RAPOPORT-ZINK SPACES OF TYPE $\mathrm{GU}(2, n-2)$

MARIA FOX, BENJAMIN HOWARD, AND NAOKI IMAI

ABSTRACT. We describes the structure of the supersingular Rapoport-Zink space associated to the group of unitary similitudes of signature $(2, n-2)$ for an unramified quadratic extension of p-adic fields. In earlier work, two of the authors described the irreducible components in the category of schemes-up-to-perfection. The goal of this work is to remove the qualifier "up-to-perfection".

CONTENTS

1. INTRODUCTION

The p-adic period domains of Rapoport-Zink [15] are defined as moduli spaces of p-divisible groups with additional structure. While general existence theorems are known for these spaces, it is a very difficult problem to determine their structures as formal schemes, or even the structure of their underlying reduced schemes, in any explicit way. The history of this problem is long, and has its origins in Drinfeld's work (before Rapoport and Zink) on the p-adic uniformization of quaternionic Shimura curves. The reader can find a thorough guide to the older literature to the introduction to Vollaard's work [16] on Rapoport-Zink spaces of type $\mathrm{GU}(1, n-1)$.

This work of Vollaard, and the subsequent work of Vollaard-Wedhorn [17], introduced significant new ideas into the subject, which were extended further in [3], [9], [10], [14], and [18]. In all of these works, the main results assert that the irreducible components of (the reduced scheme underlying) a

Date: August 4, 2023.

M.F. was supported by NSF MSPRF Grant 2103150. B.H. was supported in part by NSF grant DMS-2101636. N.I. was supported by JSPS KAKENHI Grant Number 22H00093.

particular Rapoport-Zink space are isomorphic to Deligne-Lusztig varieties associated to reductive groups over finite fields. Görtz-He [5] and Görtz-He-Nie [6] gave a classification of those Rapoport-Zink spaces for which one should expect the irreducible components to have this form, and called such Rapoport-Zink spaces *fully Hodge-Newton decomposable.*

Xiao-Zhu [19] proved quite general results on the structure of Rapoport-Zink spaces (and more general affine Deligne-Lustztig varieties). The results of Xiao-Zhu provide a parametrization of the irreducible components, but do not provide a description of their scheme-theoretic structure. The Xiao-Zhu parametrization of components has since been generalized by other authors; see [7], [8], [13], and [20].

Among the simplest examples of Rapoport-Zink spaces that are not fully Hodge-Newton decomposable are those of type $\mathrm{GU}(2, n-2)$. Because of the results of Görtz-He-Nie mentioned above, there is no expectation that the irreducible components in this setting are Deligne-Lusztig varieties. However, in earlier work two of the authors [4] provided a description of (an open dense subset of) each irreducible component by fibering it over a Deligne-Lusztig variety, and then describing the fibers.

One sense in which the descriptions of irreducible components in [4] is incomplete is that everything is understood in the category of schemes-up-to-perfection. Loosely speaking, this means that for each irreducible component, a scheme (fibered over a Deligne-Lusztig variety) is exhibited with the property that it has the same functor of points *when restricted to perfect algebras in characteristic p.* For example, when viewed as an object of this category, each irreducible component is indistinguishable from its Frobenius twists. This ambiguity in the scheme-theoretic structure of the components arises because [4] adheres closely to the framework of [19], in which Rapoport-Zink spaces are replaced by their corresponding affine Deligne-Lustzig varieties, which are thought of not as moduli spaces of p-divisible groups, but as closed subsets of the Witt vector affine Grassmannians of [21] and [1]. These Witt vector affine Grassmannians are only defined as objects in the category of (ind-)schemes-up-to-perfection.

The primary purpose of this paper is to revisit the results of [4], in order to pin down the precise scheme-theoretic structure, not just up to perfection, of the irreducible components of the $\mathrm{GU}(2, n-2)$ Rapoport-Zink space.

In practice, this requires making more systematic use of the moduli interpretation of the Rapoport-Zink space (as opposed to the interpretation as a closed subset of a Witt vector affine Grassmannian), in order to exploit the Grothendieck-Messing deformation theory of the universal p-divisible group that lives over it.

1.1. **Statement of the results.** Throughout this paper we denote by $\breve{\mathbb{Q}}_p$ the completion of the maximal unramified extension of $\mathbb{Q}_p$, and by $\breve{\mathbb{Z}}_p$ its ring of integers. Let $\sigma : \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$ be the Frobenius, inducing the p-power automorphism of the residue field

$$\breve{\mathbb{F}}_p = \breve{\mathbb{Z}}_p / p\breve{\mathbb{Z}}_p.$$

Note that $\breve{\mathbb{F}}_p$ is just an algebraic closure of $\mathbb{F}_p$.

Fix an unramified quadratic field extension E of $\mathbb{Q}_p$. We are interested in the Rapoport-Zink formal scheme

$$\mathrm{RZ} \to \mathrm{Spf}(\breve{\mathbb{F}}_p)$$

parametrizing p-divisible groups X of dimension $n \geqslant 2$ over $\mathbb{F}_p$-algebras, endowed with principal polarizations, an action of $\mathcal{O}_E$ satisfying a signature $(2, n-2)$ condition, and a quasi-isogeny $\varrho_X : X \dashrightarrow \mathbb{X}$ to a fixed framing object. See §2.2 for the precise definitions.

For our framing object $\mathbb{X}$ we make a nonstandard choice: an n-dimension p-divisible group, again with an $\mathcal{O}_E$-action and principal polarization, but satisfying a signature $(0, n)$-condition. We will show that $\mathbb{X}$ is unique up to isomorphism, not just isogeny, and has the form

$$\mathbb{X} \cong \Lambda \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}}$$

for $\overline{\mathbb{Y}}$ a supersingular p-divisible group of height 2 and dimension 1 endowed with an action of $\mathcal{O}_E$ of signature $(0, 1)$, and Λ a rank n self-dual hermitian $\mathcal{O}_E$-lattice. By the comments preceding Definition 2.2.1, our $\mathbb{X}$ is also quasi-isogenous to a p-divisible group of signature $(2, n-2)$, so this unusual choice of framing object yields the usual Rapoport-Zink space for $\mathrm{GU}(2, n-2)$.

Let $G = \mathrm{GU}(\Lambda)$ be the group of unitary similitudes of Λ, a reductive group over $\mathbb{Z}_p$. Its group of $\mathbb{Q}_p$-points acts on the framing object $\Lambda \otimes \overline{\mathbb{Y}}$ by quasi-isogenies, and hence also acts on RZ.

Denote by $\mathrm{RZ}_\Lambda \subset \mathrm{RZ}$ the projective closed subscheme parametrizing commutative diagrams

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & \downarrow{\scriptstyle \varrho_X} & \searrow & \\ p\Lambda \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}} & \longrightarrow & \Lambda \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}} & \longrightarrow & p^{-1}\Lambda \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}}, \end{array}$$

in which all solid arrows are isogenies, and the horizontal arrows are induced by the inclusions $p\Lambda \subset \Lambda \subset p^{-1}\Lambda$. It will turn out (see the proof of Corollary 6.3.5) that

$$\mathrm{RZ}^{\mathrm{red}} = \bigcup_{\gamma \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)} \gamma \cdot \mathrm{RZ}_\Lambda^{\mathrm{red}},$$

where the superscript red indicates underlying reduced scheme. (We suspect that RZ_Λ is already reduced, and that this might be proved by arguing as in Corollary 3.2.3 of [12], but we have not checked this.) Hence the irreducible components of the left hand side are precisely the $G(\mathbb{Q}_p)$-translates of irreducible components of $\mathrm{RZ}_\Lambda^{\mathrm{red}}$.

To describe these, we decompose

$$\mathrm{RZ}_\Lambda^{\mathrm{red}} = \bigsqcup_{1\leqslant k\leqslant \lfloor n/2\rfloor} \mathrm{RZ}_\Lambda^{k,\mathrm{red}}$$

into locally closed subschemes. The precise definition of the schemes on the right hand side appears in §2.4, but loosely speaking, as k increases the points of $\mathrm{RZ}_\Lambda^{k,\mathrm{red}}$ get farther from the framing object.

The following result is stated in the text as Theorem 6.3.1. The Deligne-Lusztig variety is that of Definition 3.4.1, and the vector bundle $\mathcal{V}$ on it is constructed in §4.1.

Theorem A. *If $k < n/2$ then $\mathrm{RZ}_\Lambda^{k,\mathrm{red}}$ is a smooth and irreducible $\breve{\mathbb{F}}_p$-scheme of dimension $n-2$, and its closure*

$$\overline{\mathrm{RZ}}_\Lambda^{k,\mathrm{red}} \subset \mathrm{RZ}^{\mathrm{red}}$$

is an irreducible component with stabilizer $G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. Moreover, there is a smooth morphism

$$\mathrm{RZ}_\Lambda^{k,\mathrm{red}} \to \mathrm{DL}_\Lambda^k$$

of relative dimension $k-1$ to a smooth and projective Deligne-Lusztig variety with the following property: over DL_Λ^k there is a vector bundle $\mathcal{V}$ of rank $2k-1$, endowed with a rank k local direct summand $\mathcal{V}^{(k)} \subset \mathcal{V}$ and a morphism

$$\beta : \mathcal{V} \otimes \sigma^*\mathcal{V} \to \mathcal{O}_{\mathrm{DL}_\Lambda^k},$$

such that $\mathrm{RZ}_\Lambda^{k,\mathrm{red}}$ is identified with the moduli space parametrizing complementary summands

$$\mathcal{V} = \mathcal{F} \oplus \mathcal{V}^{(k)}$$

that are totally isotropic, in the sense that $\beta(\mathcal{F}\otimes\sigma^*\mathcal{F}) = 0$. Here σ^* denotes pullback of coherent sheaves with the respect to the p-power Frobenius on the structure sheaf of DL_Λ^k.

Although the technical details obscure it, the basic idea for constructing a map from $\mathrm{RZ}_\Lambda^{k,\mathrm{red}}$ to a Deligne-Lusztig variety is quite simple. An $\breve{\mathbb{F}}_p$-valued point of $\mathrm{RZ}_\Lambda^{k,\mathrm{red}}$ corresponds to a quasi-isogeny of p-divisible groups $X \dashrightarrow \mathbb{X}$. This realizes the covariant Dieudonné modules of these p-divisible groups as lattices in a common $\breve{\mathbb{Q}}$-vector space, and the intersection $D(H) = D(X) \cap D(\mathbb{X})$ is the Dieudonné module of a p-divisible group H endowed with an isogeny $H \to \mathbb{X}$. This latter isogeny is quite small, in the sense that

$$pD(\mathbb{X}) \subset D(H) \subset D(\mathbb{X}),$$

and so is determined by the subspace

$$\frac{D(H)}{pD(\mathbb{X})} \subset \frac{D(\mathbb{X})}{pD(\mathbb{X})} \cong \Lambda \otimes_{\mathbb{Z}_p} \breve{\mathbb{F}}_p.$$

The isomorphism here comes from our particular choice of framing object, as a choice of isomorphism $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_E \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ identifies

$$D(\mathbb{X}) = D(\Lambda \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}}) = \Lambda \otimes_{\mathcal{O}_E} D(\overline{\mathbb{Y}}) \cong \Lambda \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p.$$

For the rough purposes of this introduction, one can can think of the Deligne-Lusztig variety as parametrizing all subspaces that arise from this construction.

If n is odd Theorem A completes our description of $\mathrm{RZ}^{\mathrm{red}}$, as

$$\mathrm{RZ}^{\mathrm{red}} = \bigcup_{\substack{1 \leqslant k < n/2 \\ \gamma \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)}} \gamma \cdot \overline{\mathrm{RZ}}^{k,\mathrm{red}}_{\Lambda}$$

exhibits the left hand side as the union of its irreducible components. Ideally one would like to have a description not just of $\mathrm{RZ}^{k,\mathrm{red}}_{\Lambda}$, but of its closure. This seems quite difficult for $k > 1$. When $k = 1$, Theorem A implies that

$$\mathrm{RZ}^{1,\mathrm{red}}_{\Lambda} \cong \mathrm{DL}^{1}_{\Lambda}$$

is projective, so no closure is needed.

When n is even we must also examine $\mathrm{RZ}^{n/2,\mathrm{red}}_{\Lambda}$. For every intermediate lattice $p\Lambda \subsetneq \Lambda' \subsetneq \Lambda$ such that $\Lambda'/p\Lambda \subset \Lambda/p\Lambda$ is maximal isotropic, we define

$$\mathrm{RZ}^{\heartsuit}_{\Lambda'} \subset \mathrm{RZ}_{\Lambda}$$

as the projective closed subscheme parametrizing commutative diagrams

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & \downarrow{\scriptstyle \varrho_X} & \searrow & \\ \Lambda' \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}} & \longrightarrow & \Lambda \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}} & \longrightarrow & p^{-1}\Lambda' \otimes_{\mathcal{O}_E} \overline{\mathbb{Y}}, \end{array}$$

in which all solid arrows are isogenies, and the horizontal arrows are induced by the inclusions $\Lambda' \subset \Lambda \subset p^{-1}\Lambda'$.

The following result is stated in the text as Theorem 6.3.4. The Deligne-Lusztig variety in the theorem is that of Definition 5.2.1.

Theorem B. *If n is even then*

$$\mathrm{RZ}^{n/2,\mathrm{red}}_{\Lambda} \subset \bigcup_{p\Lambda \subsetneq \Lambda' \subsetneq \Lambda} \mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'},$$

where the union is over intermediate lattices for which $\Lambda'/p\Lambda \subset \Lambda/p\Lambda$ is maximal isotropic. All of the $\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}$ appearing on the right hand side lie in the same $G(\mathbb{Q}_p)$-orbit, each is an irreducible component of $\mathrm{RZ}^{\mathrm{red}}$ with stabilizer a $G(\mathbb{Q}_p)$-conjugate of $G(\mathbb{Z}_p)$, and each is isomorphic to a smooth projective Deligne-Lusztig variety of dimension $n-2$.

When n is even, if we fix one $\Lambda' \subset \Lambda$ as above then

$$\mathrm{RZ}^{\mathrm{red}} = \Big(\bigcup_{\substack{1 \leqslant k < n/2 \\ \gamma \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)}} \gamma \cdot \overline{\mathrm{RZ}}_{\Lambda}^{k,\mathrm{red}} \Big) \cup \Big(\bigcup_{\gamma \in G(\mathbb{Q}_p)/hG(\mathbb{Z}_p)h^{-1}} \gamma \cdot \mathrm{RZ}_{\Lambda'}^{\heartsuit,\mathrm{red}} \Big)$$

exhibits the left hand side as the union of its irreducible components. Here $h \in G(\mathbb{Q}_p)$ is any element satisfying $h\Lambda = p^{-1}\Lambda'$, as in Theorem 6.3.4 and its proof.

1.2. **General notation.** Throughout the paper, E is an unramified quadratic extension of $\mathbb{Q}_p$. Denote by

$$i_0, i_1 : \mathcal{O}_E \to \breve{\mathbb{Z}}_p$$

the two embeddings. The nontrivial automorphism of E is denoted $x \mapsto \overline{x}$, so that $i_0(\overline{x}) = \sigma(i_1(x))$ and similarly with the indices 0 and 1 reversed. Abbreviate

$$\breve{E} = E \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p,$$

and $\mathcal{O}_{\breve{E}} = \mathcal{O}_E \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$. Label the orthogonal idempotents

$$e_0, e_1 \in \mathcal{O}_{\breve{E}} \cong \breve{\mathbb{Z}}_p \times \breve{\mathbb{Z}}_p$$

in such a way that for any $\breve{\mathbb{Z}}_p$-module M with a commuting action of $\mathcal{O}_E$, the actions of $\mathcal{O}_E$ on the direct summands

$$M_0 = e_0 M \quad \text{and} \quad M_1 = e_1 M \tag{1.2.1}$$

are through i_0 and i_1, respectively.

A hermitian form on an E-vector space or $\mathcal{O}_E$-module is always E-linear in the first variable and conjugate-linear in the second variable.

If $\mathcal{F}$ is a coherent sheaf on a scheme X, the notation $\mathcal{F}_x$ always means the fiber (not the stalk) at a point $x \in X$. If X is any scheme (or formal scheme) we denote by X^{red} the underlying reduced scheme.

If M is a $\mathbb{Z}_p$-module, we usually abbreviate $M[1/p] = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and

$$\breve{M} = M \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p.$$

2. The Rapoport-Zink space

In this section we define our main object of study, a Rapoport-Zink space RZ of type $\mathrm{GU}(2, n-2)$, as a formal $\breve{\mathbb{F}}_p$-scheme parametrizing p-divisible groups with additional structure. This additional structure includes a quasi-isogeny to a fixed framing object, which determines distinguished closed subscheme $\mathrm{RZ}_\Lambda \subset \mathrm{RZ}$.

We describe the $\breve{\mathbb{F}}_p$-valued points of this subscheme in terms of lattices in a free $\breve{\mathbb{Z}}_p$-module, and then express RZ_Λ as a union of locally closed subschemes. The examination of these locally closed subschemes will occupy the rest of the paper.

2.1. **Framing objects.** For any p-divisible group X over $\breve{\mathbb{F}}_p$, let $D(X)$ be the covariant Dieudonné module. It is a $\breve{\mathbb{Z}}_p$-module endowed with semi-linear operators F and V satisfying $FV = VF = p$. Under the covariant conventions,

$$\mathrm{Lie}(X) \cong D(X)/VD(X). \tag{2.1.1}$$

If X has an action of $\mathcal{O}_E$, the induced action of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ on its Dieudonné module determines, by (1.2.1), a decomposition

$$D(X) = D(X)_0 \oplus D(X)_1.$$

Suppose X is a p-divisible group over an $\breve{\mathbb{F}}_p$-scheme S, equipped with an action $\mathcal{O}_E \to \mathrm{End}(X)$. As above, the induced action of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ on $\mathrm{Lie}(X)$ determines a decomposition

$$\mathrm{Lie}(X) = \mathrm{Lie}(X)_0 \oplus \mathrm{Lie}(X)_1 \tag{2.1.2}$$

as a sum of locally free $\mathcal{O}_S$-modules.

Definition 2.1.1. The action $\mathcal{O}_E \to \mathrm{End}(X)$ has *signature* (s_0, s_1) if the summands on the right hand side of (2.1.2) are locally free of ranks s_0 and s_1, respectively.

Definition 2.1.2. A polarization $\lambda_X : X \to X^\vee$ is *conjugate $\mathcal{O}_E$-linear* if $\lambda_X \circ x = \overline{x}^\vee \circ \lambda_X$ for all $x \in \mathcal{O}_E$.

There is an analogue of Serre's tensor construction for p-divisible groups. Suppose X is any p-divisible group over an $\breve{\mathbb{F}}_p$-scheme S, endowed with an action of $\mathcal{O}_E$. If Λ is any free $\mathcal{O}_E$-module of finite rank, one can form a new p-divisible group $\Lambda \otimes X$ with $\mathcal{O}_E$-action over S with functor of points

$$(\Lambda \otimes X)(T) = \Lambda \otimes X(T)$$

for any $T \to S$. Note that both tensor products over $\mathcal{O}_E$, but we typically omit this from the notation. Of course choosing an $\mathcal{O}_E$-basis of Λ identifies $\Lambda \otimes X$ with a product of $\mathrm{rank}_{\mathcal{O}_E}(\Lambda)$ copies of X.

Suppose further that X is endowed with a conjugate $\mathcal{O}_E$-linear polarization $\lambda_X : X \to X^\vee$, while Λ is endowed with a hermitian form $h : \Lambda \times \Lambda \to \mathcal{O}_E$. This data induces a conjugate $\mathcal{O}_E$-linear polarization

$$\lambda_{\Lambda \otimes X} : \Lambda \otimes X \to (\Lambda \otimes X)^\vee$$

constructed as follows. First note that if G is any p-divisible group over S endowed with an $\mathcal{O}_E$-action, there are isomorphisms of finite flat group schemes

$$\underline{\mathrm{Hom}}_{\overline{\mathcal{O}}_E}(G[p^k], \mathcal{O}_E \otimes \mu_{p^k}) \cong \underline{\mathrm{Hom}}(G[p^k], \mu_{p^k}) = G^\vee[p^k],$$

compatible as k varies, where the first isomorphism is defined by the trace map $\mathcal{O}_E \to \mathbb{Z}_p$, and the second is the definition of $G^\vee$. The subscript $\overline{\mathcal{O}}_E$

indicates $\mathcal{O}_E$-conjugate-linear homomorphisms. The principal polarization λ_X therefore induces, for every k, an isomorphism

$$\tilde{\lambda}_X : X[p^k] \to \underline{\mathrm{Hom}}_{\overline{\mathcal{O}}_E}(X[p^k], \mathcal{O}_E \otimes \mu_{p^k}).$$

The polarization $\lambda_{\Lambda\otimes X}$ is defined, on p^k-torsion, as the composition

$$\begin{array}{ccc} (\Lambda\otimes X)[p^k] & = & \Lambda\otimes X[p^k] \\ & & \downarrow \\ \underline{\mathrm{Hom}}_{\overline{\mathcal{O}}_E}(\Lambda\otimes X[p^k], \mathcal{O}_E\otimes\mu_{p^k}) & = & (\Lambda\otimes X)^\vee[p^k] \end{array}$$

in which the vertical arrow sends $a \otimes x \in \Lambda \otimes X[p^k]$ to the $\mathcal{O}_E$-conjugate linear homomorphism $b \otimes y \mapsto h(a,b) \cdot \tilde{\lambda}_X(x)(y)$.

Let $\mathbb{Y}$ be the p-divisible group of a supersingular elliptic curve over $\breve{\mathbb{F}}_p$. In other words, $\mathbb{Y}$ is the (unique up to isomorphism) connected p-divisible group of dimension 1 and height 2. Fix an action $\mathcal{O}_E \to \mathrm{End}(\mathbb{Y})$ of signature $(1,0)$, and a principal polarization $\lambda_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{Y}^\vee$. Any such polarization is necessarily conjugate $\mathcal{O}_E$-linear.

Denote by $\overline{\mathbb{Y}}$ the p-divisible group $\mathbb{Y}$, but now endowed with the conjugate action of $\mathcal{O}_E$. Thus $\overline{\mathbb{Y}}$ has signature $(0,1)$. The principal polarization $\lambda_{\mathbb{Y}}$ can be viewed as a principal polarization $\lambda_{\overline{\mathbb{Y}}} : \overline{\mathbb{Y}} \to \overline{\mathbb{Y}}^\vee$, which is again conjugate $\mathcal{O}_E$-linear.

