle v_2, F(v_3)\rangle + x_2^q\langle v_3, F(v_2)\rangle + \langle v_3, F(v_3)\rangle = 0$$

in $(\mathbb{A}^2)^{\mathrm{pf}}$. We note that $(\langle v_1, F(v_3)\rangle, \langle v_2, F(v_3)\rangle) \neq (0,0)$ because $v_3 \notin \mathcal{E}_x/\mathcal{E}_-$.

We describe the fiber of

$$\pi_{j_1,j_2}\colon \mathring{X}_{\mu^*}^{\mathbf{b}_2,x}(\tau_2^*)_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]} \cap \mathrm{Par}_{t_{j_1,j_2}}(\mathscr{G}_{j_1,j_2};\mathscr{P}_{j_1,j_2})^{\mathrm{pf}}_{r_{j_1,j_2}} \to X_{I_6^{1,4}}([1])^{\mathrm{pf}}_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$$

when

$$\left(\mathring{X}_{\mu^*}^{\mathbf{b}_2,x}(\tau_2^*) \cap \mathring{X}_{\mu^*}^{\mathbf{b}_2,x'}(\tau_2^*)\right)_{\mathbf{P}_{x,x'},[w_{j_1,j_2}]}$$

is not empty.

10.3.1 $d_1 = 0$, $d_2 = 1$

In this case, $0 \le j_1 \le 1$ and $j_2 = 0$. We have $d_{j_1,j_2} = 2 - j_1$. The fiber of $\pi_{j_1,0}$ is given by the condition $\mathcal{E} \not\subset \mathcal{E}_{+,-}$.

10.3.2 $d_1 = 0$, $d_2 = 2$

In this case, $0 \le j_1 \le 1$ and $0 \le j_2 \le 1$. We have $d_{j_1,j_2} = 1 - j_1 + j_2$. The fiber of $\pi_{j_1,0}$ is given by the condition $\mathcal{E} \not\subset \mathcal{E}_{+,-}$. The fiber of $\pi_{j_1,1}$ is given by the condition $\mathcal{E} \subset \mathcal{E}_{+,-}$.

10.3.3 $d_1 = 1$, $d_2 = 1$

In this case, $j_1 = 0$ and $j_2 = 0$. We have $d_{j_1,j_2} = 2$. The fiber of $\pi_{0,0}$ is given by the condition $\mathcal{E} \subset \mathcal{E}_{+,-}$.

10.3.4 $d_1 = 0$, $d_2 = 3$

In this case, $j_1 = 0$ and $j_2 = 1$. We have $d_{j_1,j_2} = 1$. The fiber of $\pi_{0,1}$ is given by the condition $\mathcal{E} = \mathcal{E}_{+,-}$.

10.3.5 $d_1 = 1$, $d_2 = 2$

In this case, $j_1 = 0$ and $j_2 = 0$. We have $d_{j_1,j_2} = 1$. The fiber of $\pi_{0,0}$ is given by the condition $\mathcal{E} = \mathcal{E}_{+,-}$.

10.4 Intersection of components for ν_3

10.4.1 $l_{x,x'} = 1$

The intersection is isomorphic to the perfection of the Fermat hypersurface defined by

$$x_1 x_4^q + x_2 x_3^q + x_3 x_2^q + x_4 x_1^q = 0$$

in $\mathbb{P}^3$.

10.4.2 $l_{x,x'} = 2$

The intersection is a point given by $\mathcal{E}_x + \mathcal{E}_{x'}$.

11 Shimura variety

Let E be a quadratic imaginary field, and let V be an n-dimensional Hermitian space over E with signature $(2, n-2)$ at infinity. Fix a prime $p \neq 2$ inert in E. Further assume that $\mathsf{V} \otimes_E \mathbb{Q}_{p^2}$ contains a self-dual $\mathbb{Z}_{p^2}$ lattice Λ. Let $\mathsf{G} = \mathrm{GU}(\mathsf{V})$ be the general associated unitary group. We put $G = \mathrm{GU}(\Lambda)$ as before.