Proposition 2.1.3. *Fix an integer $n \geqslant 2$.*

(1) *If Λ is a self-dual hermitian $\mathcal{O}_E$-lattice of rank n, the natural $\mathcal{O}_E$-action on $\Lambda\otimes\overline{\mathbb{Y}}$ has signature $(0,n)$, and the natural conjugate $\mathcal{O}_E$-linear polarization is principal.*

(2) *Conversely, suppose $\mathbb{X}$ is any p-divisible group over $\breve{\mathbb{F}}_p$, equipped with an $\mathcal{O}_E$-action of signature $(0,n)$. There exists a self-dual hermitian lattice Λ of rank n, and an $\mathcal{O}_E$-linear isomorphism $\mathbb{X} \cong \Lambda\otimes\overline{\mathbb{Y}}$. Moreover, if $\mathbb{X}$ is endowed with a $\mathcal{O}_E$-conjugate linear principal polarization, this isomorphism may be chosen to identify the polarizations on source and target.*

We remark that there is a unique (up to isometry) self-dual hermitian lattice Λ of rank n, and so the $\mathbb{X}$ in (2), whether polarized or not, is unique up to isomorphism.

Proof. The first claim is obvious from the definitions.

For the second claim, the signature condition on $\mathbb{X}$ implies that its Dieudonné module satisfies

$$VD(\mathbb{X})_0 = pD(\mathbb{X})_1 \quad \text{and} \quad VD(\mathbb{X})_1 = D(\mathbb{X})_0.$$

It follows that $\tau = p^{-1}F^2$ defines a σ^2-semi-linear automorphism of $D(\mathbb{X})_0$, whose fixed points $D(\mathbb{X})_0^{\tau=\mathrm{id}}$ form a free $\mathbb{Z}_{p^2} = \breve{\mathbb{Z}}_p^{\sigma^2=\mathrm{id}}$-module with

$$D(\mathbb{X})_0^{\tau=\mathrm{id}} \otimes_{\mathbb{Z}_{p^2}} \breve{\mathbb{Z}}_p = D(\mathbb{X})_0.$$

Choose a basis

$$y_1, \ldots, y_n \in D(\mathbb{X})_0^{\tau=\mathrm{id}} \subset D(\mathbb{X})_0.$$

If we define $x_i \in D(\mathbb{X})_1$ by $Vx_i = y_i$, then each pair x_i, y_i generates a $\breve{\mathbb{Z}}_p$-submodule $D^{(i)} \subset D(\mathbb{X})$ stable under F and V. The decomposition

$$D(\mathbb{X}) = D^{(1)} \oplus \cdots \oplus D^{(n)}$$

of Dieudonné modules corresponds to a decomposition $\mathbb{X} = X^{(1)} \times \cdots \times X^{(n)}$ of p-divisible groups, with each factor $\mathcal{O}_E$-linearly isomorphic to $\overline{\mathbb{Y}}$. In other words,

$$\mathbb{X} \cong \overline{\mathbb{Y}} \times \cdots \times \overline{\mathbb{Y}}. \tag{2.1.3}$$

If we also assume that $\mathbb{X}$ admits an $\mathcal{O}_E$-conjugate linear principal polarization, there is an induced perfect alternating pairing

$$\lambda_{\mathbb{X}} : D(\mathbb{X}) \times D(\mathbb{X}) \to \breve{\mathbb{Z}}_p$$

satisfying $\lambda_{\mathbb{X}}(Fx, y) = \lambda_{\mathbb{X}}(x, Vy)^\sigma$ and $\lambda_{\mathbb{X}}(\alpha x, y) = \lambda_{\mathbb{X}}(x, \bar{\alpha} y)$ for all $\alpha \in \mathcal{O}_E$ and $x, y \in D(\mathbb{X})$. Using these properties, we see that

$$\langle x, y \rangle \stackrel{\mathrm{def}}{=} p^{-1} \lambda_{\mathbb{X}}(x, Vy)$$

defines a $\mathbb{Z}_{p^2}$-valued skew-hermitian form on $D(\mathbb{X})_0^{\tau=\mathrm{id}}$ of unit determinant. Such a form admits a diagonal basis, and we assume that $y_1, \ldots, y_n$ is chosen in such a way. The polarization on the left hand side of (2.1.3) then identified with the the product of *some* principal polarizations on the factors of the right hand side. As the principal polarization on $\overline{\mathbb{Y}}$ is unique up to scaling by $\mathbb{Z}_p^\times$, we may choose the isomorphism (2.1.3) so that the given polarization on $\mathbb{X}$ matches the product of the principal polarization of $\overline{\mathbb{Y}}$ that we originally fixed.

Thus, whether $\mathbb{X}$ was polarized or not, we may take $\Lambda = \mathcal{O}_E^n$ with the hermitian form determined by the $n \times n$ identity matrix. □

Remark 2.1.4. Why have we expressed the second claim in Proposition 2.1.3 in such a peculiar way, when the proof actually shows that $\mathbb{X}$ is isomorphic to n copies of $\overline{\mathbb{Y}}$? One reason is that such an isomorphism is not canonical, but the presentation $\mathbb{X} \cong \Lambda \otimes \overline{\mathbb{Y}}$ is. Indeed, once such a presentation is known to exist, it is easy to see that

$$\Lambda \cong \mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X}),$$

in which the right hand side is endowed with the hermitian form characterized by the relation (3.1) of [11].

2.2. **The Rapoport-Zink space.** For the remainder of §2, we work with a fixed self-dual hermitian $\mathcal{O}_E$-lattice Λ of rank n. It is unique up to isomorphism.

We next define a GU$(2, n-2)$ Rapoport-Zink space RZ, parametrizing p-divisible groups with extra structure. This extra structure would normally include a quasi-isogeny to a fixed supersingular p-divisible group $\mathbb{X}$ (the *framing object*), endowed with an $\mathcal{O}_E$-action of signature $(2, n-2)$ and an $\mathcal{O}_E$-conjugate linear principal polarization. Such a framing object is unique up to quasi-isogeny by Proposition 1.15 of [16], but that result actually proves more: any such $\mathbb{X}$ is also quasi-isogenous to the p-divisible group $\Lambda \otimes \overline{\mathbb{Y}}$ of signature $(0, n)$, which can therefore also serve as a framing object. This is what we choose to do.

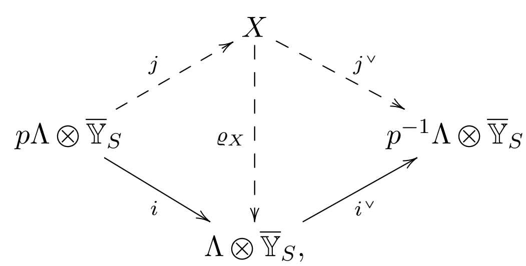
Definition 2.2.1. For an $\breve{\mathbb{F}}_p$-scheme S, denote by $\mathrm{RZ}(S)$ the set of isomorphism classes of triples $(X, \lambda_X, \varrho_X)$ in which

- X is a p-divisible group over S equipped with an $\mathcal{O}_E$-action of signature $(2, n-2)$,
- $\lambda_X : X \to X^\vee$ is a conjugate $\mathcal{O}_E$-linear principal polarization,
- $\varrho_X : X \dashrightarrow \Lambda \otimes \overline{\mathbb{Y}}_S$ is an $\mathcal{O}_E$-linear quasi-isogeny identifying polarizations up to $\mathbb{Q}_p^\times$-scaling.

The results of [15] show that the functor RZ is represented by a formal scheme over $\breve{\mathbb{F}}_p$, locally formally of finite type, and formally smooth of dimension $2n-4$. When no confusion can arise, we usually write $X \in \mathrm{RZ}(S)$ instead of $(X, \lambda_X, \varrho_X) \in \mathrm{RZ}(S)$.

For any point $X \in \mathrm{RZ}(S)$ there is a commutative diagram

(2.2.1)



in which the isogenies i and $i^\vee$ are induced by the inclusions $p\Lambda \subset \Lambda \subset p^{-1}\Lambda$, and the quasi-isogenies j and $j^\vee$ are uniquely determined by the commutativity. One can easily check that the triangle on the right is, as the notation suggests, obtained by dualizing the triangle on the left using the principal polarizations on X and $\Lambda \otimes \overline{\mathbb{Y}}_S$.

Definition 2.2.2. The *partial Rapoport-Zink space of signature* $(2, n-2)$ is the closed formal subscheme $\mathrm{RZ}_\Lambda \subset \mathrm{RZ}$ cut out by the following conditions:

- The quasi-isogeny $\varrho_X : X \dashrightarrow \mathbb{X}_S$ identifies the polarizations on source and target (not just up to scaling).
- The arrows j and $j^\vee$ in (2.2.1) are isogenies (not just quasi-isogenies).

Remark 2.2.3. By the argument of Lemma 4.2 of [17] (or §2.2 of [15]), the functor RZ_Λ on $\breve{\mathbb{F}}_p$-schemes is represented by a projective scheme (not just a formal scheme), denoted the same way.

Remark 2.2.4. Let $G = \mathrm{GU}(\Lambda)$, a reductive group scheme over $\breve{\mathbb{Z}}_p$ whose group of $\mathbb{Q}_p$-points acts on RZ. Indeed, any $\gamma \in G(\mathbb{Q}_p)$ determines an $\mathcal{O}_E$-linear quasi-isogeny from $\mathbb{X} = \Lambda \otimes \overline{\mathbb{Y}}$ to itself, respecting the polarization up to $\mathbb{Q}_p^\times$-scaling, and hence defines an automorphism of RZ by

$$(X, \lambda_X, \varrho_X) \mapsto (X, \lambda_X, \gamma \circ \varrho_X).$$

The partial Rapoport-Zink space of Definition 2.2.2 satisfies

$$\mathrm{RZ}(\breve{\mathbb{F}}_p) = \bigcup_{\gamma \in G(\mathbb{Q}_p)} \gamma \cdot \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p), \tag{2.2.2}$$

and so serves as a kind of approximate fundamental domain for this action.

The equality (2.2.2) should be thought of as a signature $(2, n-2)$ analogue of the crucial Lemma 2.1 of [16], and it is possible to give a direct (if quite technical) linear algebraic proof along the same lines. We omit this argument because (2.2.2) will fall out during the proof of Corollary 6.3.5 below, but that argument makes use of the results of [19], and so is of a far less elementary nature than the approach of [16].

2.3. **Description of the closed points.** In this subsection we give a concrete description of the closed points of the partial Rapoport-Zink space RZ_Λ of Definition 2.2.2 in terms of lattices. Endow

$$\breve{\Lambda} = \Lambda \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$$

with the σ-semi-linear automorphism $\Phi = \mathrm{id} \otimes \sigma$. Recalling (1.2.1), this operator interchanges the direct summands $\breve{\Lambda}_0$ and $\breve{\Lambda}_1$.

The hermitian form h on Λ extends $\breve{\mathbb{Z}}_p$-bilinearly to a pairing

$$h : \breve{\Lambda} \times \breve{\Lambda} \to \mathcal{O}_{\breve{E}}$$

satisfying $h(\Phi x, \Phi y) = h(x, y)^\sigma$, where the σ on the right is the Frobenius on the second factor of $\mathcal{O}_{\breve{E}} = \mathcal{O}_E \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$. We further extend h to a $\breve{\mathbb{Q}}_p$-bilinear pairing on $\breve{\Lambda}[1/p]$.

Proposition 2.3.1. *There is a bijection $X \mapsto L(X)$ from $\mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$ to the set of self-dual $\mathcal{O}_{\breve{E}}$-lattices $L \subset \breve{\Lambda}[1/p]$ lying between $p\breve{\Lambda}$ and $p^{-1}\breve{\Lambda}$, and satisfying*

$$pL_0 \overset{n-2}{\subset} \Phi^{-1} L_1 \overset{2}{\subset} L_0 \quad \textit{and} \quad pL_1 \overset{2}{\subset} p\Phi^{-1} L_0 \overset{n-2}{\subset} L_1.$$

Proof. As this is the routine identification (see especially Proposition 1.10 of [16]) of points of $\mathrm{RZ}(\breve{\mathbb{F}}_p)$ with lattices in the isocrystal of the framing object, we only explain how to account for our unusual choice of framing object.

Fix an $\mathcal{O}_{\breve{E}}$-linear isomorphism

$$D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}},$$

and use this to identify

$$D(\Lambda \otimes \overline{\mathbb{Y}}) \cong \breve{\Lambda} \otimes_{\mathcal{O}_{\breve{E}}} D(\overline{\mathbb{Y}}) \cong \breve{\Lambda}. \tag{2.3.1}$$

This identification is only well-defined up to scaling by $\mathcal{O}_{\breve{E}}^\times$, but as we are only interested in viewing lattices in the left hand side as lattices in the right hand side, this ambiguity is harmless.

The signature $(0,1)$ condition on $\overline{\mathbb{Y}}$ implies that

$$VD(\overline{\mathbb{Y}})_1 = D(\overline{\mathbb{Y}})_0 \quad \text{and} \quad VD(\overline{\mathbb{Y}})_0 = pD(\overline{\mathbb{Y}})_1.$$

Using this, one checks that the identification (2.3.1) restricts to identifications

$$\begin{array}{ccc} \breve{\Lambda}_0 & = & D(\Lambda\otimes\overline{\mathbb{Y}})_0 \\ {\scriptstyle p\Phi}\downarrow & & \downarrow{\scriptstyle F} \\ \breve{\Lambda}_1 & = & D(\Lambda\otimes\overline{\mathbb{Y}})_1 \end{array} \qquad \begin{array}{ccc} \breve{\Lambda}_1 & = & D(\Lambda\otimes\overline{\mathbb{Y}})_1 \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle F} \\ \breve{\Lambda}_0 & = & D(\Lambda\otimes\overline{\mathbb{Y}})_0 \end{array} \tag{2.3.2}$$

in which the diagrams commute up to scaling by $\breve{\mathbb{Z}}_p^\times$. In fact, one can choose the trivialization of $D(\overline{\mathbb{Y}})$ so that they commute without any scaling factor, but we have no need to do this.

Given a point $X \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$, we use the quasi-isogeny $\varrho_X : X \dashrightarrow \Lambda\otimes\overline{\mathbb{Y}}$ to define

$$L(X) = D(X) \overset{\varrho_X}{\subset} D(\Lambda\otimes\overline{\mathbb{Y}})[1/p] \overset{(2.3.1)}{=} \breve{\Lambda}[1/p]. \tag{2.3.3}$$

The signature condition on the $\mathcal{O}_E$-action is equivalent to

$$pD(X)_0 \overset{n-2}{\subset} VD(X)_1 \overset{2}{\subset} D(X)_0 \quad \text{and} \quad pD(X)_1 \overset{2}{\subset} VD(X)_0 \overset{n-2}{\subset} D(X)_1,$$

which, using (2.3.2), translates to the inclusions

$$pL_0 \overset{n-2}{\subset} \Phi^{-1}L_1 \overset{2}{\subset} L_0 \quad \text{and} \quad pL_1 \overset{2}{\subset} p\Phi^{-1}L_0 \overset{n-2}{\subset} L_1.$$

This defines the desired bijection. □

Remark 2.3.2. A more functorial characterization of $L(X)$, not requiring a choice of (2.3.1), is

$$L(X) = \mathrm{Hom}_{\mathcal{O}_{\breve{E}}}(D(\overline{\mathbb{Y}}), D(X)), \tag{2.3.4}$$

viewed as a lattice in

$$\mathrm{Hom}_{\mathcal{O}_{\breve{E}}}(D(\overline{\mathbb{Y}}), D(\Lambda\otimes\overline{\mathbb{Y}}))[1/p] = \breve{\Lambda}[1/p]$$

using the quasi-isogeny $\varrho_X : X \dashrightarrow \Lambda\otimes\overline{\mathbb{Y}}$.

We rewrite Proposition 2.3.1 in terms of lattices in $\breve{\Lambda}_0[1/p]$. The pairing

$$b : \breve{\Lambda}_0 \times \breve{\Lambda}_0 \to \breve{\mathbb{Z}}_p \tag{2.3.5}$$

defined by $b(x,y) = h(x,\Phi y) \in e_0\mathcal{O}_{\breve{E}} = \breve{\mathbb{Z}}_p$ is linear in the first variable, σ-linear in the second, and satisfies $b(\Phi^2 x, y) = b(y,x)^\sigma$. Any $\breve{\mathbb{Z}}_p$-lattice $L_0 \subset \breve{\Lambda}_0[1/p]$ has a right dual lattice

$$L_0^* = \{x \in \breve{\Lambda}_0[1/p] : b(L_0, x) \subset \breve{\mathbb{Z}}_p\}, \tag{2.3.6}$$

and one can easily verify the relations

$$L_0^{**} = \Phi^{-2}L_0 \quad \text{and} \quad \Phi^2(L_0^*) = (\Phi^2 L_0)^*. \tag{2.3.7}$$

If $L \subset \breve{\Lambda}[1/p]$ is an $\mathcal{O}_{\breve{E}}$-lattice self-dual under h, then

$$\Phi^{-1} L_1 = L_0^*. \tag{2.3.8}$$

In particular, $\breve{\Lambda}_0 = \Phi^{-1}\breve{\Lambda}_1 = \breve{\Lambda}_0^*$.

Corollary 2.3.3. *There is a bijection*

$$\mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p) \cong \left\{ \begin{array}{c} \breve{\mathbb{Z}}_p\text{-lattices } L_0 \subset \breve{\Lambda}_0[1/p] \\ \text{satisfying} \\ p\breve{\Lambda}_0 \subset L_0^* \overset{2}{\subset} L_0 \subset p^{-1}\breve{\Lambda}_0 \end{array} \right\},$$

under which $X \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$ *corresponds, as in* (2.3.3)*, to*

$$L_0^* = VD(X)_1 \quad \text{and} \quad L_0 = D(X)_0.$$

Proof. Using (2.3.8), Proposition 2.3.1 is equivalent to

$$\mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p) \cong \left\{ \begin{array}{c} \breve{\mathbb{Z}}_p\text{-lattices} \\ L_0 \subset \breve{\Lambda}_0[1/p] \end{array} : \begin{array}{c} pL_0 \subset L_0^* \overset{2}{\subset} L_0 \\ p\breve{\Lambda}_0 \subset L_0 \subset p^{-1}\breve{\Lambda}_0 \end{array} \right\}.$$

The two chains of inclusions on the right hand side are equivalent to the single chain of inclusions in the statement of the corollary. □

2.4. **Decomposition by locally closed subsets.** Now let X be the universal p-divisible group over RZ_Λ. The bijection of Proposition 2.3.1 associates to every $s \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$ an inclusion of $\mathcal{O}_{\breve{E}}$-lattices

$$p\breve{\Lambda} \subset L(X_s),$$

and hence a map of $\breve{\mathbb{F}}_p$-vector spaces

$$p\breve{\Lambda}/p^2\breve{\Lambda} \to L(X_s)/pL(X_s). \tag{2.4.1}$$

One can use Grothendieck-Messing crystals to construct a morphism of vector bundles interpolating these maps as s varies.

If S is a scheme on which p is locally nilpotent, covariant Grothendieck-Messing theory functorially associates to any p-divisible group G over S a short exact sequence

$$0 \to \mathrm{Fil}^0\mathcal{D}(G) \to \mathcal{D}(G) \to \mathrm{Lie}(G) \to 0 \tag{2.4.2}$$

of locally free $\mathcal{O}_S$-modules. When $S = \mathrm{Spec}(\breve{\mathbb{F}}_p)$, this sequence is canonically identified with

$$0 \to \frac{VD(G)}{pD(G)} \to \frac{D(G)}{pD(G)} \to \frac{D(G)}{VD(G)} \to 0.$$

Over RZ_Λ there is a universal diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{j}{\nearrow} & & \overset{j^\vee}{\searrow} & \\ p\Lambda\otimes\overline{\mathbb{Y}} & & \Big\downarrow{\scriptstyle \varrho_X} & & p^{-1}\Lambda\otimes\overline{\mathbb{Y}}_S \\ & \underset{i}{\searrow} & & \underset{i^\vee}{\nearrow} & \\ & & \Lambda\otimes\overline{\mathbb{Y}} & & \end{array} \tag{2.4.3}$$

(where we now view $\overline{\mathbb{Y}}$ as a constant p-divisible group over RZ_Λ). Mimicking (2.3.4), we define a vector bundle

$$\mathcal{L} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(X)) \tag{2.4.4}$$

with $\mathcal{O}_E$-action on RZ_Λ. The isogeny j determines an inclusion $p\Lambda \subset \mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, X)$, and hence an $\mathcal{O}_E$-linear morphism

$$p\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathrm{RZ}_\Lambda} \to \mathcal{L}$$

whose fiber at any $s \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$ is canonically identified with (2.4.1). It restricts to morphisms of vector bundles

$$p\breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}_\Lambda} \to \mathcal{L}_0 \quad \text{and} \quad p\breve{\Lambda}_1 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}_\Lambda} \to \mathcal{L}_1. \tag{2.4.5}$$

Remark 2.4.1. A choice of isomorphism $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$ determines an isomorphism $\mathcal{D}(\overline{\mathbb{Y}}) \cong \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathrm{RZ}_\Lambda}$, and hence also an isomorphism

$$\mathcal{L} \cong \mathcal{D}(X).$$

This is well-defined by to scaling by $(\mathcal{O}_{\breve{E}}/p\mathcal{O}_{\breve{E}})^\times$.

Definition 2.4.2. For any $k \geqslant 1$, define

$$\mathrm{RZ}_\Lambda^{\leqslant k} \subset \mathrm{RZ}_\Lambda$$

to be the largest closed subscheme over which the first map in (2.4.5) has rank $\leqslant k-1$. More precisely, $\mathrm{RZ}_\Lambda^{\leqslant k}$ is the closed subscheme cut out by the vanishing of

$$\bigwedge^k (p\breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}_\Lambda}) \to \bigwedge^k \mathcal{L}_0.$$

We further define $\mathrm{RZ}_\Lambda^k = \mathrm{RZ}_\Lambda^{\leqslant k} \smallsetminus \mathrm{RZ}_\Lambda^{\leqslant k-1}$, so that we have a decomposition

$$\mathrm{RZ}_\Lambda^{\mathrm{red}} = \bigsqcup_{k \geqslant 1} \mathrm{RZ}_\Lambda^{k,\mathrm{red}}$$

into locally closed subschemes.

Definition 2.4.3. Two $\breve{\mathbb{Z}}_p$-lattices L and L' in an n-dimensional $\breve{\mathbb{Q}}_p$-vector space (e.g. $\breve{\Lambda}_0[1/p]$), have *relative position invariant*

$$\mathrm{inv}(L, L') = (a_1, \dots, a_n) \in \mathbb{Z}^n$$

if there is a $\breve{\mathbb{Z}}_p$-basis $x_1, \dots, x_n \in L'$ such that $p^{a_1}x_1, \dots, p^{a_n}x_n$ is a basis of L, and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$.