We take a basis of $\mathsf{V}_{\mathbb{C}} = \mathsf{V} \otimes_E \mathbb{C}$ over $\mathbb{C}$ such that the Hermitian form is given by the matrix $\mathrm{diag}(1_2, -1_{n-2})$. Let $h \colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}\mathbb{C}} \to \mathsf{G}_{\mathbb{R}}$ be the morphism of algebraic groups over $\mathbb{R}$ such that $h(z)$ corresponds to $\mathrm{diag}(z \cdot 1_2, \bar{z} \cdot 1_{n-2})$ for $z \in \mathbb{C}^\times$ under

$$\mathsf{G}(\mathbb{R}) \subset \mathrm{Aut}_{\mathbb{C}}(\mathsf{V}_{\mathbb{C}}) \cong \mathrm{GL}_n(\mathbb{C}),$$

where the last isomorphism is given by the basis taken above. Let X be the $\mathsf{G}(\mathbb{R})$-conjugacy class of h. Then (G, X) is a Shimura datum.

We have an isomorphism

$$(\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}\mathbb{C}})_{\mathbb{C}} \simeq \mathbb{G}_{\mathrm{m}\mathbb{C}} \times \mathbb{G}_{\mathrm{m}\mathbb{C}}$$

of algebraic groups over $\mathbb{C}$ induced by the isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$; $a \otimes b \mapsto (ab, \bar{a}b)$. We define μ_h by the composition

$$\mathbb{G}_{\mathrm{m}\mathbb{C}} \hookrightarrow \mathbb{G}_{\mathrm{m}\mathbb{C}} \times \mathbb{G}_{\mathrm{m}\mathbb{C}} \simeq (\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}\mathbb{C}})_{\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} \mathsf{G}_{\mathbb{C}},$$

where the first morphism is the inclusion into the first factor. Let $\mu \colon \mathbb{G}_{\mathrm{m}E} \to \mathsf{G}_E$ be the morphism of algebraic over E such that $\mu(z)$ corresponds to $(\mathrm{diag}(z \cdot 1_2, 1_{n-2}), z)$ for $z \in E^\times$ under the isomorphism

$$\mathsf{G}_E \simeq \mathrm{GL}_n(E) \times \mathbb{G}_{\mathrm{m}E}$$

given by taking a basis of V over E. Then μ_h and $\mu_{\mathbb{C}}$ are in the same $\mathsf{G}(\mathbb{C})$-conjugacy class. We note that the reflex field $E(\mathsf{G},\mathsf{X})$ of (G,X) is E if $n \neq 4$ and $\mathbb{Q}$ if $n = 4$.

Let $K^p \subset \mathsf{G}(\mathbb{A}_{\mathrm{f}}^p)$ be a sufficiently small open compact subgroup. Let $K_p \subset \mathsf{G}(\mathbb{Q}_p)$ be a hyperspecial subgroup. We put $K = K^p K_p \subset \mathsf{G}(\mathbb{A}_{\mathrm{f}})$. Let $\mathrm{Sh}_K(\mathsf{G},\mathsf{X})$ be the canonical model over $E(\mathsf{G},\mathsf{X})$ of the Shimura variety attached to (G,X) and K. Let $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ be the canonical integral model of $\mathrm{Sh}_K(\mathsf{G},\mathsf{X})$ over $\mathcal{O}_{E(\mathsf{G},\mathsf{X}),(p)}$ constructed in [Kis10].

Let $\mathbf{S}_K(\mathsf{G},\mathsf{X})$ be the perfection of $\mathscr{S}_K(\mathsf{G},\mathsf{X}) \otimes \overline{\mathbb{F}}_p$. We have the Newton map

$$\mathcal{N} \colon \mathbf{S}_K(\mathsf{G},\mathsf{X})(\overline{\mathbb{F}}_p) \to B(G,\mu^*)$$

as in [XZ17, 7.2.7]. Let $[b] \in B(G,\mu^*)$ be the basic element. We write $\mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]}$ for the closed perfect subscheme of $\mathbf{S}_K(\mathsf{G},\mathsf{X})$ defined by $\mathcal{N}^{-1}([b])$. We call $\mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]}$ the supersingular locus of $\mathbf{S}_K(\mathsf{G},\mathsf{X})$.