We now give a concrete description of the $\breve{\mathbb{F}}_p$-valued points of RZ^k_Λ, in terms of lattices in $\breve{\Lambda}_0[1/p]$. In the notation of Definition 2.4.3, the bijection of Corollary 2.3.3 becomes

$$\text{(2.4.6)} \qquad \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p) \cong \left\{ \begin{array}{c} \breve{\mathbb{Z}}_p\text{-lattices} \\ L_0 \subset \breve{\Lambda}_0[1/p] \end{array} : \begin{array}{c} p\breve{\Lambda}_0 \subset L_0 \subset p^{-1}\breve{\Lambda}_0 \\ \mathrm{inv}(L_0^*, L_0) = (1,1,0,\ldots,0) \end{array} \right\}.$$

Proposition 2.4.4. *Under the above bijection, for any $1 \leqslant k \leqslant \lfloor n/2 \rfloor$ we have*

$$\mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p) \cong \{L_0 \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p) : \mathrm{inv}(L_0, \breve{\Lambda}_0) = \lambda_k\},$$

in which

$$\lambda_k = (\underbrace{1,\ldots,1}_{k-1 \text{ times}}, 0, \ldots, 0, \underbrace{-1,\ldots,-1}_{k \text{ times}}) \in \mathbb{Z}^n.$$

If $k > \lfloor n/2 \rfloor$ then $\mathrm{RZ}^k_\Lambda = \varnothing$.

Proof. Fix any $L_0 \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$. The conditions of (2.4.6) imply

$$\mathrm{inv}(L_0, \breve{\Lambda}_0) = (b_1, \ldots, b_n)$$

with all $-1 \leqslant b_i \leqslant 1$, and dualizing shows that

$$\mathrm{inv}(L_0^*, \breve{\Lambda}_0) = \mathrm{inv}(L_1, \breve{\Lambda}_1) = (-b_n, \ldots, -b_1).$$

These imply the equalities of lattices

$$\bigwedge^n L_0^* = p^{-(b_1+\cdots+b_n)} \bigwedge^n \breve{\Lambda}_0 = p^{-2(b_1+\cdots+b_n)} \bigwedge^n L_0$$

in $\bigwedge^n \breve{W}_0$. Combining this with $\mathrm{inv}(L_0^*, L_0) = (1,1,0,\ldots,0)$ shows that

$$b_1 + \cdots + b_n = -1,$$

which is equivalent to $\mathrm{inv}(L_0, \breve{\Lambda}_0) = \lambda_k$ for some $1 \leqslant k \leqslant \lfloor n/2 \rfloor$.

The fiber at $L_0 \in \mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p)$ of the first morphism in (2.4.5) is identified with the linear map

$$\frac{p\breve{\Lambda}_0}{p^2\breve{\Lambda}_0} \to \frac{L_0}{pL_0}$$

of rank $\dim_{\breve{\mathbb{F}}_p}(\breve{\Lambda}_0/(\breve{\Lambda}_0 \cap L_0)) = k-1$, proving that $L_0 \in \mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$. ☐

3. Enhancing the moduli problem

We continue to study the partial Rapoport-Zink space RZ_Λ associated to a self-dual hermitian $\mathcal{O}_E$-lattice Λ of rank $n \geqslant 2$, and the locally closed subset $\mathrm{RZ}^k_\Lambda \subset \mathrm{RZ}_\Lambda$ determined by a fixed integer $1 \leqslant k \leqslant \lfloor n/2 \rfloor$. Our goal is to construct a morphism from RZ^k_Λ to a Deligne-Lustzig variety. The fibers of this morphism will then be studied in §4.

3.1. **New moduli spaces.** In this subsection we will construct a commutative diagram

$$\begin{array}{ccccc} \mathrm{RZ}^k_\Lambda & \xleftarrow{\pi_0} & \mathbf{RZ}^k_\Lambda & \xrightarrow{\pi_1} & Y^k_\Lambda \\ \downarrow & & \downarrow & & \| \\ \mathrm{RZ}^{\leqslant k}_\Lambda & \xleftarrow{\pi_0} & \mathbf{RZ}^{\leqslant k}_\Lambda & \xrightarrow{\pi_1} & Y^k_\Lambda \end{array} \tag{3.1.1}$$

of $\breve{\mathbb{F}}_p$-schemes, in which the schemes in the bottom row are projective, the vertical arrows are open immersions, and the vertical arrow on the left is that of Definition 2.4.2. The definitions are somewhat elaborate; they are concocted so that the π_0 in the top row induces an isomorphism of underlying reduced schemes (Proposition 3.3.2), while the reduced scheme underlying Y^k_Λ is isomorphic to a Deligne-Lusztig variety (Theorem 3.4.5)

Define $\widetilde{Y}^k_\Lambda$ to be the $\breve{\mathbb{F}}_p$-scheme whose S-valued points are commutative diagrams

$$\begin{array}{c} p\Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{a} H \xrightarrow{d} G \xrightarrow{d^\vee} H^\vee \xrightarrow{a^\vee} p^{-1}\Lambda\otimes\overline{\mathbb{Y}}_S \\ p\Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{i} \Lambda\otimes\overline{\mathbb{Y}}_S,\quad H \xrightarrow{b} \Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{b^\vee} H^\vee,\quad \Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{i^\vee} p^{-1}\Lambda\otimes\overline{\mathbb{Y}}_S \end{array} \tag{3.1.2}$$

in which H and G are p-divisible groups over S equipped with $\mathcal{O}_E$-actions, H has signature $(1, n-1)$, G has signature $(2k, n-2k)$ and is equipped with an $\mathcal{O}_E$-linear principal polarization, the isogenies a, b, and c are $\mathcal{O}_E$-linear with $\mathrm{ht}(a) = 2n - 2k + 1$ and $\mathrm{ht}(b) = \mathrm{ht}(d) = 2k - 1$, and

$$\ker(d^\vee \circ d : H \to H^\vee) \subset H[p]. \tag{3.1.3}$$

Define $\widetilde{\mathbf{RZ}}^{\leqslant k}_\Lambda$ to be the $\breve{\mathbb{F}}_p$-scheme whose S-valued points consist of a point (2.4.3) of $\mathrm{RZ}^{\leqslant k}_\Lambda(S)$, a point (3.1.2) of $\widetilde{Y}^k_\Lambda(S)$, and an $\mathcal{O}_E$-linear isogeny $c : H \to X$ making the diagram

$$\begin{array}{c} p\Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{j} X \xrightarrow{j^\vee} p^{-1}\Lambda\otimes\overline{\mathbb{Y}}_S,\quad H \xrightarrow{c} X \xrightarrow{c^\vee} H^\vee \\ p\Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{a} H \xrightarrow{d} G \xrightarrow{d^\vee} H^\vee \xrightarrow{a^\vee} p^{-1}\Lambda\otimes\overline{\mathbb{Y}}_S \\ p\Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{i} \Lambda\otimes\overline{\mathbb{Y}}_S,\quad H \xrightarrow{b} \Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{b^\vee} H^\vee,\quad \Lambda\otimes\overline{\mathbb{Y}}_S \xrightarrow{i^\vee} p^{-1}\Lambda\otimes\overline{\mathbb{Y}}_S \end{array} \tag{3.1.4}$$

commute.

The moduli spaces we have constructed are related to RZ_Λ by morphisms

$$\mathrm{RZ}^{\leqslant k}_\Lambda \xleftarrow{\pi_0} \widetilde{\mathbf{RZ}}^{\leqslant k}_\Lambda \xrightarrow{\pi_1} \widetilde{Y}^k_\Lambda,$$

where π_0 exacts from (3.1.4) the diagram (2.4.3), while π_1 extracts (3.1.2). To define the bottom row of (3.1.1), we single out certain open and closed subschemes of the these moduli spaces. This will require the following general lemma concerning the vector bundles of (2.4.2)

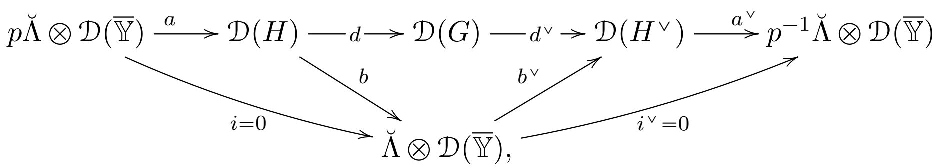
Lemma 3.1.1. *Suppose $\phi : G \to H$ is an isogeny of p-divisible groups over a scheme S satisfying $p\mathcal{O}_S = 0$. If the kernel of ϕ is annihilated by p, then the cokernel of*

$$\phi : \mathcal{D}(G) \to \mathcal{D}(H)$$

is a locally free $\mathcal{O}_S$-module of rank $\mathrm{ht}(\phi)$. In particular, its kernel and image are local direct summands of $\mathcal{D}(G)$ and $\mathcal{D}(H)$, respectively.

Proof. This is well-known. See Proposition 4.6 of [17], for example. □

Applying the functor $\mathcal{D}$ to the universal diagram over $\widetilde{Y}^k_\Lambda$ yields a diagram of vector bundles

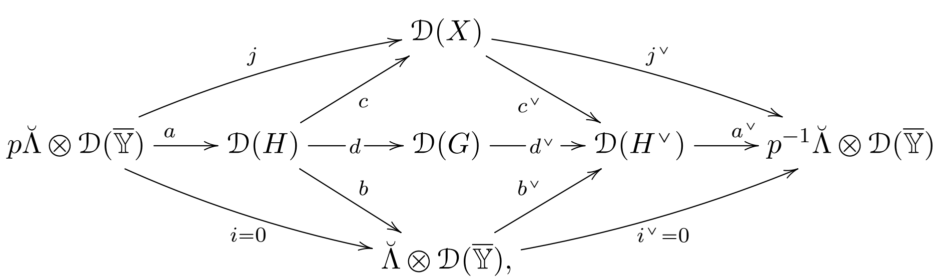


where the tensor products are over $\mathcal{O}_{\breve{E}}$. Define an open and closed subscheme $Y^k_\Lambda \subset \widetilde{Y}^k_\Lambda$ by imposing the conditions

$$\begin{aligned} \mathrm{rank}(b : \mathcal{D}(H)_0 \to \breve{\Lambda}_0 \otimes \mathcal{D}(\overline{\mathbb{Y}})_0) &= n-k+1 \\ \mathrm{rank}(b : \mathcal{D}(H)_1 \to \breve{\Lambda}_1 \otimes \mathcal{D}(\overline{\mathbb{Y}})_1) &= n-k \\ \mathrm{rank}(d : \mathcal{D}(H)_0 \to \mathcal{D}(G)_0) &= n-2k+1 \\ \mathrm{rank}(d^\vee : \mathcal{D}(G)_1 \to \mathcal{D}(H^\vee)_1) &= n-2k+1, \end{aligned} \tag{3.1.5}$$

where *rank* means the rank of the image as a vector bundle. Note we are using Lemma 3.1.1: the image of every arrow in the diagram is a locally free sheaf, so the ranks here are well-defined locally constant functions on $\widetilde{Y}^k_\Lambda$.

Similarly, over $\widetilde{\mathbf{RZ}}^{\leqslant k}_\Lambda$ there is a diagram of vector bundles



and we denote by $\mathbf{RZ}^{\leqslant k}_\Lambda \subset \widetilde{\mathbf{RZ}}^{\leqslant k}_\Lambda$ the open and closed subscheme defined by imposing the conditions (3.1.5) as well as

$$\begin{aligned} \mathrm{rank}(c : \mathcal{D}(H)_0 \to \mathcal{D}(X)_0) &= n-k \\ \mathrm{rank}(c : \mathcal{D}(H)_1 \to \mathcal{D}(X)_1) &= n-k+1. \end{aligned} \tag{3.1.6}$$

The above definitions complete the construction of the bottom row of (3.1.1). All three of the schemes appearing there are projective, by the same reasoning as in Remark 2.2.3.

The decomposition $\mathrm{RZ}_\Lambda^{\leqslant k} = \mathrm{RZ}_\Lambda^1 \sqcup \cdots \sqcup \mathrm{RZ}_\Lambda^k$ of §2.4 induces a decomposition

$$\mathbf{RZ}_\Lambda^{\leqslant k} = \mathbf{RZ}_\Lambda^1 \sqcup \cdots \sqcup \mathbf{RZ}_\Lambda^k,$$

in which the locally closed subscheme

$$\mathbf{RZ}_\Lambda^i \overset{\mathrm{def}}{=} \mathrm{RZ}_\Lambda^i \times_{\mathrm{RZ}_\Lambda^{\leqslant k}} \mathbf{RZ}_\Lambda^{\leqslant k}$$

is the preimage of RZ_Λ^i under $\pi_0 : \mathbf{RZ}_\Lambda^{\leqslant k} \to \mathrm{RZ}_\Lambda^{\leqslant k}$. Taking $i = k$, we obtain the top row of (3.1.1).

3.2. **Description of the closed points.** We now provide a description of the closed points of the schemes in (3.1.1), analogous the description of $\mathrm{RZ}_\Lambda^k(\breve{\mathbb{F}}_p)$ from Proposition 2.4.4.

Proposition 3.2.1. *There is a bijection from $Y_\Lambda^k(\breve{\mathbb{F}}_p)$ to the set of pairs of $\breve{\mathbb{Z}}_p$-lattices (M_0, N_0) in $\breve{\Lambda}_0[1/p]$ such that*

$$p\breve{\Lambda}_0 \overset{k-1}{\subset} pM_0^* \overset{1}{\subset} pN_0 \overset{n-2k}{\subset} N_0^* \overset{1}{\subset} M_0 \overset{k-1}{\subset} \breve{\Lambda}_0.$$

Here M_0^* and N_0^* are the right duals, as in (2.3.6)*, of M_0 and N_0 with respect to the pairing* (2.3.5)*. Under this bijection:*

(1) *A point of $Y_\Lambda^k(\breve{\mathbb{F}}_p)$, represented by a diagram* (3.1.2)*, corresponds to*

$$M_0^* = VD(H^\vee)_1 \qquad N_0^* = VD(H)_1$$
$$M_0 = D(H)_0 \qquad N_0 = D(H^\vee)_0$$

all viewed as lattices in $\breve{\Lambda}_0[1/p]$ as in (2.3.3).

(2) *The points of $\mathbf{RZ}_\Lambda^{\leqslant k}(\breve{\mathbb{F}}_p)$ above a given pair (M_0, N_0) are in bijection with $\breve{\mathbb{Z}}_p$-lattices $L_0 \subset \breve{\Lambda}_0[1/p]$ satisfying*

$$L_0^* \overset{2}{\subset} L_0 \quad \textit{and} \quad M_0 \overset{k}{\subset} L_0 \overset{k-1}{\subset} N_0.$$

(3) *The points of $\mathbf{RZ}_\Lambda^k(\breve{\mathbb{F}}_p)$ above a given pair (M_0, N_0) are in bijection with $\breve{\mathbb{Z}}_p$-lattices $L_0 \subset \breve{\Lambda}_0[1/p]$ satisfying the conditions of (2) and, in the notation of Proposition 2.4.4,* $\mathrm{inv}(L_0, \breve{\Lambda}_0) = \lambda_k$.

Proof. An element of $Y_\Lambda^k(\breve{\mathbb{F}}_p)$, corresponding to a diagram (3.1.2) of p-divisible groups over $\breve{\mathbb{F}}_p$ satisfying (3.1.5), determines inclusions of Dieudonné modules

$$\begin{array}{ccccccccc} p\breve{\Lambda} \otimes D(\overline{\mathbb{Y}}) & \longrightarrow & D(H) & \xrightarrow{2k-1} & D(G) & \xrightarrow{2k-1} & D(H^\vee) & \longrightarrow & p^{-1}\breve{\Lambda} \otimes D(\overline{\mathbb{Y}}) \\ & \searrow^{2n} & & \searrow^{2k-1} & & \nearrow^{2k-1} & & \nearrow^{2n} & \\ & & & & \breve{\Lambda} \otimes D(\overline{\mathbb{Y}}) & & & & \end{array}$$

of the indicated colengths, and with $pD(H^\vee) \subset D(H)$. Here all tensor products are over $\mathcal{O}_{\breve{E}}$.

The following lemma shows that, at least on the level of $\breve{\mathbb{F}}_p$-points, the p-divisible group G in the moduli problem defining Y_Λ^k can be recovered from H and $H^\vee$. In some sense G plays only an auxiliary role, imposing constraints on the polarization $H \to H^\vee$.

Lemma 3.2.2. *We have* $D(G) = D(H^\vee)_0 \oplus D(H)_1$.

Proof. The final two conditions in (3.1.5) imply that the inclusions

$$D(H)_0 \subset D(G)_0 \quad \text{and} \quad D(G)_1 \subset D(H^\vee)_1$$

each have colength $2k-1$. Comparing with the colengths in the diagram above, we deduce that the inclusions

$$D(H)_1 \subset D(G)_1 \quad \text{and} \quad D(G)_0 \subset D(H^\vee)_0$$

are equalities. □

Using (2.3.1), we view the Dieudonné modules in the diagram above as $\mathcal{O}_{\breve{E}}$-lattices

(3.2.1)
$$\begin{array}{ccccccccc} p\breve{\Lambda} & \longrightarrow & M & \longrightarrow & N_0 \oplus M_1 & \longrightarrow & N & \longrightarrow & p^{-1}\breve{\Lambda} \\ & \searrow & & \searrow & & \nearrow & & \nearrow & \\ & & & & \breve{\Lambda} & & & & \end{array}$$

in the hermitian space $\breve{\Lambda}[1/p]$, with M and N dual to one another. As in (2.3.4), one can characterize these lattices, without fixing (2.3.1), by

(3.2.2) $$M = \mathrm{Hom}_{\mathcal{O}_{\breve{E}}}(D(\overline{\mathbb{Y}}), D(H)) \quad \text{and} \quad N = \mathrm{Hom}_{\mathcal{O}_{\breve{E}}}(D(\overline{\mathbb{Y}}), D(H^\vee))$$

viewed as lattices in

$$\mathrm{Hom}_{\mathcal{O}_{\breve{E}}}(D(\overline{\mathbb{Y}}), \Lambda \otimes D(\overline{\mathbb{Y}}))[1/p] \cong \breve{\Lambda}[1/p].$$

As M and N are dual to one another, each of $M_0 \oplus N_1$ and $N_0 \oplus M_1$ is self-dual under the hermitian form on $\breve{\Lambda}[1/p]$. Hence, by (2.3.8),

(3.2.3) $$\Phi^{-1} N_1 = M_0^* \quad \text{and} \quad \Phi^{-1} M_1 = N_0^*.$$

The signature $(1, n-1)$ conditions on H and $H^\vee$ imply

$$VD(H)_1 \overset{1}{\subset} D(H)_0 \quad \text{and} \quad VD(H^\vee)_1 \overset{1}{\subset} D(H^\vee)_0.$$

Using (2.3.2), these translate to

$$N_0^* \overset{(3.2.3)}{=} \Phi^{-1} M_1 \overset{1}{\subset} M_0 \quad \text{and} \quad M_0^* \overset{(3.2.3)}{=} \Phi^{-1} N_1 \overset{1}{\subset} N_0.$$

Similarly, the signature $(2k, n-2k)$ condition on G implies

$$pD(H^\vee)_0 = pD(G)_0 \overset{n-2k}{\subset} VD(G)_1 = VD(H)_1,$$

(the outer equalities are by Lemma 3.2.2), which translates to

$$pN_0 \overset{n-2k}{\subset} \Phi^{-1} M_1 = N_0^*.$$

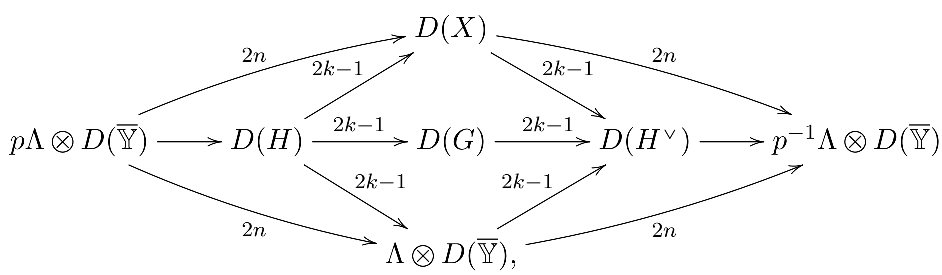
The first condition in (3.1.5) implies $D(H)_0 \subset \breve{\Lambda}_0 \otimes D(\overline{\mathbb{Y}})_0$, with colength $k-1$, which translates to $M_0 \subset \breve{\Lambda}_0$ with colength $k-1$, and dualizing shows

$$\breve{\Lambda}_0 = \Phi^{-1}\breve{\Lambda}_1 \overset{(2.3.8)}{=} \breve{\Lambda}_0^* \overset{k-1}{\subset} M_0^*.$$

All of this shows that the pair (M_0, N_0) satisfies the chain of inclusions in the statement of the proposition.

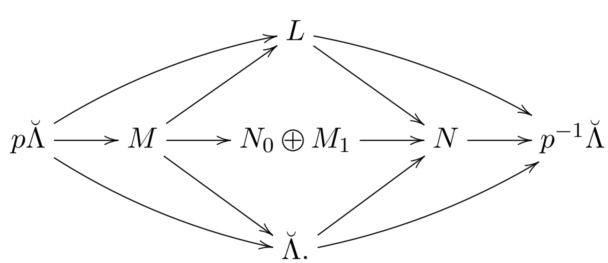
This process can be reversed. Starting from a pair (M_0, N_0) one uses (3.2.3) to define M_1 and N_1, thereby obtaining a diagram (3.2.1) of lattices in $\breve{\Lambda}[1/p]$. Converting these to lattices in $D(\mathbb{X})[1/p]$ using (2.3.1), one finds inclusions of Dieudonné modules whose corresponding p-divisible groups define a point of $Y^k_\Lambda(\breve{\mathbb{F}}_p)$.

The analysis of $\mathbf{RZ}^{\leqslant k}_\Lambda$ is entirely similar. A point of $\mathbf{RZ}^{\leqslant k}_\Lambda(\breve{\mathbb{F}}_p)$, corresponding to a diagram of p-divisible groups (3.1.4), determines inclusions of Dieudonné modules



and we use (2.3.1) to convert these into $\mathcal{O}_{\breve{E}}$-lattices

(3.2.4)



The first condition in (3.1.6) implies $D(H)_0 \subset D(X)_0$ with colength k, which translates to $M_0 \subset L_0$ with colength k. As the inclusion $L_0 \subset N_0$ is obvious, and $L_0^* \subset L_0$ with colength 2 by (2.4.6), the triple (L_0, M_0, N_0) satisfies the properties stated in (2).

The description of $\mathbf{RZ}^k_\Lambda$ in part (3) follows immediately from Proposition 2.4.4 and the description of $\mathbf{RZ}^{\leqslant k}_\Lambda$, completing the proof of Proposition 3.2.1. □

Corollary 3.2.3. *The map* $\pi_0 : \mathbf{RZ}^k_\Lambda \to \mathrm{RZ}^k_\Lambda$ *is bijective on* $\breve{\mathbb{F}}_p$*-points.*

Proof. Recalling the bijection

$$\mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p) \cong \left\{ \begin{array}{c} \breve{\mathbb{Z}}_p\text{-lattices} \\ L_0 \subset \breve{\Lambda}_0[1/p] \end{array} : \begin{array}{c} \mathrm{inv}(L_0^*, L_0) = (1,1,0,\ldots,0) \\ \mathrm{inv}(L_0, \breve{\Lambda}_0) = \lambda_k \end{array} \right\}.$$

of Proposition 2.4.4, fix a point $L_0 \in \mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$. If $(L_0, M_0, N_0) \in \mathbf{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$ lies above it (using the identifications of Proposition 3.2.1), then $M_0 \subset L_0 \cap \breve{\Lambda}_0$ and $L_0 + \breve{\Lambda}_0 \subset N_0$. The first inclusion is an equality because both lattices have colength $k-1$ in $\breve{\Lambda}_0$. The second inclusion is an equality because both lattices contain $\breve{\Lambda}_0$ with colength k. In other words,

$$M_0 = L_0 \cap \breve{\Lambda}_0 \quad \text{and} \quad N_0 = L_0 + \breve{\Lambda}_0, \tag{3.2.5}$$

showing that there is a unique point above L_0. This proves the injectivity of the map in question.

For surjectivity we again start with a point $L_0 \in \mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$, and now define M_0 and N_0 by (3.2.5). An exercise in linear algebra, using Proposition 3.2.1, shows that the triple (L_0, M_0, N_0) defines a point of $\mathbf{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$ above L_0. □

The proof of Corollary 3.2.3 provides an description of the inverse to

$$\pi_0 : \mathbf{RZ}^k_\Lambda(\breve{\mathbb{F}}_p) \to \mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p).$$

By continuing the line of reasoning a bit further, one can describe the inverse in the language of the original moduli problems defining the source and target.

Recall that for any point of $\mathbf{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$, represented by a diagram (3.1.4), the corresponding lattices M and N in (3.2.4) are dual to one another under the hermitian form on $\breve{\Lambda}[1/p]$, while L and $\breve{\Lambda}$ are each self-dual. Thus dualizing the second equality in (3.2.5) shows $M_1 = L_1 \cap \breve{\Lambda}_1$, while dualizing the first shows $N_1 = L_1 + \breve{\Lambda}_1$. We deduce that

$$M = L \cap \breve{\Lambda} \quad \text{and} \quad N = L + \breve{\Lambda}.$$

Recalling how (3.2.4) was constructed from the diagram of Dieudonné modules above it, we find that the inverse to the above function π_0 sends a point $X \in \mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$, corresponding to a diagram (2.4.3), to the diagram (3.1.4) determined by

$$\begin{aligned} D(H) &= D(X) \cap D(\Lambda \otimes \overline{\mathbb{Y}}) \\ D(H^\vee) &= D(X) + D(\Lambda \otimes \overline{\mathbb{Y}}), \end{aligned} \tag{3.2.6}$$

and $D(G) = D(H^\vee)_0 \oplus D(H)_1$, as per Lemma 3.2.2.

3.3. **Analysis of π_0.** Our goal in this subsection is to prove that the arrow

$$\pi_0 : \mathbf{RZ}^k_\Lambda \to \mathrm{RZ}^k_\Lambda$$

in (3.1.1) is a closed immersion inducing an isomorphism of underlying reduced schemes. This will use the following general result of Grothendieck-Messing theory, in which $\breve{\mathbb{F}}_p[\epsilon]$ denotes the usual ring of infinitesimals defined by $\epsilon^2 = 0$.

Lemma 3.3.1. *Let $f' : H_1' \to H_2'$ be a morphism of p-divisible groups over $\breve{\mathbb{F}}_p[\epsilon]$, and denote by $f : H_1 \to H_2$ its reduction to $\breve{\mathbb{F}}_p$. Assume that the induced morphism*

$$f : \mathcal{D}(H_1) \to \mathcal{D}(H_2)$$

of $\breve{\mathbb{F}}_p$-vector spaces satisfies

$$\mathrm{Fil}^0\mathcal{D}(H_1) = \ker\left(\mathcal{D}(H_1) \xrightarrow{f} \frac{\mathcal{D}(H_2)}{\mathrm{Fil}^0\mathcal{D}(H_2)}\right). \tag{3.3.1}$$

If H_2' is isomorphic to the constant deformation of H_2, then H_1' is isomorphic to the constant deformation of H_1.

Proof. Let $f^\circ : H_1^\circ \to H_2^\circ$ be the constant deformation of $f : H_1 \to H_2$ to $\breve{\mathbb{F}}_p[\epsilon]$. Grothendieck-Messing theory provides us with a canonical commutative diagram

$$\begin{array}{ccc} \mathcal{D}(H_1') & \xrightarrow{f'} & \mathcal{D}(H_2') \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{D}(H_1^\circ) & \xrightarrow[f^\circ]{} & \mathcal{D}(H_2^\circ) \end{array}$$

of $\breve{\mathbb{F}}_p[\epsilon]$-modules. The vertical isomorphism on the right identifies the Hodge filtrations on source and target, as it is induced by an isomorphism of deformations $H_2^\circ \cong H_2'$, and hence there is an induced diagram

$$\begin{array}{ccc} \mathcal{D}(H_1') & \xrightarrow{f'} & \mathcal{D}(H_2')/\mathrm{Fil}^0\mathcal{D}(H_2') \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{D}(H_1^\circ) & \xrightarrow[f^\circ]{} & \mathcal{D}(H_2^\circ)/\mathrm{Fil}^0\mathcal{D}(H_2^\circ). \end{array} \tag{3.3.2}$$

The top horizontal arrow in (3.3.2), being induced by the morphism of p-divisible groups $f' : H_1' \to H_2'$, satisfies

$$\mathrm{Fil}^0\mathcal{D}(H_1') \subset \ker\left(\mathcal{D}(H_1') \xrightarrow{f'} \frac{\mathcal{D}(H_2')}{\mathrm{Fil}^0\mathcal{D}(H_2')}\right).$$

On the other hand, as $f^\circ : H_1^\circ \to H_2^\circ$ is the constant deformation, (3.3.1) implies

$$\mathrm{Fil}^0\mathcal{D}(H_1^\circ) = \ker\left(\mathcal{D}(H_1^\circ) \xrightarrow{f^\circ} \frac{\mathcal{D}(H_2^\circ)}{\mathrm{Fil}^0\mathcal{D}(H_2^\circ)}\right).$$

It follows that the left vertical isomorphism in (3.3.2) restricts to a map

$$\mathrm{Fil}^0\mathcal{D}(H_1') \to \mathrm{Fil}^0\mathcal{D}(H_1^\circ),$$

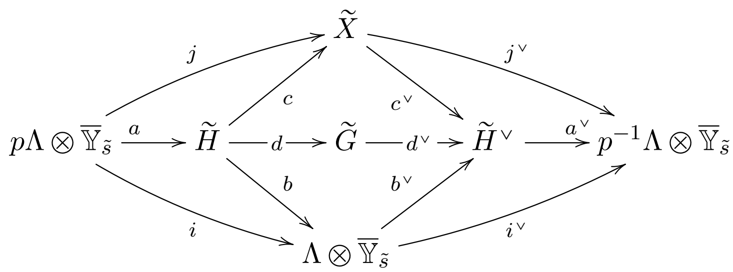
which is an isomorphism because the Hodge filtrations are local direct summands of the same rank. By Grothendieck-Messing theory there is an isomorphism of deformations $H_1' \cong H_1^\circ$. □

Proposition 3.3.2. *The map* $\pi_0 : \mathbf{RZ}^k_\Lambda \to \mathrm{RZ}^k_\Lambda$ *is a closed immersion inducing an isomorphism of underlying reduced schemes.*

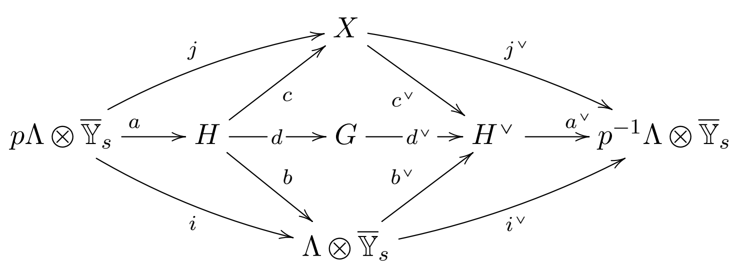
Proof. The key step is to show that the map in question is unramified. For this, it suffices to show that the induced map on tangent spaces is injective. Abbreviate

$$s = \mathrm{Spec}(\breve{\mathbb{F}}_p) \quad \text{and} \quad \tilde{s} = \mathrm{Spec}(\breve{\mathbb{F}}_p[\epsilon]).$$

Given a point of $\mathbf{RZ}^k_\Lambda(\tilde{s})$, represented by a diagram



of p-divisible groups over $\tilde{s}$, and deforming a diagram



of p-divisible groups over S. We must show that if $\widetilde{X}$ is the constant deformation, then so are $\widetilde{H}$ and $\widetilde{G}$.

The constancy of $\widetilde{H}$ follows by applying Lemma 3.3.1 to the morphism

$$H \xrightarrow{c \times b} X \times (\Lambda \otimes \overline{\mathbb{Y}}_s).$$

The only thing to check is that the hypothesis

$$\mathrm{Fil}^0 \mathcal{D}(H) = \ker\left(\mathcal{D}(H) \to \frac{\mathcal{D}(X)}{\mathrm{Fil}^0 \mathcal{D}(X)} \times \frac{\mathcal{D}(\Lambda \otimes \overline{\mathbb{Y}})}{\mathrm{Fil}^0 \mathcal{D}(\Lambda \otimes \overline{\mathbb{Y}})} \right)$$

of that lemma is satisfied. This hypothesis is equivalent to

$$\frac{VD(H)}{pD(H)} = \ker\left(\frac{D(H)}{pD(H)} \to \frac{D(X)}{VD(X)} \times \frac{D(\Lambda \otimes \overline{\mathbb{Y}})}{VD(\Lambda \otimes \overline{\mathbb{Y}})} \right),$$

which is clear from (3.2.6).

It remains to show that $\widetilde{G}$ is the constant deformation of G. Denote by H° and G° be the constant deformations of H and G to $\tilde{s}$. Grothendieck-Messing theory provides us with a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(H^\circ) & = & \mathcal{D}(\widetilde{H}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathcal{D}(G^\circ) & = & \mathcal{D}(\widetilde{G}) \end{array}$$

of vector bundles on $\widetilde{s}$. As we have already know $H^\circ \cong \widetilde{H}$, the top isomorphism respects the Hodge filtrations. The vertical arrows respect Hodge filtrations, because they arise from morphisms of p-divisible groups. over $\tilde{s}$. We must show that the bottom isomorphism also respects Hodge filtrations.

The essential point is that the vertical arrows restrict to surjections

$$\mathrm{Fil}^0\mathcal{D}(H^\circ)_0 \to \mathrm{Fil}^0\mathcal{D}(G^\circ)_0 \quad \text{and} \quad \mathrm{Fil}^0\mathcal{D}(\widetilde{H})_0 \to \mathrm{Fil}^0\mathcal{D}(\widetilde{G})_0.$$

Indeed, surjectivity can be checked on fibers. On fibers both are equivalent to the surjectivity of

$$\frac{VD(H)_1}{pD(H)_0} \to \frac{VD(G)_1}{pD(G)_0},$$

which is clear from Lemma 3.2.2.

It follows that under the canonical identification $\mathcal{D}(G^\circ) = \mathcal{D}(\widetilde{G})$

$$\begin{aligned} \mathrm{Fil}^0\mathcal{D}(G^\circ)_0 &= \mathrm{Image}(\mathrm{Fil}^0\mathcal{D}(H^\circ)_0 \to \mathcal{D}(G^\circ)_0) \\ &= \mathrm{Image}(\mathrm{Fil}^0\mathcal{D}(\widetilde{H})_0 \to \mathcal{D}(\widetilde{G})_0) \\ &= \mathrm{Fil}^0\mathcal{D}(\widetilde{G})_0. \end{aligned}$$

A similar argument, using the $d^\vee : G \to H^\vee$ in place of $d : H \to G$, shows that $\mathrm{Fil}^0\mathcal{D}(G^\circ)_1 = \mathrm{Fil}^0\mathcal{D}(\widetilde{G})_1$. Hence $G^\circ \cong \widetilde{G}$ as deformations of G. □

3.4. **A Deligne-Lusztig variety.** In this subsection we identify the reduced scheme $Y_\Lambda^{k,\mathrm{red}}$ underlying Y_Λ^k with a Deligne-Lusztig variety.

Suppose S is an $\breve{\mathbb{F}}_p$-scheme, and let $\sigma : \mathcal{O}_S \to \mathcal{O}_S$ be the p-power Frobenius. The pairing (2.3.5) induces a pairing of $\mathcal{O}_S$-modules

$$(\breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S) \times (\breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S) \to \mathcal{O}_S$$

that is linear in the first variable and σ-linear in the second. For any local direct summand $\mathcal{F} \subset \breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S$, its left annihilator

$$\mathcal{F}^\perp = \{x \in \breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S : b(x, \mathcal{F}) = 0\}$$

is again a local direct summand. Taking into account the switch from right dual to left annihilator, the analogue of (2.3.7) is

$$\mathcal{F}^{\perp\perp} = \Phi^2\mathcal{F} \quad \text{and} \quad \Phi^2(\mathcal{F}^\perp) = (\Phi^2\mathcal{F})^\perp.$$

Definition 3.4.1. Define DL_Λ^k to be the proper $\breve{\mathbb{F}}_p$-scheme whose functor of points assigns to any $\breve{\mathbb{F}}_p$-scheme S the set of flags of $\mathcal{O}_S$-module local direct summands

$$0 \overset{k-1}{\subset} \mathcal{J} \overset{1}{\subset} \mathcal{K}^\perp \overset{n-2k}{\subset} \mathcal{K} \overset{1}{\subset} \mathcal{J}^\perp \overset{k-1}{\subset} \breve{\Lambda}_0 \otimes \mathcal{O}_S$$

of the indicated coranks.

Proposition 3.4.2. *The scheme* DL_Λ^k *of Definition 3.4.1 is a Deligne-Lusztig variety for the unitary group of the finite hermitian space* $\Lambda/p\Lambda$*. It is smooth of dimension* $n-k-1$*, and is irreducible if* $k < n/2$*.*

Proof. Let H be the base change to $\breve{\mathbb{F}}_p$ of the unitary group of $\Lambda/p\Lambda$. Fix an $\mathcal{O}_E$-basis $x_1, \ldots, x_n \in \Lambda$ with respect to which the hermitian form satisfies the anti-diagonal relation $h(x_i, x_j) = \delta_{i,n-i+1}$, and denote by $y_1, \ldots, y_n \in \breve{\Lambda}_0$ the projections of $x_1, \ldots, x_n$ to the first factor in the decomposition $\breve{\Lambda} = \breve{\Lambda}_0 \oplus \breve{\Lambda}_1$. Use the reductions $y_1, \ldots, y_n \in \breve{\Lambda}_0/p\breve{\Lambda}_0$ to identify

$$H \cong \mathrm{GL}(\breve{\Lambda}_0/p\breve{\Lambda}_0) \cong \mathrm{GL}_n .$$

The Weyl group W of the diagonal torus is generated by the set of simple reflections $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i \in H(\breve{\mathbb{F}}_p)$ is the transposition matrix interchanging $y_i \leftrightarrow y_{i+1}$. The action of Frobenius on S is $\sigma(s_i) = s_{n-i}$. Every subset $I \subset S$ determines a standard parabolic subgroup

$$P_I = BW_IB \subset H,$$

where $W_I \subset W$ is the subgroup generated by I, and $B \subset H$ is the upper-triangular Borel.

We are interested in the subset $I = S \smallsetminus \{s_{k-1}, s_{n-k}\}$ whose corresponding parabolic P_I is the stabilizer of the standard flag

$$0 \overset{k-1}{\subset} \mathcal{J} \overset{n-2k+1}{\subset} \mathcal{K} \overset{k}{\subset} \breve{\Lambda}_0/p\breve{\Lambda}_0$$

defined by

$$\mathcal{J} = \mathrm{Span}\{y_1, \ldots, y_{k-1}\} \quad \text{and} \quad \mathcal{K} = \mathrm{Span}\{y_1, \ldots, y_{n-k}\}.$$

The parabolic $P_{\sigma(I)}$ defined by $\sigma(I) = S \smallsetminus \{s_k, s_{n-k+1}\}$ is the stabilizer of

$$0 \overset{k}{\subset} \mathcal{K}^\perp \overset{n-2k+1}{\subset} \mathcal{J}^\perp \overset{k-1}{\subset} \breve{\Lambda}_0/p\breve{\Lambda}_0.$$

Using Lemma 2.12 of [16], one sees that DL_Λ^k sits in a cartesian diagram

$$\begin{array}{ccc} \mathrm{DL}_\Lambda^k & \longrightarrow & H/P_I \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}\times\sigma} \\ H/P_{I\cap\sigma(I)} & \longrightarrow & H/P_I \times H/P_{\sigma(I)}, \end{array}$$

and hence, using the notation of §4.4 of [17],

$$\mathrm{DL}_\Lambda^k \cong X_{I\cap\sigma(I)}(\mathrm{id}).$$

All parts of the proposition now follow from the discussion of *loc. cit.*. Note that the irreducibility claim (a result of Bonnafe and Rouquier [2]) requires $k < n/2$ because this ensures $I \cup \sigma(I) = S$. $\square$

The remainder of this subsection is devoted to proving $Y_\Lambda^{k,\mathrm{red}} \cong \mathrm{DL}_\Lambda^k$. We need two elementary lemmas from commutative algebra.

Lemma 3.4.3. *Let k be an algebraically closed field, let $\pi : X' \to X$ be a proper unramified morphism between k-schemes of finite type, and suppose π is bijective on closed points.*

(1) *The morphism π is a closed immersion, and induces an isomorphism of underlying reduced schemes.*
(2) *If X is reduced then so is X', and π is an isomorphism.*

Proof. A proper and quasi-finite morphism is finite, hence affine. This reduces to the case where $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(B)$ with $A \to B$ a finite morphism of finite type k-algebras.

The unramifiedness assumption implies that $\mathfrak{m}B \subset B$ is maximal for any maximal ideal $\mathfrak{m} \subset A$. In particular

$$k \cong A/\mathfrak{m} \to B/\mathfrak{m}B \cong k$$

is an isomorphism of A-modules, and Nakayama's lemma implies that $A \to B$ is surjective. In other words, π is a closed immersion.

The bijectivity of $A \to B$ on maximal ideals implies that $I = \ker(A \to B)$ is contained in every maximal ideal of A. As A is a Jacobson ring, the intersection of its maximal ideals is equal to the nilradical $\mathfrak{n} \subset A$. Thus $I \subset \mathfrak{n}$, which implies that $\mathfrak{n}/I$ is the nilradical of $A/I = B$. All claims of the lemma follow immediately. $\square$

Lemma 3.4.4. *Let k be an algebraically closed field, suppose X is a reduced scheme of finite type over k, and $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module. If the k-dimension of the fiber $\mathcal{E}_x$ is constant as $x \in X$ varies over all closed points, then $\mathcal{E}$ is locally free.*

Proof. Let r be the common dimension of all fibers. At any closed point $x \in X$, one can use Nakayama's lemma to find an open neighborhood $U \ni x$ over which there exists a surjection $\mathcal{O}_U^r \to \mathcal{E}_U$. By the constancy of fiber dimensions, such a map must induce an isomorphism fiber-by-fiber, and hence have kernel contained in the subsheaf $\mathcal{J}\mathcal{O}_U^r$, where $\mathcal{J} \subset \mathcal{O}_U$ is the Jacobson radical. As X is reduced of finite type over a field, $\mathcal{J} = 0$, proving $\mathcal{O}_U^r \cong \mathcal{E}_U$. $\square$

Theorem 3.4.5. *There is an isomorphism*

$$Y_\Lambda^{k,\mathrm{red}} \cong \mathrm{DL}_\Lambda^k$$

sending a pair $(M_0, N_0) \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$ as in Proposition 3.2.1 to the flag

$$0 \subset \frac{pM_0^*}{p\breve{\Lambda}_0} \subset \frac{pN_0}{p\breve{\Lambda}_0} \subset \frac{N_0^*}{p\breve{\Lambda}_0} \subset \frac{M_0}{p\breve{\Lambda}_0} \subset \frac{\breve{\Lambda}_0}{p\breve{\Lambda}_0}.$$

Proof. As in (2.4.2), Grothendieck-Messing theory provides us with a diagram of vector bundles

$$\begin{tikzcd} p\breve{\Lambda}\otimes \mathcal{D}(\overline{\mathbb{Y}}) \arrow[r, "a"] \arrow[drr, "i=0"'] & \mathcal{D}(H) \arrow[r, "d"] \arrow[dr, "b"] & \mathcal{D}(G) \arrow[r, "d^\vee"] & \mathcal{D}(H^\vee) \arrow[r, "a^\vee"] & p^{-1}\breve{\Lambda}\otimes \mathcal{D}(\overline{\mathbb{Y}}) \\ & & \breve{\Lambda}\otimes \mathcal{D}(\overline{\mathbb{Y}}) \arrow[ur, "b^\vee"] \arrow[urr, "i^\vee=0"'] & & \end{tikzcd}$$

on $Y_\Lambda^{k,\mathrm{red}}$. Applying the functor $\mathrm{Hom}_{\mathcal{O}_{\breve{E}}}(\mathcal{D}(\overline{\mathbb{Y}}), -)$ yields a diagram

$$\begin{tikzcd} p\breve{\Lambda}\otimes \mathcal{O}_{Y_\Lambda^{\mathrm{red}}} \arrow[r, "a"] \arrow[drr, "i=0"'] & \mathcal{M} \arrow[r] \arrow[dr, "b"] & \mathcal{N}_0\oplus\mathcal{M}_1 \arrow[r] & \mathcal{N} \arrow[r, "a^\vee"] & p^{-1}\breve{\Lambda}\otimes \mathcal{O}_{Y_\Lambda^{\mathrm{red}}} \\ & & \breve{\Lambda}\otimes \mathcal{O}_{Y_\Lambda^{\mathrm{red}}} \arrow[ur, "b^\vee"] \arrow[urr, "i^\vee=0"'] & & \end{tikzcd}$$

of vector bundles on $Y_\Lambda^{k,\mathrm{red}}$, in which

$$\mathcal{M} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H)) \quad \text{and} \quad \mathcal{N} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee))$$

are defined exactly as in (2.4.4). Note that in the first diagram the tensor products are over $\mathcal{O}_{\breve{E}}$, while in the second they are over $\breve{\mathbb{Z}}_p$. Note also that we are making use of the fact that the natural maps

$$\mathcal{D}(H)_1 \to \mathcal{D}(G)_1 \quad \text{and} \quad \mathcal{D}(G)_0 \to \mathcal{D}(H^\vee)_0$$

are isomorphisms, so that $\mathcal{N}_0 \oplus \mathcal{M}_0 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(G))$. Indeed, this can be checked on fibers, where it follows from Lemma 3.2.2.