Remark 11.1. *In [Kot92], a moduli space of abelian schemes with additional structures is constructed. It is isomorphic to a finite union of integral models of Shimura varieties. Under the isomorphism, a point of $\mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]}$ corresponds to a supersingular abelian variety.*

We take a point $x \in \mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]}(\overline{\mathbb{F}}_p)$. We put $L = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$. Then we have a basic element $b_x \in G(L)$ and an algebraic group I_x over $\mathbb{Q}$ as in [XZ17, 7.2.9]. We have embeddings $I_x(\mathbb{Q}) \subset \mathsf{G}(\mathbb{A}_{\mathrm{f}}^p)$ and $I_x(\mathbb{Q}) \subset J_{b_x}(\mathbb{Q}_p)$ as in [XZ17, 7.2.13]. Then we have the isomorphism

$$I_x(\mathbb{Q})\backslash X_{\mu^*}(b_x) \times \mathsf{G}(\mathbb{A}_{\mathrm{f}}^p)/K^p \xrightarrow{\sim} \mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]} \tag{11.1}$$

by [XZ17, Corollary 7.2.16]. We use notations in §8 for $F = \mathbb{Q}_p$.

Proposition 11.2. *We have $\dim \mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]} = n-2$. The irreducible components of $\mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]}$ are parametrized by*

$$\coprod_{1 \leq i \leq r} I_x(\mathbb{Q})\backslash(G(\mathbb{Q}_p)/G(\mathbb{Z}_p)) \times \mathsf{G}(\mathbb{A}_{\mathrm{f}}^p)/K^p.$$

For sufficiently small K^p, a non-empty open subscheme of each irreducible component of $\mathbf{S}_K(\mathsf{G},\mathsf{X})_{[b]}$ is isomorphic to a non-empty open subscheme of $\mathring{X}_{\mu^*}^{\mathbf{b}_i,x_0}(\tau_i^*)$ for some i, which is described in §8.

Proof. The first two claims follow from Proposition 8.1 and (11.1). The last claim is proved in the same way as [Vol10, Theorem 6.1]. □

References

[BR06] C. Bonnafé and R. Rouquier, On the irreducibility of Deligne-Lusztig varieties, C. R. Math. Acad. Sci. Paris 343 (2006), no. 1, 37–39.

[BS17] B. Bhatt and P. Scholze, Projectivity of the Witt vector affine Grassmannian, Invent. Math. 209 (2017), no. 2, 329–423.

[Gas10] Q. R. Gashi, On a conjecture of Kottwitz and Rapoport, Ann. Sci. Éc. Norm. Supér. (4) 43 (2010), no. 6, 1017–1038.

[GHKR06] U. Görtz, T. J. Haines, R. E. Kottwitz and D. C. Reuman, Dimensions of some affine Deligne-Lusztig varieties, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 3, 467–511.

[GHN19] U. Görtz, X. He and S. Nie, Fully Hodge-Newton decomposable Shimura varieties, Peking Math. J. 2 (2019), no. 2, 99–154.

[Hai06] T. J. Haines, Equidimensionality of convolution morphisms and applications to saturation problems, Adv. Math. 207 (2006), no. 1, 297–327.

[HP14] B. Howard and G. Pappas, On the supersingular locus of the $\mathrm{GU}(2,2)$ Shimura variety, Algebra Number Theory 8 (2014), no. 7, 1659–1699.

[Kis10] M. Kisin, Integral models for Shimura varieties of abelian type, J. Amer. Math. Soc. 23 (2010), no. 4, 967–1012.

[Kot92] R. E. Kottwitz, Points on some Shimura varieties over finite fields, J. Amer. Math. Soc. 5 (1992), no. 2, 373–444.

[Kot97] R. E. Kottwitz, Isocrystals with additional structure. II, Compositio Math. 109 (1997), no. 3, 255–339.

[KR11] S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties I. Unramified local theory, Invent. Math. 184 (2011), no. 3, 629–682.

[MV07] I. Mirković and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[SGA3-3] Schémas en groupes. III: Structure des schémas en groupes réductifs, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin-New York, 1970.

[Vol10] I. Vollaard, The supersingular locus of the Shimura variety for $\mathrm{GU}(1,s)$, Canad. J. Math. 62 (2010), no. 3, 668–720.

[VW11] I. Vollaard and T. Wedhorn, The supersingular locus of the Shimura variety of $\mathrm{GU}(1,n-1)$ II, Invent. Math. 184 (2011), no. 3, 591–627.

[XZ17] L. Xiao and X. Zhu, Cycles on Shimura varieties via geometric Satake, 2017, arXiv:1707.05700.

[Zhu17] X. Zhu, Affine Grassmannians and the geometric Satake in mixed characteristic, Ann. of Math. (2) 185 (2017), no. 2, 403–492.

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