The vector bundles above come with filtrations

$$0 \subset \mathrm{Fil}^1\mathcal{M} \subset \mathrm{Fil}^0\mathcal{M} \subset \mathcal{M} \quad \text{and} \quad 0 \subset \mathrm{Fil}^1\mathcal{N} \subset \mathrm{Fil}^0\mathcal{N} \subset \mathcal{N}$$

induced by the Hodge filtrations (2.4.2) on $\mathcal{D}(\overline{\mathbb{Y}})$, $\mathcal{D}(H)$, and $\mathcal{D}(H^\vee)$. More precisely, $\mathrm{Fil}^0\,\mathcal{M}$ consists of morphisms that carry $\mathrm{Fil}^0\,\mathcal{D}(\overline{\mathbb{Y}})$ into $\mathrm{Fil}^0\,\mathcal{D}(H)$, while $\mathrm{Fil}^1\,\mathcal{M}$ consists of morphisms that carry $\mathcal{D}(\overline{\mathbb{Y}})$ into $\mathrm{Fil}^0\,\mathcal{D}(H)$ and kill $\mathrm{Fil}^0\,\mathcal{D}(\overline{\mathbb{Y}})$. The filtration on $\mathcal{N}$ is defined in the same way.

As in Remark 2.4.1, a choice of $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$ determines isomorphisms

$$\mathcal{M} \cong \mathcal{D}(H) \quad \text{and} \quad \mathcal{N} \cong \mathcal{D}(H^\vee). \tag{3.4.1}$$

While these do not respect the filtrations, they do identify

$$\mathrm{Fil}^0\,\mathcal{M}_0 \cong \mathrm{Fil}^0\,\mathcal{D}(H)_0 \quad \text{and} \quad \mathrm{Fil}^0\,\mathcal{N}_0 \cong \mathrm{Fil}^0\,\mathcal{D}(H^\vee)_0. \tag{3.4.2}$$

This follows from the signature $(0,1)$ condition on $\overline{\mathbb{Y}}$, which is equivalent to $\mathrm{Fil}^0\,\mathcal{D}(\overline{\mathbb{Y}})_0 = \mathcal{D}(\overline{\mathbb{Y}})_0$.

Lemma 3.4.6. *Fix a point $s \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$ corresponding to a lattice chain*

$$p\breve{\Lambda}_0 \overset{k-1}{\subset} pM_0^* \overset{1}{\subset} pN_0 \overset{n-2k}{\subset} N_0^* \overset{1}{\subset} M_0 \overset{k-1}{\subset} \breve{\Lambda}_0$$

as in Proposition 3.2.1. There are canonical identifications

$$\frac{N_0^*}{pM_0} = \mathrm{Fil}^0\,\mathcal{M}_{0,s} \subset \mathcal{M}_{0,s} = \frac{M_0}{pM_0}$$

and

$$\frac{M_0^*}{pM_0} = \mathrm{Fil}^0\,\mathcal{N}_{0,s} \subset \mathcal{N}_{0,s} = \frac{N_0}{pN_0}.$$

Proof. Fix an isomorphism $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$, so that

$$\mathcal{M}_{0,s} \overset{(3.4.1)}{\cong} \mathcal{D}(H_s)_0 = \frac{D(H_s)_0}{pD(H_s)_0} \overset{(3.2.2)}{\cong} \frac{M_0}{pM_0}.$$

The first and third isomorphisms depend on the choice of $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$, but the composition does not. Similarly,

$$\mathrm{Fil}^0\,\mathcal{M}_{0,s} \overset{(3.4.2)}{\cong} \mathrm{Fil}^0\,\mathcal{D}(H)_{0,s} = \frac{VD(H_s)_1}{pD(H_s)_0} \overset{2.3.2}{\cong} \frac{\Phi^{-1}M_1}{pM_0} \overset{(3.2.3)}{=} \frac{N_0^*}{pM_0}.$$

Once again, the first and third isomorphisms depend on the choice of $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$, but the composition does not. The claims about $\mathcal{N}_{0,s}$ are proved in the same way, using $H_s^\vee$ in place of H_s. □

Now consider the subsheaves of $\breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$ defined by

$$\begin{aligned}\mathcal{J} &= \mathrm{Image}(\mathrm{Fil}^0\mathcal{N}_0 \to p^{-1}\breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \cong \breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}})\\ \mathcal{K} &= \mathrm{Image}(\mathrm{Fil}^0\mathcal{M}_0 \to \breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}).\end{aligned}$$

Lemma 3.4.6 implies that the quotients by these subsheaves have constant fiber dimension, hence by Lemma 3.4.4 the quotients are locally free, and hence $\mathcal{J}$ and $\mathcal{K}$ are local direct summands. Again by checking on fibers, using Lemma 3.4.6, one sees that these subsheaves have left annihilators

$$\begin{aligned}\mathcal{J}^\perp &= \mathrm{Image}(\mathcal{M}_0 \to \breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}})\\ \mathcal{K}^\perp &= \mathrm{Image}(\mathcal{N}_0 \to p^{-1}\breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \cong \breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}).\end{aligned}$$

Yet again checking on fibers, one finds using Lemma 3.4.6 that these satisfy

$$0 \overset{k-1}{\subset} \mathcal{J} \overset{1}{\subset} \mathcal{K}^\perp \overset{n-2k}{\subset} \mathcal{K} \overset{1}{\subset} \mathcal{J}^\perp \overset{k-1}{\subset} \breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}. \tag{3.4.3}$$

The flag (3.4.3) defines the desired morphism

$$Y_\Lambda^{k,\mathrm{red}} \to \mathrm{DL}_\Lambda^k, \tag{3.4.4}$$

and it remains to show that it is an isomorphism. They key to this is the following lemma, which tells us how to recover the filtration (3.4.2) from the flag (3.4.3), using the quotient maps $\mathcal{M}_0 \to \mathcal{J}^\perp$ and $\mathcal{N}_0 \to \mathcal{K}^\perp$.

Lemma 3.4.7. *The above vector bundles on $Y_\Lambda^{k,\mathrm{red}}$ satisfy*

$$\mathrm{Fil}^0\mathcal{M}_0 = \ker(\mathcal{M}_0 \to \mathcal{J}^\perp/\mathcal{K}) \quad \textit{and} \quad \mathrm{Fil}^0\mathcal{N}_0 = \ker(\mathcal{N}_0 \to \mathcal{K}^\perp/\mathcal{J}).$$

Proof. For the first equality, one uses Lemma 3.4.6 to check that the cokernel of $\mathcal{M}_0 \to \mathcal{J}^{\perp}/\mathcal{K}$ has constant fiber dimension, and so is locally free by Lemma 3.4.4. This implies that the kernel of this morphism is a local direct summand of $\mathcal{M}_0$. The desired equality can therefore be checked on fibers, which is again done using Lemma 3.4.6. Indeed, at a point $(M_0, N_0) \in Y_{\Lambda}^{k,\mathrm{red}}$ the desired equality is precisely

$$\frac{N_0^*}{pM_0} = \ker\left(\frac{M_0}{pM_0} \to \frac{M_0/p\breve{\Lambda}_0}{N_0^*/p\breve{\Lambda}_0}\right),$$

which is clear. The second equality is proved in exactly the same way. ☐

Lemma 3.4.8. *The morphism* (3.4.4) *is unramified.*

Proof. We show that the morphism in question is formally unramified. Suppose we are given a diagram

$$\begin{array}{ccc} S & \longrightarrow & Y_{\Lambda}^{k,\mathrm{red}} \\ \downarrow & \nearrow & \downarrow \\ \widetilde{S} & \longrightarrow & \mathrm{DL}_{\Lambda}^{k} \end{array}$$

of solid arrows such that the left vertical arrow is a square-zero thickening of $\breve{\mathbb{F}}_p$-schemes. We must show there is at most one dotted arrow making the diagram commute.

Suppose we have two such arrows $a, b : \widetilde{S} \to Y_{\Lambda}^{k,\mathrm{red}}$, and consider the p-divisible groups a^*H and b^*H on $\widetilde{S}$. We do not yet know that these are isomorphic, but they have the same restriction to $S \subset \widetilde{S}$, and so Grothendieck-Messing deformation theory provides us with canonical isomorphisms of vector bundles

$$a^*\mathcal{D}(H) \cong \mathcal{D}(a^*H) \cong \mathcal{D}(b^*H) \cong b^*\mathcal{D}(H)$$

making the diagram

$$\begin{array}{ccc} a^*\mathcal{D}(H) & = & b^*\mathcal{D}(H) \\ & \searrow \quad \swarrow & \\ & \mathcal{D}(\breve{\Lambda}_0 \otimes \overline{\mathbb{Y}}_{\widetilde{S}}) & \end{array}$$

commute. Here the diagonal arrows are the pullbacks (via a and b) of the morphism $\mathcal{D}(H) \to \mathcal{D}(\breve{\Lambda}_0 \otimes \overline{\mathbb{Y}})$ induced by the universal isogeny $H \to \breve{\Lambda}_0 \otimes \overline{\mathbb{Y}}$ over $Y_{\Lambda}^{k,\mathrm{red}}$.

Fixing a choice of $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$, and hence isomorphisms (3.4.1) and (3.4.2) of vector bundles on $Y_\Lambda^{k,\mathrm{red}}$, we find a diagram

$$\begin{array}{ccc} a^*\mathcal{M}_0 & = & b^*\mathcal{M}_0 \\ & \searrow \quad \swarrow & \\ & \breve{\Lambda}_0 \otimes \mathcal{O}_{\widetilde{S}}. & \end{array}$$

As a and b induce the same map $\widetilde{S} \to \mathrm{DL}_\Lambda^k$, we have equalities $a^*\mathcal{J}^\perp = b^*\mathcal{J}^\perp$ and $a^*\mathcal{K} = b^*\mathcal{K}$ as local direct summands of $\breve{\Lambda}_0 \otimes \mathcal{O}_{\widetilde{S}}$, and hence a commutative diagram

$$\begin{array}{ccccccc} a^*\mathcal{D}(H)_0 & = & a^*\mathcal{M}_0 & = & b^*\mathcal{M}_0 & = & b^*\mathcal{D}(H)_0 \\ & & \downarrow & & \downarrow & & \\ & & a^*(\mathcal{J}^\perp/\mathcal{K}) & = & b^*(\mathcal{J}^\perp/\mathcal{K}). & & \end{array}$$

Combining this diagram with Lemma 3.4.7 and (3.4.2), and applying the same reasoning to $H^\vee$, we find that the canonical isomorphisms

$$\mathcal{D}(a^*H)_0 \cong \mathcal{D}(b^*H)_0 \quad \text{and} \quad \mathcal{D}(a^*H^\vee)_0 \cong \mathcal{D}(b^*H^\vee)_0$$

of Grothendieck-Messing theory respect Hodge filtrations. By the duality between $\mathcal{D}(H)_1$ and $\mathcal{D}(H^\vee)_0$, under which the Hodge filtrations are annihilators of one another, the second of these implies that the isomorphism

$$\mathcal{D}(a^*H)_1 \cong \mathcal{D}(b^*H)_1$$

also respects Hodge filtrations, and it follows that $a^*H \cong b^*H$.

It only remains to show that $a^*G \cong b^*G$, but this follows from $a^*H \cong b^*H$, exactly as in the proof of Proposition 3.3.2. □

At last we complete the proof of Theorem 3.4.5. The morphism (3.4.4) is bijective on closed points, by comparing the definition of DL_Λ^k with Proposition 3.2.1. It is unramified by Lemma 3.4.8. It is proper, as the source is a projective $\breve{\mathbb{F}}_p$-scheme by the same reasoning as in Remark 2.2.3. We conclude now from Lemma 3.4.3 that (3.4.4) is an isomorphism. □

4. Fiber varieties over Deligne-Lusztig varieties

We continue to work with a fixed self-dual hermitian $\mathcal{O}_E$-lattice Λ of rank $n \geqslant 2$, and a fixed $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, as in §3. Using the morphism

$$\pi_1 : \mathbf{RZ}_\Lambda^{\leqslant k} \to Y_\Lambda^k$$

from (3.1.1), we will realize (the reduced scheme underlying) $\mathbf{RZ}_\Lambda^{\leqslant k}$ as a closed subscheme of the relative Grassmannian parametrizing rank $k-1$ local direct summands of a rank $2k-1$ vector bundle $\mathcal{V}$ on Y_Λ^k.

4.1. **A special vector bundle.** Recall the scheme Y_Λ^k of §3.1, whose underlying reduced scheme $Y_\Lambda^{k,\mathrm{red}}$ we have identified with a Deligne-Lusztig variety in Theorem 3.4.5.

Our first goal is to construct a filtered vector bundle

$$0 \subset \mathcal{V}^{(1)} \subset \mathcal{V}^{(k)} \subset \mathcal{V} \tag{4.1.1}$$

on $Y_\Lambda^{k,\mathrm{red}}$ from the filtered vector bundles

$$\mathcal{M} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H))$$
$$\mathcal{N} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee))$$

used in the proof of Theorem 3.4.5. Here $H \to H^\vee$ is the isogeny of p-divisible groups appearing in the universal diagram (3.1.2) over $Y_\Lambda^{k,\mathrm{red}}$.

Remark 4.1.1. Recall from (3.4.1) that a choice of $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$ determines isomorphisms

$$\mathcal{M}_0 \cong \mathcal{D}(H)_0 \quad \text{and} \quad \mathcal{N}_0 \cong \mathcal{D}(H^\vee)_0,$$

each of which identifies the subsheaves Fil^0 on source and target.

Remark 4.1.2. At a point $s \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$, corresponding to a pair of lattices (M_0, N_0) under Proposition 3.2.1, there are canonical identifications

$$\mathrm{Fil}^0 \mathcal{M}_{0,s} = \frac{N_0^*}{pM_0} \subset \frac{M_0}{pM_0} = \mathcal{M}_{0,s}$$
$$\mathrm{Fil}^0 \mathcal{N}_{0,s} = \frac{M_0^*}{pN_0} \subset \frac{N_0}{pN_0} = \mathcal{N}_{0,s}.$$

This is just a reminder of Lemma 3.4.6.

Define coherent sheaves on $Y_\Lambda^{k,\mathrm{red}}$ by

$$\mathcal{V} = \mathrm{coker}(\mathrm{Fil}^0 \mathcal{M}_0 \to \mathrm{Fil}^0 \mathcal{N}_0) \tag{4.1.2}$$
$$\mathcal{W} = \mathrm{coker}(\mathcal{M}_0 \to \mathcal{N}_0).$$
$$\mathcal{V}^{(1)} = \ker(\mathcal{V} \to \mathcal{W}).$$

The arrow $b^\vee$ in the universal diagram (3.1.2) determines a tautological inclusion $\Lambda \subset \mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, H^\vee)$. This induces a morphism of vector bundles

$$\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \to \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee)),$$

which in turn restricts to a morphism

$$\breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \to \mathrm{Fil}^0 \mathcal{N}_0.$$

This allows us to define

$$\mathcal{V}^{(k)} = \mathrm{Im}(\breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \to \mathrm{Fil}^0 \mathcal{N} \to \mathcal{V}). \tag{4.1.3}$$

Proposition 4.1.3. *At a point $s \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$, represented by lattices (M_0, N_0) as in Proposition 3.2.1, there are canonical identifications*

$$\mathcal{V}_s = M_0^*/N_0^* \quad \textit{and} \quad \mathcal{W}_s = N_0/M_0.$$

Moreover, the fiber at s of (4.1.1) is identified with

$$0 \subset \frac{M_0}{N_0^*} \subset \frac{\breve{\Lambda}_0}{N_0^*} \subset \frac{M_0^*}{N_0^*}.$$

Proof. This is clear from Remark 4.1.2. □

Corollary 4.1.4. *The coherent sheaves on $Y_\Lambda^{k,\mathrm{red}}$ defined above satisfying the following properties.*

1. *$\mathcal{V}$ and $\mathcal{W}$ are locally free of rank $2k-1$,*
2. *$\mathcal{V}^{(1)}$ and $\mathcal{V}^{(k)}$ are local direct summands of $\mathcal{V}$ of ranks 1 and k, respectively.*

Proof. Proposition 4.1.3 implies that $\mathcal{V}$ and $\mathcal{W}$ have constant fiber dimension $2k-1$, and so the first claim follows from Lemma 3.4.4. The same reasoning shows that the cokernel of $\mathcal{V}^{(i)} \to \mathcal{V}$ is locally free of rank $2k-1-i$, and so $\mathcal{V}^{(i)}$ is a local direct summand of rank i. □

We next endow $\mathcal{V}$ with additional structure, derived from the pairing

$$b : \breve{\Lambda}_0 \times \breve{\Lambda}_0 \to \breve{\mathbb{Z}}_p$$

of (2.3.5). For any $\mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$-module $\mathcal{F}$, denote by

$$\sigma^*\mathcal{F} = \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \otimes_{\sigma, \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}} \mathcal{F} \tag{4.1.4}$$

the pullback by the p-power Frobenius $\sigma : \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}} \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$.

Proposition 4.1.5. *There is a unique $\mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$-linear map*

$$\beta : \mathcal{V} \otimes \sigma^*\mathcal{V} \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$$

that, when viewed as a pairing

$$\beta : \mathcal{V} \times \mathcal{V} \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$$

(linear in the first variable and σ-linear in the second), satisfies the following property: At a point $s \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$, represented by a pair of lattices (M_0, N_0), as in Proposition 3.2.1, the fiber

$$\beta_s : \frac{M_0^*}{N_0^*} \times \frac{M_0^*}{N_0^*} \to \breve{\mathbb{F}}_p$$

is identified with the reduction of

$$pb : M_0^* \times M_0^* \to \breve{\mathbb{Z}}_p.$$

Proof. The uniqueness claim is clear, because we have specified the pairing on fibers. The existence will follow from the next two lemmas.

Lemma 4.1.6. *There is an $\mathcal{O}_{Y_\Lambda^{\mathrm{red}}}$-linear morphism*

$$\sigma^* \operatorname{Fil}^0 \mathcal{N}_0 \to \mathcal{M}_1$$

whose fiber at any $s \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$ is identified with

$$\sigma^*(M_0^*/pN_0) \xrightarrow{1\otimes x \mapsto p\Phi x} M_1/pM_1.$$

Here M and N are the lattices in $\breve{\Lambda}[1/p]$ associated to s as in (3.2.1)*.*

Proof. Over $Y_\Lambda^{k,\mathrm{red}}$ there is a universal diagram (3.1.2) of p-divisible groups. By (3.1.3), there is a unique isogeny $\varpi : H^\vee \to H$ making the diagram

$$\begin{array}{ccc} H^\vee & \xrightarrow{\varpi} & H \\ & \searrow^{p} & \downarrow{\scriptstyle d^\vee \circ d} \\ & & H^\vee \end{array}$$

commute. This induces an $\mathcal{O}_E$-linear morphism of vector bundles

$$\mathcal{N} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee)) \xrightarrow{\varpi\circ} \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H)) = \mathcal{M},$$

whose fiber a point of $Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$ is identified with

$$N/pN \xrightarrow{p} M/pM.$$

Now consider the Frobenius and Verscheibung morphisms

$$\mathrm{Fr} : H \to \sigma^* H \quad \text{and} \quad \mathrm{Ver} : \sigma^* H \to H.$$

These induce morphisms of vector bundles

$$\mathcal{D}(H) \xrightarrow{V = \mathcal{D}(\mathrm{Fr})} \sigma^*\mathcal{D}(H) \quad \text{and} \quad \sigma^*\mathcal{D}(H) \xrightarrow{F = \mathcal{D}(\mathrm{Ver})} \mathcal{D}(H),$$

and similarly with H replaced by $\overline{\mathbb{Y}}$ or $H^\vee$. We use this to define a morphism

$$\sigma^*\mathcal{N} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\sigma^*\mathcal{D}(\overline{\mathbb{Y}}), \sigma^*\mathcal{D}(H^\vee)) \xrightarrow{\alpha} \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee)) = \mathcal{N}$$

by $x \mapsto F \circ x \circ V$. The fiber of α at a geometric point is identified with (the linearization of)

$$p\Phi : N/pN \to N/pN.$$

The maps α and $\varpi\circ$ above sit in a commutative diagram

$$\begin{array}{ccc} \sigma^*\mathcal{N} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\sigma^*\mathcal{D}(\overline{\mathbb{Y}}), \sigma^*\mathcal{D}(H^\vee)) & \xrightarrow{\alpha} & \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee)) = \mathcal{N} \\ \downarrow{\scriptstyle F\circ} & & \downarrow{\scriptstyle \varpi\circ} \\ \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\sigma^*\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H^\vee)) & \xrightarrow[\beta]{} & \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{D}(\overline{\mathbb{Y}}), \mathcal{D}(H)) = \mathcal{M} \\ \downarrow{\scriptstyle V\circ} & & \\ \sigma^*\mathcal{N} = \underline{\mathrm{Hom}}_{\mathcal{O}_E}(\sigma^*\mathcal{D}(\overline{\mathbb{Y}}), \sigma^*\mathcal{D}(H^\vee)) & & \end{array}$$

in which the arrow labeled β sends $x \mapsto \varpi \circ x \circ V$. The vertical column on the left is exact, and the image of the final vertical arrow is

$$\sigma^* \operatorname{Fil}^0 \mathcal{N} \subset \sigma^* \mathcal{N}.$$

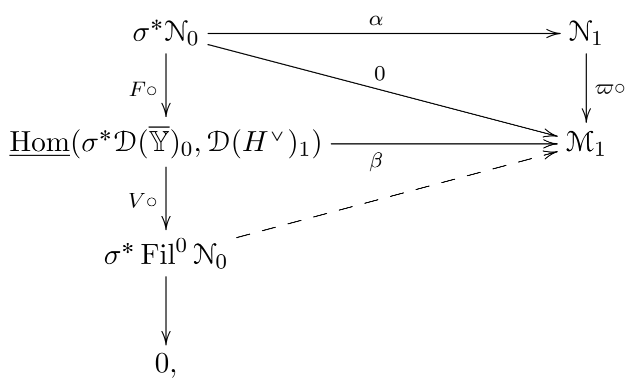
On fibers, the composition from the upper left corner to the lower right corner of the square is (the linearization of)

$$N/pN \xrightarrow{p^2\Phi} M/pM.$$

This last map need not be 0, but the relation $pN_0 \subset N_0^* = \Phi^{-1}M_1$ of Proposition 3.2.1 (and (3.2.3), for the equality) implies that its restriction

$$N_0/pN_0 \xrightarrow{p^2\Phi} M_1/pM_1$$

vanishes. Thus the diagram above restricts to a commutative diagram of solid arrows



and the exactness of the vertical column on the left implies the existence of a unique dotted arrow making the diagram commute. This dotted arrow is the morphism we seek. □

Lemma 4.1.7. *There is a $\mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$-linear map*

$$\mathcal{N}_0 \otimes \sigma^* \operatorname{Fil}^0 \mathcal{N}_0 \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$$

whose fiber at any $s \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$ is identified with the $\breve{\mathbb{F}}_p$-linear map

$$\frac{N_0}{pN_0} \otimes \sigma^* \left(\frac{M_0^*}{pN_0} \right) \to \breve{\mathbb{F}}_p$$

obtained by reducing $x \otimes (1 \otimes y) \mapsto pb(x,y) \in \breve{\mathbb{Z}}_p$ for $x \in N_0$ and $y \in M_0^*$.

Proof. The polarization on $\overline{\mathbb{Y}}$ induces a perfect bilinear pairing on the constant vector bundle $\mathcal{D}(\overline{\mathbb{Y}})$, which induces an isomorphism

$$\mathcal{D}(\overline{\mathbb{Y}})_0 \otimes \mathcal{D}(\overline{\mathbb{Y}})_1 \cong \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}.$$

Similarly, there is a canonical perfect bilinear pairing

$$\mathcal{D}(H^\vee)_0 \otimes \mathcal{D}(H)_1 \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$$

on the vector bundles associated to the universal p-divisible groups H and $H^\vee$ over $Y_\Lambda^{k,\mathrm{red}}$. These induce a perfect bilinear pairing

$$\begin{aligned}\mathcal{N}_0 \otimes \mathcal{M}_1 &= \underline{\mathrm{Hom}}(\mathcal{D}(\overline{\mathbb{Y}})_0, \mathcal{D}(H^\vee)_0) \otimes \underline{\mathrm{Hom}}(\mathcal{D}(\overline{\mathbb{Y}})_1, \mathcal{D}(H)_1) \\ &\cong \underline{\mathrm{Hom}}(\mathcal{D}(\overline{\mathbb{Y}})_0 \otimes \mathcal{D}(\overline{\mathbb{Y}})_1, \mathcal{D}(H^\vee)_0 \otimes \mathcal{D}(H)_1) \\ &\to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}.\end{aligned}$$

Pulling this back via the morphism $\sigma^* \mathrm{Fil}^0 \mathcal{N}_0 \to \mathcal{M}_1$ of Lemma 4.1.6 yields a pairing $\mathcal{N}_0 \otimes \sigma^* \mathrm{Fil}^0 \mathcal{N}_0 \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$ having the desired form on $\breve{\mathbb{F}}_p$-valued points. ☐

We now complete the proof of Proposition 4.1.5. By construction, there is a canonical map $\mathcal{M} \to \mathcal{N}$ respecting filtrations, and we claim that the composition

$$\mathcal{M}_0 \otimes \sigma^* \mathrm{Fil}^0 \mathcal{M}_0 \to \mathcal{N}_0 \otimes \sigma^* \mathrm{Fil}^0 \mathcal{N}_0 \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$$

is trivial. Indeed, taking fibers at a point $(M_0, N_0) \in Y_\Lambda^{k,\mathrm{red}}(\breve{\mathbb{F}}_p)$, the composition is identified with the pairing

$$\frac{M_0}{pM_0} \times \frac{N_0^*}{pM_0} \to \frac{N_0}{pN_0} \times \frac{M_0^*}{pN_0} \xrightarrow{pb(-,-)} \breve{\mathbb{F}}_p,$$

which is trivial as $b(M_0, N_0^*) \subset b(M_0, M_0^*) \subset \breve{\mathbb{Z}}_p$.

Recalling the definitions (4.1.2), the triviality of the above composition implies that the morphism of Lemma 4.1.7 descends to a morphism

$$\mathcal{W} \otimes \sigma^* \mathcal{V} \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}.$$

The composition

$$\mathcal{V} \otimes \sigma^* \mathcal{V} \to \mathcal{W} \otimes \sigma^* \mathcal{V} \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$$

(the first arrow is the canonical map $\mathcal{V} \to \mathcal{W}$ on the first tensor factor, and the identity on the second) at last defines the desired pairing β. ☐

Modulo the fact that the pairing β is not bilinear, the following proposition essentially says that the radical of β is $\mathcal{V}^{(1)}$, and

$$\mathcal{V}^{(k)}/\mathcal{V}^{(1)} \subset \mathcal{V}/\mathcal{V}^{(1)}$$

is a maximal isotropic subbundle.

Proposition 4.1.8. *We have*

$$\beta(\mathcal{V}^{(k)} \otimes \sigma^* \mathcal{V}^{(k)}) = 0 \quad \text{and} \quad \beta(\mathcal{V}^{(1)} \otimes \sigma^* \mathcal{V}) = 0.$$

Moreover, the induced map

$$\mathcal{V}^{(k)}/\mathcal{V}^{(1)} \xrightarrow{v \mapsto (w \mapsto \beta(v \otimes w))} \underline{\mathrm{Hom}}(\sigma^* \mathcal{V}/\sigma^* \mathcal{V}^{(k)}, \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}})$$

is an isomorphism.

Proof. All claims can be checked on fibers, where they follow from Propositions 4.1.3 and 4.1.5. ☐

Recalling Theorem 3.4.5, over $Y_\Lambda^{k,\mathrm{red}} \cong \mathrm{DL}_\Lambda^k$ we have the universal flag

$$0 \overset{k-1}{\subset} \mathcal{J} \overset{1}{\subset} \mathcal{K}^\perp \overset{n-2k}{\subset} \mathcal{K} \overset{1}{\subset} \mathcal{J}^\perp \overset{k-1}{\subset} \breve{\Lambda}_0 \otimes \mathcal{O}_S$$

of Definition 3.4.1. The relation between this flag and the filtered vector bundle $\mathcal{V}$ is a bit unclear. It is not hard to see that there is a short exact sequence

$$0 \to \mathcal{V}^{(k)} \to \mathcal{V} \to \mathcal{J} \to 0,$$

and isomorphisms

$$\mathcal{V}^{(1)} \cong \frac{\mathcal{J}^\perp}{\mathcal{K}} \quad \text{and} \quad \mathcal{V}^{(k)} \cong \frac{\breve{\Lambda}_0 \otimes \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}}{\mathcal{K}}.$$

In particular, $\mathcal{V}$ can be realized as an extension of two vector bundles that are each intrinsic to the Deligne-Lusztig variety DL_Λ^k, in the sense that their construction does not rely on the isomorphism $Y_\Lambda^{k,\mathrm{red}} \cong \mathrm{DL}_\Lambda^k$. It is not obvious to authors how one can express $\mathcal{V}$ itself in a way intrinsic to DL_Λ^k.

4.2. **Analysis of π_1.** We can now make explicit the structure of the morphism

$$\pi_1 : \mathbf{RZ}_\Lambda^{\leqslant k} \to Y_\Lambda^k,$$

at least on the level of underlying reduced schemes, in terms of the morphism $\beta : \mathcal{V} \otimes \sigma^*\mathcal{V} \to \mathcal{O}_{Y_\Lambda^{k,\mathrm{red}}}$ of Proposition 4.1.5.

To this end, consider the scheme

$$R_\Lambda^{\leqslant k} \to Y_\Lambda^{k,\mathrm{red}}$$

whose functor of points assigns to any $Y_\Lambda^{k,\mathrm{red}}$-scheme S the set

$$R_\Lambda^{\leqslant k}(S) = \left\{ \begin{array}{l} \text{rank } k-1 \text{ local direct summands} \\ \mathcal{F} \subset \mathcal{V}_S \text{ satisfying } \beta(\mathcal{F} \otimes \sigma^*\mathcal{F}) = 0 \end{array} \right\}.$$

Denote by $R_\Lambda^k \subset R_\Lambda^{\leqslant k}$ the open subscheme with functor of points

$$R_\Lambda^k(S) = \{\mathcal{F} \in R_\Lambda^{\leqslant k}(S) : \mathcal{V}_S = \mathcal{F} \oplus \mathcal{V}_S^{(k)}\},$$

where $\mathcal{V}^{(k)} \subset \mathcal{V}$ is the rank k local direct summand defined by (4.1.3).

Proposition 4.2.1. *The morphism $R_\Lambda^k \to Y_\Lambda^{k,\mathrm{red}}$ is smooth of relative dimension $k-1$.*

Proof. We will show that the morphism in question is formally smooth. Suppose we have a commutative diagram of solid arrows

$$\begin{array}{ccc} \mathrm{Spec}(B/I) & \longrightarrow & R_\Lambda^k \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & Y_\Lambda^{k,\mathrm{red}} \end{array}$$

in which the left vertical arrow is a square-zero thickening of affine $\breve{\mathbb{F}}_p$-schemes. We must show that, Zariski locally on Spec(B), there exists a dotted arrow making the diagram commute.

The pullbacks to Spec(B) of the vector bundles $\mathcal{V}^{(1)} \subset \mathcal{V}^{(k)} \subset \mathcal{V}$ on $Y_\Lambda^{k,\mathrm{red}}$ correspond to inclusions of locally free B-modules

$$V^{(1)} \subset V^{(k)} \subset V,$$

with V endowed with a function $\beta : V \times V \to B$ that is B-linear in the first variable and σ-linear in second. The top horizontal arrow in the diagram corresponds to a choice of complementary summand

$$F \subset V/IV$$

to $V^{(k)}/IV^{(k)}$, satisfying the isotropy condition $\beta(F, F) = 0$.

Lemma 4.2.2. *The set of dotted arrows making the diagram commute admits a simply transitive action of (the additive group underlying) the B/I-module* $\mathrm{Hom}(F, IV^{(1)})$.

Proof. Consider the set $\mathcal{X}$ of all lifts of F to B-submodules $F' \subset V$ satisfying

$$V = F' \oplus V^{(k)}$$

(with no isotropy condition on F'). By standard arguments, this is a principal homogenous space under $\mathrm{Hom}(F, IV^{(k)})$, where the action is defined as follows: given a point $F' \in \mathcal{X}$ and a $\phi : F \to IV^{(k)}$, we let ϕ' denote the composition

$$F' \to F'/IF' = F \xrightarrow{\phi} IV^{(k)},$$

and define

$$\phi + F' = \{x - \phi'(x) : x \in F'\}.$$

The set of dotted arrow making the diagram commute is in bijection with the set of all $F' \in \mathcal{X}$ that satisfy the isotropy condition $\beta(F', F') = 0$. Let us now fix one $F' \in \mathcal{X}$, and parametrize those $\phi \in \mathrm{Hom}(F, IV^{(k)})$ for which $\phi + F'$ is isotropic.

To say that $\phi + F'$ is isotropic is equivalent to saying that

$$\beta(x - \phi'(x), y - \phi'(y)) = 0$$

for all $x, y \in F'$. By assumption $I^2 = 0$, and so the σ-linearity of β in the second variable implies

$$\beta(F', \phi'(F')) \subset \beta(F', IV^{(k)}) = I^p \beta(F', V^{(k)}) = 0.$$

The isotropy condition on $\phi + F'$ therefore simplifies to

$$\beta(x, y) = \beta(\phi'(x), y) \tag{4.2.1}$$

for all $x, y \in F'$.

Proposition 4.1.8 tells us that the natural map

$$V^{(k)}/V^{(1)} \xrightarrow{x \mapsto (y \mapsto \beta(x,y))} \mathrm{Hom}_\sigma(V/V^{(k)}, B) = \mathrm{Hom}_\sigma(F', B)$$

is an isomorphism, where Hom_σ means σ-linear homomorphisms of B-modules. In particular, there is a unique homomorphism

$$\psi' : F' \to V^{(k)}/V^{(1)}$$

satisfying $\beta(x,y) = \beta(\psi'(x),y)$ for all $x,y \in F'$. Using the isotropy of the original $F \subset V$, we see that ψ' takes values in $IV^{(k)}/IV^{(1)}$, and hence factors through a morphism

$$F = F'/IF' \xrightarrow{\psi} IV^{(k)}/IV^{(1)}.$$

What we have shown is that the $\phi \in \mathrm{Hom}(F, IV^{(k)})$ for which $\phi + F'$ is isotropic are precisely those that satisfy (4.2.1), and these are precisely the ϕ that lift $\psi \in \mathrm{Hom}(F, IV^{(k)}/IV^{(1)})$. This is clearly a principal homogeneous space under $\mathrm{Hom}(F, IV^{(1)})$. □

The lemma shows first that the set of dotted arrows making the diagram commute is nonempty, and hence $R^k_\Lambda \to Y^{k,\mathrm{red}}_\Lambda$ is formally smooth.

By taking $B = \breve{\mathbb{F}}_p[\epsilon]$ to be the ring of dual numbers, the lemma also implies that the relative tangent space to this morphism at a point $s \in R^k_\Lambda(\breve{\mathbb{F}}_p)$, corresponding to an isotropic direct summand $\mathcal{F}_s \subset \mathcal{V}_s$, is isomorphic to the $\breve{\mathbb{F}}_p$-vector space $\mathrm{Hom}(\mathcal{F}_s, \mathcal{V}^{(1)}_s)$ of dimension $k-1$. Hence all closed fibers of the morphism in question have dimension $k-1$. □

Theorem 4.2.3. *There is a canonical isomorphism of $Y^{k,\mathrm{red}}_\Lambda$-schemes*

$$\mathbf{RZ}^{\leqslant k,\mathrm{red}}_\Lambda \cong R^{\leqslant k}_\Lambda,$$

restricting to an isomorphism $\mathbf{RZ}^{k,\mathrm{red}}_\Lambda \cong R^k_\Lambda$.

Proof. Over $\mathbf{RZ}^{\leqslant k,\mathrm{red}}_\Lambda$, the universal diagram (3.1.4) determines morphisms of filtered vector bundles

$$\mathcal{D}(H) \to \mathcal{D}(X) \to \mathcal{D}(H^\vee).$$

These induce morphisms $\mathcal{M}_0 \to \mathcal{L}_0 \to \mathcal{N}_0$ between the filtered vector bundles from (2.4.4) and the proof of Theorem 3.4.5.

Recall from (4.1.2) that $\mathcal{V}$ is a quotient of $\mathrm{Fil}^0 \mathcal{N}_0$. The key step of the proof is to note that the coherent sheaf

$$\mathcal{F} \stackrel{\mathrm{def}}{=} \mathrm{Image}(\mathrm{Fil}^0 \mathcal{L}_0 \to \mathrm{Fil}^0 \mathcal{N}_0 \to \mathcal{V})$$

on $\mathbf{RZ}^{\leqslant k,\mathrm{red}}_\Lambda$ is a rank $k-1$ local direct summand of $\mathcal{V}$ satisfying $b(\mathcal{F} \otimes \sigma^*\mathcal{F}) = 0$, while the coherent sheaf

$$\mathcal{H} \stackrel{\mathrm{def}}{=} \ker(\mathrm{Fil}^0 \mathcal{N}_0 \to \mathcal{V}/\mathcal{F})$$

is a local direct summand of $\mathrm{Fil}^0 \mathcal{N}_0$ satisfying

$$\mathrm{Fil}^0 \mathcal{L}_0 = \ker(\mathcal{L}_0 \to \mathcal{N}_0/\mathcal{H}). \tag{4.2.2}$$

All of these claims can be verified on fibers at $\breve{\mathbb{F}}_p$-valued points, where (under the identifications of Proposition 3.2.1) we have

$$\begin{aligned} \mathrm{Fil}^0 \mathcal{L}_{0,s} &= L_0^*/pL_0 \subset L_0/pL_0 = \mathcal{L}_{0,s} \\ \mathcal{F}_s &= L_0^*/N_0^* \subset M_0^*/N_0^* = \mathcal{V}_s \\ \mathcal{H}_s &= L_0^*/pN_0 \subset M_0^*/pN_0 = \mathrm{Fil}^0 \mathcal{N}_{0,s}. \end{aligned}$$

In particular, the isotropic subbundle $\mathcal{F} \subset \mathcal{V}$ just constructed defines a morphism

$$\mathbf{RZ}_\Lambda^{\leqslant k,\mathrm{red}} \to R_\Lambda^{\leqslant k}, \tag{4.2.3}$$

which is bijective on $\breve{\mathbb{F}}_p$-valued points.

We will use the auxiliary vector bundle $\mathcal{H} \subset \mathrm{Fil}^0 \mathcal{N}_0$ to show that the morphism we have constructed is formally unramified. To this end, assume we have a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{RZ}_\Lambda^{\leqslant k,\mathrm{red}} \\ \downarrow & \nearrow & \downarrow \\ \widetilde{S} & \longrightarrow & R_\Lambda \end{array}$$

of solid arrows, in which the left vertical arrow is a square-zero thickening. To show that our morphism is formally unramified we must show that there is at most one dotted arrow making the diagram commute, so suppose we have two such arrows $a, b : \widetilde{S} \to \mathbf{RZ}_\Lambda^{\leqslant k,\mathrm{red}}$.

Because a and b define the same $\widetilde{S}$-point of R_Λ, there are canonical identifications (respecting filtrations)

$$\begin{array}{ccccccc} a^*\mathcal{M}_0 & \longrightarrow & a^*\mathcal{N}_0 & \longrightarrow & a^*\mathcal{V} & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ b^*\mathcal{M}_0 & \longrightarrow & b^*\mathcal{N}_0 & \longrightarrow & b^*\mathcal{V} & \longrightarrow & 0, \end{array}$$

of vector bundles on $\widetilde{S}$, the last of which identifies $a^*\mathcal{F} = b^*\mathcal{F}$. It follows that there are canonical identifications

$$\begin{array}{ccccccc} a^*\mathcal{H} & \longrightarrow & a^*\,\mathrm{Fil}^0 \mathcal{N}_0 & \longrightarrow & a^*(\mathcal{V}/\mathcal{F}) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ b^*\mathcal{H} & \longrightarrow & b^*\,\mathrm{Fil}^0 \mathcal{N}_0 & \longrightarrow & b^*(\mathcal{V}/\mathcal{F}) & \longrightarrow & 0. \end{array}$$

of vector bundles on $\widetilde{S}$.

Now consider the p-divisible groups a^*X and b^*X over $\widetilde{S}$. We do not yet know that these are isomorphic, but they are deformations of the same p-divisible group over S. By Grothendieck-Messing theory there is a canonical isomorphism

$$a^*\mathcal{D}(X) = \mathcal{D}(a^*X) \cong \mathcal{D}(b^*X) = b^*\mathcal{D}(X) \tag{4.2.4}$$

of vector bundles on $\widetilde{S}$. By Remark 2.4.1 this induces an isomorphism $a^*\mathcal{L} \cong b^*\mathcal{L}$, and we now have canonical identifications

$$\begin{array}{ccc} a^*\mathcal{L}_0 & \longrightarrow & a^*(\mathcal{N}_0/\mathcal{H}) \\ \| & & \| \\ b^*\mathcal{L}_0 & \longrightarrow & b^*(\mathcal{N}_0/\mathcal{H}). \end{array}$$

of vector bundles on $\widetilde{S}$. It follows from (4.2.2) that the isomorphism on the left identifies $a^*\operatorname{Fil}^0\mathcal{L}_0 = b^*\operatorname{Fil}^0\mathcal{L}_0$. As with (3.4.2), the isomorphism of Remark 2.4.1 identifies the Fil^0 on source and target, and hence (4.2.4) identifies

$$a^*\operatorname{Fil}^0\mathcal{D}(X)_0 = b^*\operatorname{Fil}^0\mathcal{D}(X)_0.$$

As the principal polarization on X induces a perfect bilinear pairing between $\mathcal{D}(X)_0$ and $\mathcal{D}(X)_1$, under which the Hodge filtrations are exact annihilators of each other, the same identification holds if we replace $\mathcal{D}(X)_0$ with $\mathcal{D}(X)_1$. In other words, (4.2.4) respects Hodge filtrations, and so $a^*X \cong b^*X$ by Grothendieck-Messing theory. From this it follows easily that $a = b$.

The unramified morphism (4.2.3) is proper (its source is projective over $\breve{\mathbb{F}}_p$) and bijective on closed points, so is an isomorphism by Lemma 3.4.3.

For the final claim, fix a point

$$s \in \mathbf{RZ}_\Lambda^{\leqslant k,\mathrm{red}}(\breve{\mathbb{F}}_p) \cong R_\Lambda^{\leqslant k}(\breve{\mathbb{F}}_p)$$

represented by a triple (L_0, M_0, N_0) as in Proposition 3.2.1. In particular,

$$L_0 \cap \breve{\Lambda}_0 \subset M_0 \overset{k-1}{\subset} \breve{\Lambda}_0. \tag{4.2.5}$$

By definition, $s \in R_\Lambda^k(\breve{\mathbb{F}}_p)$ holds if and only if equality holds in

$$\mathcal{F}_s + \mathcal{V}_s^{(k)} = \frac{L_0^* + \breve{\Lambda}_0}{N_0^*} \subset \frac{M_0^*}{N_0^*} = \mathcal{V}_s.$$

Dualizing, this is equivalent to $L_0 \cap \breve{\Lambda}_0 = M_0$, which is equivalent to

$$L_0 \cap \breve{\Lambda}_0 \overset{k-1}{\subset} \breve{\Lambda}_0$$

by (4.2.5), which is equivalent to $s \in \mathbf{RZ}_\Lambda^k(\breve{\mathbb{F}}_p)$ by Proposition 2.4.4 and the final claim of Proposition 3.2.1. □

Corollary 4.2.4. *There is a smooth morphism*

$$\mathrm{RZ}_\Lambda^{k,\mathrm{red}} \cong R_\Lambda^k \to Y_\Lambda^{k,\mathrm{red}} \cong \mathrm{DL}_\Lambda^k$$

of relative dimension $k-1$. In particular, $\mathrm{RZ}_\Lambda^{k,\mathrm{red}}$ is itself smooth of dimension $n-2$.

Proof. The first isomorphism is the composition

$$\mathrm{RZ}_\Lambda^{k,\mathrm{red}} \cong \mathbf{RZ}_\Lambda^{k,\mathrm{red}} \cong R_\Lambda^k$$

of the isomorphisms of Proposition 3.3.2 and Theorem 4.2.3. The arrow in the middle is the smooth morphism of Proposition 4.2.1. The final isomorphism is that of Theorem 3.4.5. The final claim of the corollary now follows from Proposition 3.4.2. □

Remark 4.2.5. In the special case $k = 1$ we have

$$\mathrm{RZ}_\Lambda^{1,\mathrm{red}} \cong R_\Lambda^1 = Y_\Lambda^{1,\mathrm{red}} \cong \mathrm{DL}_\Lambda^1.$$

5. Supplementary results when n is even

Intuitively, points of the closed subscheme $\mathrm{RZ}_\Lambda \subset \mathrm{RZ}$ parametrize p-divisible groups that are relatively close to the framing object $\Lambda \otimes \overline{\mathbb{Y}}$ of signature $(0, n)$. Proposition 2.4.4 can be understood as saying that in the decomposition

$$\mathrm{RZ}_\Lambda^{\mathrm{red}} = \bigsqcup_{1 \leqslant k \leqslant \lfloor n/2 \rfloor} \mathrm{RZ}_\Lambda^{k,\mathrm{red}},$$

the locally closed subschemes indexed by smaller k parametrize points that are closer to $\Lambda \otimes \overline{\mathbb{Y}}$ than those parametrized by larger k.

It turns out that in the extremal case in which n is even and $k = n/2$, each point of $\mathrm{RZ}_\Lambda^{n/2}$, while as far from $\Lambda \otimes \overline{\mathbb{Y}}$ as it allowed to be, is very close to one of finitely many other framing objects. These other framing objects, which again have signature $(0, n)$ but are endowed with non-principal polarizations, provide a different way to parametrize the points of $\mathrm{RZ}_\Lambda^{n/2}$. This is what we explore in this section.

5.1. **Another partial Rapoport-Zink space.**

As always, Λ is a self-dual hermitian $\mathcal{O}_E$-lattice of rank $n \geqslant 2$. the hermitian form is denoted $h(-,-)$.

Definition 5.1.1. An $\mathcal{O}_E$-lattice $\Lambda' \subset \Lambda[1/p]$ is *scalar-self-dual* if there exists a $c \in \mathbb{Q}_p^\times$ such that

$$c\Lambda' = \{x \in \Lambda[1/p] : h(\Lambda', x) \subset \mathcal{O}_E\}.$$

We are especially interested in scalar-self-dual lattices Λ' satisfying

$$p\Lambda \subsetneq \Lambda' \subsetneq \Lambda. \tag{5.1.1}$$

The self-duality of Λ then forces both $\mathrm{ord}_p(c) = -1$ and

$$\mathrm{length}_{\mathcal{O}_E}(\Lambda/\Lambda') = \frac{n}{2}.$$

In particular, such lattices can only exist when n is even, and we assume this for the remainder of §5. The scalar self-duality condition on Λ' can then be interpreted as saying that $\Lambda'/p\Lambda \subset \Lambda/p\Lambda$ is maximal isotropic under the natural $\mathcal{O}_E/p\mathcal{O}_E$-valued hermitian form

Definition 5.1.2. For a scalar-self-dual $\mathcal{O}_E$-lattice Λ' as in (5.1.1), and an $\breve{\mathbb{F}}_p$-scheme S, define $\mathrm{RZ}^{\heartsuit}_{\Lambda'}(S)$ to be the set of isomorphism classes of quadruples $(X, \lambda_X, \alpha_X, \beta_X)$ in which

- X is a p-divisible group over S equipped with an $\mathcal{O}_E$-action of signature $(2, n-2)$,
- $\lambda_X : X \to X^\vee$ is a conjugate $\mathcal{O}_E$-linear principal polarization,
- α_X and β_X are $\mathcal{O}_E$-linear isogenies
$$\Lambda' \otimes \overline{\mathbb{Y}}_S \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} p^{-1}\Lambda' \otimes \overline{\mathbb{Y}}_S$$
whose composition is induced by the inclusion $\Lambda' \subset p^{-1}\Lambda'$, and such that $\alpha_X^* \lambda_X$ agrees with the canonical (non-principal) polarization on $\Lambda' \otimes \overline{\mathbb{Y}}_S$.

.

The functor $\mathrm{RZ}^{\heartsuit}_{\Lambda'}$ of Definition 5.1.2 is represented by a projective $\breve{\mathbb{F}}_p$-scheme, denoted the same way. For any point $(X, \lambda_X, \alpha_X, \beta_X) \in \mathrm{RZ}^{\heartsuit}_{\Lambda'}(S)$ there is a unique quasi-isogeny ϱ_X making the diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{\alpha_X}{\nearrow} & \big\downarrow{\scriptstyle \varrho_X} & \overset{\beta_X}{\searrow} & \\ \Lambda' \otimes \overline{\mathbb{Y}}_S & \longrightarrow & \Lambda \otimes \overline{\mathbb{Y}}_S & \longrightarrow & p^{-1}\Lambda' \otimes \overline{\mathbb{Y}}_S \end{array}$$

commute, and this determines a point $(X, \lambda_X, \varrho_X) \in \mathrm{RZ}_\Lambda(S)$. We use this construction to regard
$$\mathrm{RZ}^{\heartsuit}_{\Lambda'} \subset \mathrm{RZ}_\Lambda$$
as a closed subscheme.

Proposition 5.1.3. *For any scalar self-dual lattice Λ' satisfying (5.1.1), the bijection of Corollary 2.3.3 restricts to a bijection*
$$\mathrm{RZ}^{\heartsuit}_{\Lambda'}(\breve{\mathbb{F}}_p) \cong \left\{ \begin{array}{c} \breve{\mathbb{Z}}_p\text{-lattices} \\ L_0 \subset \breve{\Lambda}_0[1/p] \end{array} : \begin{array}{c} pL_0 \subset L_0^* \overset{2}{\subset} L_0 \\ \breve{\Lambda}'_0 \subset L_0 \subset p^{-1}\breve{\Lambda}'_0 \end{array} \right\}.$$
Moreover, the two chains of inclusions on the right hand side are equivalent to the single chain condition
$$\breve{\Lambda}'_0 \overset{\frac{n}{2}-1}{\subset} L_0^* \overset{2}{\subset} L_0 \overset{\frac{n}{2}-1}{\subset} p^{-1}\breve{\Lambda}'_0.$$

Proof. The first claim follows directly from the construction of the bijection of Corollary 2.3.3. The scalar-self-duality assumption on Λ' implies that $\breve{\Lambda}'_0 \oplus p^{-1}\breve{\Lambda}'_1$ is self-dual. Using this and (2.3.8), we deduce that the right dual operator (2.3.6) interchanges the lattices $\breve{\Lambda}'_0$ and $p^{-1}\breve{\Lambda}'_0$. Applying this operator throughout
$$\breve{\Lambda}'_0 \subset L_0 \overset{r}{\subset} p^{-1}\breve{\Lambda}'_0$$
(this is the definition of r) therefore results in
$$\breve{\Lambda}'_0 \overset{r}{\subset} L_0^* \subset p^{-1}\breve{\Lambda}'_0,$$

and the second claim follows immediately. □

Proposition 5.1.4. *For every point* $s \in \mathrm{RZ}^{n/2}_{\Lambda}(\breve{\mathbb{F}}_p)$ *there exists a scalar-self-dual lattice* Λ' *as in* (5.1.1) *such that* $s \in \mathrm{RZ}^{\heartsuit}_{\Lambda'}(\breve{\mathbb{F}}_p)$.

Proof. Under the bijection of Proposition 3.2.1, the point s corresponds to a chain of lattices

$$p\breve{\Lambda}_0 \overset{\frac{n}{2}-1}{\subset} pM_0^* \overset{1}{\subset} pN_0 = N_0^* \overset{1}{\subset} M_0 \overset{\frac{n}{2}-1}{\subset} \breve{\Lambda}_0,$$

together with a lattice $L_0 \subset \breve{\Lambda}_0$ satisfying

$$pL_0 \subset L_0^* \overset{2}{\subset} L_0 \quad \text{and} \quad M_0 \overset{\frac{n}{2}}{\subset} L_0 \overset{\frac{n}{2}-1}{\subset} N_0.$$

The essential thing is the middle equality $pN_0 = N_0^*$, which implies

$$\Phi^{-2} N_0 \overset{(2.3.7)}{=} N_0^{**} = (pN_0)^* = N_0.$$

It follows that the $\mathcal{O}_{\breve{E}}$-lattice $p(N_0 \oplus \Phi N_0) \subset \breve{\Lambda}_0 \oplus \breve{\Lambda}_1$ is fixed by Φ, and we define an $\mathcal{O}_E$-lattice

$$\Lambda' = p(N_0 \oplus \Phi N_0)^{\Phi = \mathrm{id}} \subset \Lambda.$$

Using (2.3.8) we see that $N_0 \oplus \Phi N_0^* \subset \breve{\Lambda}[1/p]$ is self-dual under the hermitian pairing. This implies that the dual lattice of $N_0 \oplus \Phi N_0$ is $p(N_0 \oplus \Phi N_0)$, and taking Φ-fixed points shows that the dual lattice of Λ' is $p^{-1}\Lambda'$. It follows that Λ' is scalar-self-dual and satisfies (5.1.1).

The inclusions $M_0 \subset L_0 \subset N_0$ imply

$$\breve{\Lambda}'_0 = pN_0 \subset L_0 \subset N_0 = p^{-1}\breve{\Lambda}'_0,$$

and Proposition 5.1.3 shows that $s \in \mathrm{RZ}^{\heartsuit}_{\Lambda'}(\breve{\mathbb{F}}_p)$. □

5.2. **Another Deligne-Lusztig variety.** We continue to assume that n is even, and fix a scalar-self dual lattice $\Lambda' \subset \Lambda[1/p]$ satisfying (5.1.1).

The pairing b of (2.3.5) determines a pairing

$$b' = p^{-1} b : \breve{\Lambda}'_0 \times \breve{\Lambda}'_0 \to \breve{\mathbb{Z}}_p.$$

Exactly as in §3.4, this induces a pairing of $\mathcal{O}_S$-modules

$$(\breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S) \times (\breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S) \to \mathcal{O}_S,$$

linear in the first variable and σ-linear in the second, for any $\breve{\mathbb{F}}_p$-scheme S. Once again, for any local direct summand $\mathcal{F} \subset \breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S$, we denote its left annihilator under this pairing by

$$\mathcal{F}^{\perp} = \{x \in \breve{\Lambda}_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S : b'(x, \mathcal{F}) = 0\}.$$

Definition 5.2.1. Define $\mathrm{DL}^{\heartsuit}_{\Lambda'}$ to be the projective $\breve{\mathbb{F}}_p$-scheme whose functor of points assigns to any $\breve{\mathbb{F}}_p$-scheme S the set of flags of $\mathcal{O}_S$-module local direct summands

$$0 \overset{\frac{n}{2}-1}{\subset} \mathcal{F} \overset{2}{\subset} \mathcal{F}^{\perp} \overset{\frac{n}{2}-1}{\subset} \breve{\Lambda}'_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_S$$

of the indicated coranks.

Proposition 5.2.2. *The scheme* $\mathrm{DL}^{\heartsuit}_{\Lambda'}$ *of Definition 5.2.1 is a Deligne-Lusztig variety for the unitary group of the finite hermitian space* $\Lambda'/p\Lambda'$ *(where the hermitian form on* Λ' *is first multiplied by* p^{-1} *to make it self-dual). It is irreducible and smooth of dimension* $n-2$.

Proof. The proof is the same as for Proposition 3.4.2. □

Theorem 5.2.3. *There is a isomorphism*

$$\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'} \cong \mathrm{DL}^{\heartsuit}_{\Lambda'}$$

sending an $\breve{\mathbb{F}}_p$*-valued point of the left hand side to the flag*

$$0 \subset \frac{pL_0^*}{p\breve{\Lambda}'_0} \subset \frac{pL_0}{p\breve{\Lambda}'_0} \subset \frac{\breve{\Lambda}'_0}{p\breve{\Lambda}'_0}$$

determined by the bijection of Proposition 5.1.3.

Proof. This is similar to the proof of Theorem 3.4.5. The universal isogeny $\beta_X : X \to p^{-1}\Lambda' \otimes \overline{\mathbb{Y}}$ over $\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}$ induces a morphism of vector bundles

$$\mathcal{D}(X) \to \mathcal{D}(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}}),$$

and we define coherent sheaves

$$0 \subset \mathcal{G} \subset \mathcal{G}^{\dagger} \subset \mathcal{D}(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}})_0 \tag{5.2.1}$$

on $\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}$ by

$$\mathcal{G} = \mathrm{Image}\big(\operatorname{Fil}^0 \mathcal{D}(X)_0 \to \mathcal{D}(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}})_0\big)$$
$$\mathcal{G}^{\dagger} = \mathrm{Image}\big(\mathcal{D}(X)_0 \to \mathcal{D}(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}})_0\big).$$

Fix an isomorphism $\mathcal{O}_{\breve{E}}$-modules $D(\overline{\mathbb{Y}}) \cong \mathcal{O}_{\breve{E}}$. This determines isomorphisms of $\mathcal{O}_{\breve{E}}$-modules

$$D(\Lambda' \otimes \overline{\mathbb{Y}}) \cong \breve{\Lambda}'$$

as in (2.3.1), and an isomorphism of vector bundles

$$\mathcal{D}(\Lambda' \otimes \overline{\mathbb{Y}}) \cong \breve{\Lambda}' \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}}.$$

Using the multiplication-by-p isomorphism $p^{-1}\Lambda' \cong \Lambda'$, we now identify

$$\mathcal{D}(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}}) \cong \breve{\Lambda}' \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}},$$

and identify (5.2.1) with a flag of coherent sheaves

$$0 \subset \mathcal{F} \subset \mathcal{F}^{\dagger} \subset \breve{\Lambda}'_0 \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}}. \tag{5.2.2}$$

At a point of $\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}(\breve{\mathbb{F}}_p)$, corresponding to a chain of lattices

$$\breve{\Lambda}'_0 \overset{\frac{n}{2}-1}{\subset} L_0^* \overset{2}{\subset} L_0 \overset{\frac{n}{2}-1}{\subset} p^{-1}\breve{\Lambda}'_0,$$

the fibers of $\mathcal{F}$ and $\mathcal{F}^\dagger$ are identified with the images of

$$\frac{pL_0^*}{p^2L_0} \to \frac{\breve{\Lambda}_0'}{p\breve{\Lambda}_0'} \quad \text{and} \quad \frac{pL_0}{p^2L_0} \to \frac{\breve{\Lambda}_0'}{p\breve{\Lambda}_0'},$$

respectively. See the proof of Theorem 3.4.5, and especially Lemma 3.4.6), In particular these coherent sheaves have constant fiber dimension, and one can deduce using Lemma 3.4.4 that they are local direct summands of $\breve{\Lambda}_0' \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathrm{RZ}_{\Lambda'}^{\heartsuit,\mathrm{red}}}$. The equality $\mathcal{F}^\dagger = \mathcal{F}^\perp$ can also be checked on fibers, where it is clear from the definition of the pairing b'.

All of this shows that (5.2.2) defines a morphism

$$\mathrm{RZ}_{\Lambda'}^{\heartsuit,\mathrm{red}} \to \mathrm{DL}_{\Lambda'}^{\heartsuit}$$

with the desired form on $\breve{\mathbb{F}}_p$-valued points, and it remains to show that it is an isomorphism. For this, once again by Lemma 3.4.3, it suffices to show that the morphism in question is unramified. The proof is essentially the same as that of Lemma 3.4.8, replacing the use of Lemma 3.4.7 with the equality

$$\mathrm{Fil}^0\, \mathcal{D}(X) = \ker(\mathcal{D}(X) \to \mathcal{G}^\dagger/\mathcal{G} \cong \mathcal{F}^\perp/\mathcal{F}).$$

This last equality can be verified on fibers at point $s \in \mathrm{RZ}_{\Lambda'}^{\heartsuit,\mathrm{red}}(\breve{\mathbb{F}}_p)$, where it is equivalent to the obvious equality

$$\frac{L_0^*}{pL_0} = \ker\left(\frac{L_0}{pL_0} \xrightarrow{p} \frac{pL_0/p\breve{\Lambda}_0'}{pL_0^*/p\breve{\Lambda}_0'}\right). \qquad \square$$

6. Irreducible components of the Rapoport-Zink space

In this section we prove our main results on the structure of the reduced scheme $\mathrm{RZ}^{\mathrm{red}}$ underlying the formal $\breve{\mathbb{F}}_p$-scheme RZ of Definition 2.2.1. The key point is to explain the relation between the locally closed subschemes $\mathrm{RZ}_{\Lambda}^{k,\mathrm{red}} \subset \mathrm{RZ}^{\mathrm{red}}$ of Definition 2.4.2, and the irreducible components of $\mathrm{RZ}^{\mathrm{red}}$, as described in [19] and [4].

6.1. **The affine Deligne-Luszig variety.** Fix an integer $n \geqslant 2$. Let W be an n-dimensional vector space over E equipped with a hermitian form $h : W \times W \to E$. Up to isomorphism there are two such W, distinguished by the value of

$$\det(W) \in \mathbb{Q}_p^\times/\mathrm{Nm}_{E/\mathbb{Q}_p}(E^\times).$$

We assume that $\det(W) = 1$, which implies the existence of an E-basis $x_1, \ldots, x_n \in W$ such that the hermitian form is given by the matrix with 1's on the antidiagonal, and 0's elsewhere. In other words

$$h(x_i, x_j) = \begin{cases} 1 & \text{if } i + j = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

The group of unitary similitudes $G = \mathrm{GU}(W)$ is an unramified reductive group over $\mathbb{Q}_p$, and our choice of basis determines subgroups

$$T \subset B \subset G,$$

in which the Borel B is the stabilizer of the flag $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = V$ defined by $\mathcal{F}_i = \mathrm{Span}_E\{x_1, \ldots, x_i\}$, and T is the maximal torus that acts through scalars on every x_i.

Regard $x_1, \ldots, x_n \in \breve{W} \stackrel{\mathrm{def}}{=} W \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$ as an $\breve{E}$-basis, and denote by

$$y_1, \ldots, y_n \in \breve{W}_0 \quad \text{and} \quad z_1, \ldots, z_n \in \breve{W}_1$$

the projections of these basis vectors to the two summands in the decomposition of (1.2.1). Thus $x_i = y_i + z_i$, the Frobenius operator $\sigma : \breve{W} \to \breve{W}$ interchanges y_i with z_i, and we have a $\breve{\mathbb{Q}}_p$-basis

$$y_1, \ldots, y_n, z_1, \ldots, z_n \in \breve{W}.$$

We use this last basis to identify $G(\breve{\mathbb{Q}}_p) \subset \mathrm{GL}_{2n}(\breve{\mathbb{Q}}_p)$. The Borel $B(\breve{\mathbb{Q}}_p)$ is then identified with the upper triangular matrices in $G(\breve{\mathbb{Q}}_p)$, while $T(\breve{\mathbb{Q}}_p)$ is the subgroup of diagonal matrices of the form

$$[t_0, t_1, \ldots, t_n] \stackrel{\mathrm{def}}{=} \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1}t_0 & & \\ & & & & \ddots & \\ & & & & & t_1^{-1}t_0 \end{pmatrix}.$$

The cocharacter lattice of T has a $\mathbb{Z}$-basis $\epsilon_0, \epsilon_1, \ldots, \epsilon_n \in X_*(T)$ given by

$$\begin{aligned} \epsilon_0(t) &= [t, 1, 1, \ldots, 1] \\ \epsilon_1(t) &= [1, t, 1, \ldots, 1] \\ &\vdots \\ \epsilon_n(t) &= [1, 1, \ldots, 1, t], \end{aligned}$$

Definition 6.1.1. A cocharacter

$$\lambda = a_0\epsilon_0 + a_1\epsilon_1 \cdots + a_n\epsilon_n \in X_*(T)$$

is *minuscule* if $|a_i - a_j| \leqslant 1$ for all $i, j \in \{1, \ldots, n\}$, and is *dominant* (with respect to B) if $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$.

Remark 6.1.2. The action of the Frobenius σ on $X_*(T)$ is given by

$$\begin{aligned} \epsilon_0^\sigma &= \epsilon_0 + \epsilon_1 + \cdots + \epsilon_n \\ \epsilon_i^\sigma &= -\epsilon_{i^\vee} \text{ for } 1 \leqslant i \leqslant n, \end{aligned}$$

where we abbreviate $i^\vee = n + 1 - i$.

Remark 6.1.3. The center $Z \subset G$ is isomorphic to the Weil restriction $\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$, and its cocharacter lattice is

$$X_*(Z) = \mathrm{Span}_{\mathbb{Z}}\{\epsilon_0, \epsilon_0^\sigma\} \subset X_*(T).$$

The basis vectors $x_1, \ldots, x_n \in W$ span an $\mathcal{O}_E$-lattice

$$\Lambda = \mathrm{Span}_{\mathcal{O}_E}\{x_1, \ldots, x_n\} \subset W$$

self-dual under the hermitian form. The group of unitary similitudes of Λ determines an extension of G to a reductive group scheme over $\mathbb{Z}_p$, denoted the same way. The subgroup $G(\breve{\mathbb{Z}}_p) \subset G(\breve{\mathbb{Q}}_p)$ is the stabilizer of the $\mathcal{O}_{\breve{E}}$-lattice

$$\mathbb{D} \stackrel{\text{def}}{=} \Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}} = \mathrm{Span}_{\breve{\mathbb{Z}}_p}\{y_1, \ldots, y_n, z_1, \ldots, z_n\} \subset \breve{W}. \tag{6.1.1}$$

We now define the particular affine Deligne-Lusztig variety of interest.

Definition 6.1.4. The *affine Deligne-Lusztig variety* is the set

$$X_\mu(b) = \{g \in G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Z}}_p) : g^{-1}bg^\sigma \in G(\breve{\mathbb{Z}}_p)\mu(p)G(\breve{\mathbb{Z}}_p)\}$$

where $b = \epsilon_0(p) \in Z(\breve{\mathbb{Q}}_p)$ and $\mu = \epsilon_0 + \epsilon_1 + \epsilon_2 \in X_*(T)$.

Remark 6.1.5. Because our chosen $b \in G(\breve{\mathbb{Q}}_p)$ is central, its twisted centralizer J_b is canonically identified with G, and the Deligne-Lusztig variety $X_\mu(b)$ is stable under left multiplication by $G(\mathbb{Q}_p)$.

Proposition 6.1.6. *If we use the lattice* Λ *above to define the Rapoport-Zink space of §2.2, there is a bijection of sets*

$$\mathrm{RZ}(\breve{\mathbb{F}}_p) \cong X_\mu(b)$$

identifying

$$\mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p) \cong \{g \in X_\mu(b) : p\mathbb{D} \subset g\mathbb{D} \subset p^{-1}\mathbb{D}\}.$$

Proof. First note that $F = b \circ \sigma$ defines an isocrystal structure on $\breve{W}$ with $Fy_i = pz_i$ and $Fz_i = y_i$. The lattice $\mathbb{D}$ is stable under F and V, and it is easy to see from (2.1.1) that it is the Dieudonné module $\mathbb{D} = D(\mathbb{X})$ of a p-divisible group $\mathbb{X}$ with an $\mathcal{O}_E$-action of signature $(0, n)$. Fixing a $u \in \mathcal{O}_E^\times$ with $\overline{u} = -u$, the alternating form

$$\lambda_W(x, y) \stackrel{\text{def}}{=} \mathrm{Tr}_{E/\mathbb{Q}_p} h(ux, y)$$

on W extends $\breve{\mathbb{Q}}_p$-bilinearly to a polarization of the isocrystal $\breve{W}$. The lattice $\mathbb{D} \subset \breve{W}$ is self-dual under this alternating form, which determines an $\mathcal{O}_E$-conjugate linear principal polarization of $\mathbb{X}$.

We also have the operator $\Phi = \sigma$ on $\breve{W}$, which makes it into a slope 0 isocrystal. Under (6.1.1), there is a unique σ-semi-linear operator F on $\mathcal{O}_{\breve{E}}$ such that the operator F on $\mathbb{D}$ agrees with the operator $\Phi \otimes F$ on $\Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$. This operator makes $\mathcal{O}_{\breve{E}}$ into a Dieudonné module isomorphic to $D(\overline{\mathbb{Y}})$, and a choice of such an isomorphism identifies

$$\mathbb{X} = \Lambda \otimes \overline{\mathbb{Y}}.$$

The rest is routine. For any $g \in X_\mu(b)$ the $\mathcal{O}_{\breve{E}}$-lattice $g\mathbb{D} = D(X)$ is the Dieudonné module of a p-divisible group X with an $\mathcal{O}_E$-action, and the cocharacter μ was chosen to ensure that X has signature $(2, n-2)$. The inclusion

$$D(X) = g\mathbb{D} \subset \mathbb{D}[1/p] = D(\mathbb{X})[1/p]$$

corresponds to an $\mathcal{O}_E$-linear quasi-isogeny $\varrho_X : X \dashrightarrow \mathbb{X}$, and the pullback of the principal polarization on the target can be rescaled by a unique power of p to obtain a principal polarization λ_X of X. The triple $(X, \lambda_X, \varrho_X)$ defines a point of $\mathrm{RZ}(\breve{\mathbb{F}}_p)$, and this is the desired bijection. □

6.2. **Labeling the components.** We now invoke the parametrization of irreducible components of Rapoport-Zink spaces due to Xiao-Zhu [19], and its refinement in the case of $\mathrm{GU}(2, n-2)$ worked out in [4].

Given cosets $g_1, g_2 \in G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Z}}_p)$, the Cartan decomposition implies the existence of a unique dominant $\beta_{g_1,g_2} \in X_*(T)$ such that

$$g_2^{-1} g_1 \in G(\breve{\mathbb{Z}}_p)\beta_{g_1,g_2}(p)G(\breve{\mathbb{Z}}_p).$$

We call this cocharacter the *relative position invariant* of the lattices $g_1\mathbb{D}$ and $g_2\mathbb{D}$, and denote it by

$$\mathrm{inv}_G(g_1\mathbb{D}, g_2\mathbb{D}) \stackrel{\mathrm{def}}{=} \beta_{g_1,g_2} \in X_*(T).$$

The subscript G is included to distinguish this from the invariant of Definition 2.4.3. In this terminology, our affine Deligne-Lusztig variety becomes

$$X_\mu(b) = \{g \in G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Z}}_p) : \mathrm{inv}_G(bg^\sigma\mathbb{D}, g\mathbb{D}) = \mu\}.$$

Define cocharacters $\alpha_1, \ldots, \alpha_{\lfloor n/2 \rfloor} \in X_*(T)$ by (recall $k^\vee = n + 1 - k$)

$$\alpha_k = \begin{cases} (\epsilon_1 + \cdots + \epsilon_{k-1}) - (\epsilon_{k^\vee} + \cdots + \epsilon_{1^\vee}) & \text{if } k < n/2 \\ \epsilon_0 + \epsilon_1 + \cdots + \epsilon_{k-1} & \text{if } k = n/2. \end{cases} \tag{6.2.1}$$

Using the bijection $X_\mu(b) \cong \mathrm{RZ}(\breve{\mathbb{F}}_p)$ of Proposition 6.1.6, for any $\gamma \in G(\mathbb{Q}_p)$ and $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, denote by

$$\mathrm{RZ}_{(k,\gamma)} \subset \mathrm{RZ}^{\mathrm{red}}$$

the locally closed subset (endowed with its reduced scheme structure) whose $\breve{\mathbb{F}}_p$-points are

$$\mathrm{RZ}_{(k,\gamma)}(\breve{\mathbb{F}}_p) = \{g \in X_\mu(b) : \mathrm{inv}_G(g\mathbb{D}, \gamma\mathbb{D}) = \alpha_k\}. \tag{6.2.2}$$

This only depends on the coset $\gamma \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, and satisfies

$$\mathrm{RZ}_{(k,\gamma)} = \gamma \cdot \mathrm{RZ}_{(k,\mathrm{id})}. \tag{6.2.3}$$

Remark 6.2.1. The cocharacters α_1 and (if n is even) $\alpha_{n/2}$ are minuscule, but the other α_i's are not. This is closely tied up with the fact that $\mathrm{RZ}_{(1,\gamma)}$ and $\mathrm{RZ}_{(n/2,\gamma)}$ are isomorphic to Deligne-Lusztig varieties, as we soon see.

Theorem 6.2.2. *Abbreviate* $r = \lfloor n/2 \rfloor$. *There is a bijection*

$$\{1, \ldots, r\} \times G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \cong \{\textit{irreducible components of } \mathrm{RZ}\}$$

sending the pair (k, γ) *to the closure of* $\mathrm{RZ}_{(k,\gamma)}$.

Proof. This is the parametrization of irreducible components from [19], made explicit in [4] in the special case of GU$(2, n-2)$. For the reader's benefit, we provide a very rough sketch.

The triple $T \subset B \subset G$ has Langlands dual

$$\widehat{T} \subset \widehat{B} \subset \widehat{G} \cong \mathrm{GL}_{2n} \times \mathbb{G}_m,$$

where $\widehat{B}$ is the subgroup of matrices upper triangular in the first factor, and $\widehat{T}$ is the subgroup of matrices diagonal in the first factor. Let V_μ be the representation of $\widehat{G}$ of heighest weight

$$\mu \in X_*(T) = X^*(\widehat{T}).$$

Every $\lambda \in X_*(T) = X^*(\widehat{T})$ determines a weight space $V_\mu(\lambda)$, and because μ is minuscule the nonzero weight spaces lie in a single orbit under the action of the Weyl group. In other words

$$\dim V_\mu(\lambda) = \begin{cases} 1 & \text{if } \lambda = \epsilon_0 + \epsilon_i + \epsilon_j \text{ for some } i \neq j \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.4}$$

For any $\lambda \in X_*(T)$ denote by $[\lambda] \in X_*(T)/(\sigma - 1)X_*(T)$ its image under the quotient map. By Theorem 4.4.14 of [19], and recalling the equality $J_b = G$ of Remark 6.1.5, the irreducible components of $X_\mu(b)$ are in bijection with

$$\bigsqcup_{\substack{\lambda \in X_*(T) \\ [\lambda] = [\epsilon_0]}} \mathbb{MV}_\mu(\lambda) \times G(\mathbb{Q}_p)/G(\mathbb{Z}_p), \tag{6.2.5}$$

where $\mathbb{MV}_\mu(\lambda)$ is a finite set of cardinality (6.2.4).

As $[\epsilon_0 + \epsilon_i + \epsilon_j] = [\epsilon_0]$ if and only if $j = i^\vee$, the calculation (6.2.4) shows that the λ contributing to (6.2.5) are precisely those of the form

$$\lambda_i \stackrel{\text{def}}{=} \epsilon_0 + \epsilon_i + \epsilon_{i^\vee} \in X_*(T)$$

with $1 \leqslant i \leqslant r$. Thus Theorem 4.4.14 of [19] establishes a bijection

$$\{\lambda_1, \ldots, \lambda_r\} \times G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \cong \{\text{irreducible components of } \mathrm{RZ}^{\mathrm{red}}\}, \tag{6.2.6}$$

which we must make explicit.

For every $1 \leqslant i \leqslant r$, Lemma 4.4.3 of [19] associates a dominant cocharacter $\nu_i \in X_*(T)$ to the unique element of the set $\mathbb{MV}_\mu(\lambda_i)$. This cocharacter is not uniquely determined by this recipe, but once it is chosen one defines

$$\tau_i = \lambda_i + \nu_i - \nu_i^\sigma \in X_*(T),$$

and chooses any $\delta_i \in X_*(T)$ satisfying $\tau_i = \epsilon_0 + \delta_i - \delta_i^\sigma$. In the case at hand we make the following choices. For $1 \leqslant i < n/2$ set

$$\begin{aligned}\nu_i &= (\epsilon_1 + \cdots + \epsilon_{i-1}) - (\epsilon_{i^\vee} + \cdots + \epsilon_{1^\vee})\\ \tau_i &= \epsilon_0\\ \delta_i &= 0.\end{aligned}$$

When n is even, so that $r = n/2$, set

$$\begin{aligned}\nu_r &= \epsilon_1 + \cdots + \epsilon_{r-1}\\ \tau_r &= \epsilon_0 + \epsilon_1 + \cdots + \epsilon_n\\ \delta_r &= \epsilon_1 + \cdots + \epsilon_r.\end{aligned}$$

Each of the minuscule cocharacters τ_i determines a $b_i = \tau_i(p) \in G(\breve{\mathbb{Q}}_p)$, with its own affine Deligne-Lusztig variety

$$X_\mu(b_i) = \{g \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p) : \mathrm{inv}_G(b_i g^\sigma \mathbb{D}, g\mathbb{D}) = \mu\}.$$

The locally closed subset

$$\mathring{X}_{\mu,\nu_i}(b_i) = \{g \in X_\mu(b_i) : \mathrm{inv}_G(g\mathbb{D}, \mathbb{D}) = \nu_i\}$$

is irreducible (see the proof of Lemma 7.2 of [4]) and its closure in $X_\mu(b_i)$ is an irreducible component.

There is a bijection

$$\Delta_i : X_\mu(b_i) \to X_\mu(b)$$

defined by $\Delta_i(g) = \delta_i(p^{-1})g$. Note that $\delta_i(p^{-1}) \in G(\breve{\mathbb{Q}}_p)$ may not be central, and so this bijection need not respect the natural left actions of $G(\mathbb{Q}_p)$ on the source and target. As $\delta_i - \delta_i^\sigma$ is a central cocharacter, conjugation by $\delta_i(p^{-1})$ defines an automorphism of $G(\mathbb{Q}_p)$. All hyperspecial subgroups of $G(\mathbb{Q}_p)$ are conjugate, so we may fix a $k_i \in G(\mathbb{Q}_p)$ such that

$$\delta_i(p^{-1})G(\mathbb{Z}_p)\delta_i(p) = k_i^{-1}G(\mathbb{Z}_p)k_i. \tag{6.2.7}$$

The closure of

$$k_i\Delta_i(\mathring{X}_{\mu,\nu_i}(b_i)) = \big\{g \in X_\mu(b) : \mathrm{inv}_G(g\mathbb{D}, k_i\delta_i(p^{-1})\mathbb{D}) = \nu_i\big\} \tag{6.2.8}$$

is an irreducible component of $\mathrm{RZ}^{\mathrm{red}}$, and is the image of (i, id) under (6.2.6). Indeed, this is the definition of the bijection (6.2.6).

If $1 \leqslant i < n/2$ then

$$\begin{aligned}(6.2.8) &= \{g \in X_\mu(b) : \mathrm{inv}_G(g\mathbb{D}, \mathbb{D}) = \nu_i\}\\ &= \{g \in X_\mu(b) : \mathrm{inv}_G(g\mathbb{D}, \mathbb{D}) = \alpha_i\}\\ &= \mathrm{RZ}_{(i,\mathrm{id})},\end{aligned}$$

as desired. Now suppose n is even, and $i = n/2$. The cocharacter $\epsilon_0 + \delta_i$ is fixed by σ, and taking $k_i = (\epsilon_0 + \delta_i)(p) \in G(\mathbb{Q}_p)$ we find

$$\begin{aligned}(6.2.8) &= \{g \in X_\mu(b) : \mathrm{inv}_G(g\mathbb{D}, \epsilon_0(p)\mathbb{D}) = \nu_i\} \\ &= \{g \in X_\mu(b) : \mathrm{inv}_G(g\mathbb{D}, \mathbb{D}) = \alpha_i\} \\ &= \mathrm{RZ}_{(i,\mathrm{id})},\end{aligned}$$

as desired. □

6.3. **Main results.** We can now put everything together to describe our main results on the structure of $\mathrm{RZ}^{\mathrm{red}}$.

This amounts to describing the structure of the locally closed subsets $\mathrm{RZ}_{(k,\gamma)}$ appearing in Theorem 6.2.2, which we do by comparing them with the locally closed subsets (Definition 2.4.2) appearing in the decomposition

$$\mathrm{RZ}^{\mathrm{red}}_\Lambda = \bigsqcup_{k \geqslant 1} \mathrm{RZ}^{k,\mathrm{red}}_\Lambda \tag{6.3.1}$$

determined by our choice of framing object $\Lambda \otimes \overline{\mathbb{Y}}$.

By Proposition 2.4.4, the subschemes on the right hand side of (6.3.1) are nonempty only when $1 \leqslant k \leqslant \lfloor n/2 \rfloor$. The cases $k < n/2$ and $k = n/2$ will require separate treatment, as did the definition of the cocharacter α_k appearing in (6.2.2).

Theorem 6.3.1. *For any $k < n/2$ we have*

$$\mathrm{RZ}_{(k,\mathrm{id})} = \mathrm{RZ}^{k,\mathrm{red}}_\Lambda.$$

Moreover, for any $\gamma \in G(\mathbb{Q}_p)$ there is a smooth morphism

$$\mathrm{RZ}_{(k,\gamma)} \to \mathrm{DL}^k_\Lambda$$

to the smooth and proper Deligne-Lusztig variety of Definition 3.4.1. Over this Deligne-Lusztig variety there is a rank $2k-1$ vector bundle $\mathcal{V}$ equipped with a morphism

$$\beta : \mathcal{V} \otimes \sigma^*\mathcal{V} \to \mathcal{O}_{\mathrm{DL}^k_\Lambda}$$

and a rank k local direct summand $\mathcal{V}^{(k)} \subset \mathcal{V}$ such that

$$\mathrm{RZ}_{(k,\gamma)}(S) \cong \left\{ \begin{array}{c} \textit{rank } k-1 \textit{ local direct summands } \mathcal{F} \subset \mathcal{V}_S \\ \textit{satisfying } \beta(\mathcal{F} \otimes \sigma^*\mathcal{F}) = 0 \textit{ and } \mathcal{V}_S = \mathcal{F} \oplus \mathcal{V}^{(k)}_S \end{array} \right\}$$

for any DL^k_Λ-scheme S. Recall that $\sigma^*\mathcal{V}$ means the Frobenius twist (4.1.4).

Proof. Recall the relative position invariant of Definition 2.4.3. Under the bijection of Proposition 6.1.6, an element $g \in X_\mu(b)$ corresponds to a p-divisible group $X \in \mathrm{RZ}(\breve{\mathbb{F}}_p)$ satisfying

$$\begin{aligned}\mathrm{inv}(g\mathbb{D}_0, \mathbb{D}_0) &= \mathrm{inv}(D(X)_0, D(\Lambda \otimes \overline{\mathbb{Y}})_0) \\ \mathrm{inv}(g\mathbb{D}_1, \mathbb{D}_1) &= \mathrm{inv}(D(X)_1, D(\Lambda \otimes \overline{\mathbb{Y}})_1),\end{aligned}$$

which corresponds, under the bijection of Proposition 2.3.1, to a lattice $L \subset \breve{\Lambda}[1/p]$ satisfying

$$\mathrm{inv}(D(X)_0, D(\Lambda \otimes \overline{\mathbb{Y}})_0) = \mathrm{inv}(L_0, \breve{\Lambda}_0)$$
$$\mathrm{inv}(D(X)_1, D(\Lambda \otimes \overline{\mathbb{Y}})_1) = \mathrm{inv}(L_1, \breve{\Lambda}_1).$$

Directly from the definition (6.2.2), the element g above lies in the subset $\mathrm{RZ}_{(k,\mathrm{id})}$ if and only if

$$\mathrm{inv}(g\mathbb{D}_0, \mathbb{D}_0) = (\overbrace{1, \dots, 1}^{k-1 \text{ times}}, 0, \dots, 0, \overbrace{-1, \dots, -1}^{k \text{ times}})$$
$$\mathrm{inv}(g\mathbb{D}_1, \mathbb{D}_1) = (\underbrace{1, \dots, 1}_{k \text{ times}}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{k-1 \text{ times}}).$$

By Proposition 2.4.4 and the previous paragraph, these conditions are equivalent to $X \in \mathrm{RZ}^k_\Lambda(\breve{\mathbb{F}}_p)$. In other words, $\mathrm{RZ}_{(k,\mathrm{id})} = \mathrm{RZ}^{k,\mathrm{red}}_\Lambda$.

Given this last equality and (6.2.3), the rest of the theorem is a restatement of Corollary 4.2.4. □

Remark 6.3.2. We remind the reader that the vector bundle $\mathcal{V}$ on

$$Y^{k,\mathrm{red}}_\Lambda \cong \mathrm{DL}^k_\Lambda$$

was constructed in Section 4.1, which contains more information about its filtration and the morphism β. See especially Proposition 4.1.8.

Remark 6.3.3. Theorem 6.3.1 implies that

$$\mathrm{RZ}_{(1,\gamma)} \cong \mathrm{RZ}^{1,\mathrm{red}}_\Lambda \cong \mathrm{DL}^1_\Lambda$$

is itself a Deligne-Lusztig variety.

Now we turn to the case $k = n/2$, so suppose n is even. In this case we have inclusions (Proposition 5.1.4)

$$\mathrm{RZ}^{n/2,\mathrm{red}}_\Lambda \subset \bigcup_{p\Lambda \subsetneq \Lambda' \subsetneq \Lambda} \mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'} \subset \mathrm{RZ}^{\mathrm{red}} \tag{6.3.2}$$

in which the union is over all scalar-self-dual (Definition 5.1.1) $\mathcal{O}_E$-lattices Λ' lying between $p\Lambda$ and Λ, and each

$$\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'} \subset \mathrm{RZ}^{\mathrm{red}}_\Lambda$$

is the closed subscheme of Definition 5.1.2. The following implies that every such $\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}$ is an irreducible component of $\mathrm{RZ}^{\mathrm{red}}$.

Theorem 6.3.4. *Assume that $k = n/2$, and fix a Λ' as in* (6.3.2)

1. *There is an $h \in G(\mathbb{Q}_p)$ such that $\mathrm{RZ}_{(k,h)} = \mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}$.*
2. *For any $\gamma \in G(\mathbb{Q}_p)$ there is an isomorphism*

$$\mathrm{RZ}_{(k,\gamma)} \cong \mathrm{DL}^{\heartsuit}_{\Lambda'}$$

where the smooth and proper Deligne-Lusztig variety on the right is that of Definition 5.2.1.

Proof. Recall that a hermitian space over E is determined up to isometry by its dimension and determinant, viewed as an element of $\mathbb{Q}_p^\times/\mathrm{Nm}_{E/\mathbb{Q}_p}(E^\times)$. As $n = \dim(W)$ is even, its determinant (hence its isometry classes) is unchanged if we rescale the hermitian form by an element of $\mathbb{Q}_p^\times$. In other words, the similitude character $G(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is surjective. Combining this with the observation that all self-dual lattices in W are isometric, it follows that any two scalar self-dual lattices in W lie in the same $G(\mathbb{Q}_p)$-orbit.

By the previous paragraph, we may now fix an $h \in G(\mathbb{Q}_p)$ satisfying

$$h\Lambda = p^{-1}\Lambda'.$$

Directly from the definition (6.2.2), an element $g \in X_\mu(b)$ lies in the subset $\mathrm{RZ}_{(k,h)}(\breve{\mathbb{F}}_p)$ if and only if

$$ph\mathbb{D}_0 \overset{k+1}{\subset} g\mathbb{D}_0 \overset{k-1}{\subset} h\mathbb{D}_0 \quad \text{and} \quad ph\mathbb{D}_1 \overset{k-1}{\subset} g\mathbb{D}_1 \overset{k+1}{\subset} h\mathbb{D}_1.$$

By our choice of h, this is equivalent to the corresponding p-divisible group $X \in \mathrm{RZ}(\breve{\mathbb{F}}_p)$ satisfying

$$D(\Lambda' \otimes \overline{\mathbb{Y}})_0 \overset{k+1}{\subset} D(X)_0 \overset{k-1}{\subset} D(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}})_0$$
$$D(\Lambda' \otimes \overline{\mathbb{Y}})_1 \overset{k-1}{\subset} D(X)_1 \overset{k+1}{\subset} D(p^{-1}\Lambda' \otimes \overline{\mathbb{Y}})_1.$$

Such p-divisible groups lie in the subset $\mathrm{RZ}_\Lambda(\breve{\mathbb{F}}_p) \subset \mathrm{RZ}(\breve{\mathbb{F}}_p)$, by the inclusions

$$p^{-1}\Lambda \subset \Lambda' \subset p^{-1}\Lambda' \subset \Lambda,$$

and correspond under the bijection of Corollary 2.3.3 to lattices $L_0 \subset \breve{\Lambda}_0[1/p]$ satisfying

$$\breve{\Lambda}'_0 \overset{k-1}{\subset} L_0^* \subset L_0 \overset{k-1}{\subset} p^{-1}\breve{\Lambda}'_0.$$

This is exactly the characterization of $\mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}(\breve{\mathbb{F}}_p)$ from Proposition 5.1.3, proving that

$$\mathrm{RZ}_{(k,h)}(\breve{\mathbb{F}}_p) = \mathrm{RZ}^{\heartsuit,\mathrm{red}}_{\Lambda'}(\breve{\mathbb{F}}_p).$$

This completes the proof of the first claim.

Given the first claim and (6.2.3), the second claim follows immediately from Theorem 5.2.3. □

Recall that Theorem 6.2.2 presents the irreducible components of $\mathrm{RZ}^{\mathrm{red}}$ as Zariski closures of certain locally closed subsets $\mathrm{RZ}_{(k,\gamma)}$. A priori, these locally closed subsets could be rather small; indeed, deleting a proper closed subset from any of them would not change the statement of that theorem. The following result says that they are large enough to cover the entire Rapoport-Zink space, without needing to take their Zariski closures. It would be interesting to know if this phenomenon is particular to the GU$(2, n-2)$ Rapoport-Zink space, or if it holds in the greater generality of [19].

Corollary 6.3.5. *The locally closed subschemes of* (6.2.2) *satisfy*

$$\mathrm{RZ}^{\mathrm{red}} = \bigcup_{\substack{1 \leqslant k \leqslant \lfloor n/2 \rfloor \\ \gamma \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)}} \mathrm{RZ}_{(k,\gamma)}.$$

Proof. Combining (6.2.3) with Theorems 6.3.1 and 6.3.4, we find that every $\mathrm{RZ}_{(k,\gamma)}$ is contained in some $G(\mathbb{Q}_p)$-translate of $\mathrm{RZ}_\Lambda^{\mathrm{red}}$. As $\mathrm{RZ}_\Lambda^{\mathrm{red}} \subset \mathrm{RZ}^{\mathrm{red}}$ is closed, it follows now from Theorem 6.2.2 that every irreducible component of $\mathrm{RZ}^{\mathrm{red}}$ is also contained in such a translate. This proves that

$$\mathrm{RZ}^{\mathrm{red}} = \bigcup_{\gamma \in G(\mathbb{Q}_p)} \gamma \cdot \mathrm{RZ}_\Lambda^{\mathrm{red}}.$$

Combining this with (6.3.1), we find that

$$\mathrm{RZ}^{\mathrm{red}} = \bigcup_{\substack{\gamma \in G(\mathbb{Q}_p) \\ 1 \leqslant k \leqslant \lfloor n/2 \rfloor}} \gamma \cdot \mathrm{RZ}_\Lambda^{k,\mathrm{red}}.$$

Now fix a point $s \in \mathrm{RZ}^{\mathrm{red}}$. The paragraph above shows that s is contained in some $\gamma \cdot \mathrm{RZ}_\Lambda^{k,\mathrm{red}}$, and we consider two cases. If $k < n/2$ then Theorem 6.3.1 implies

$$s \in \gamma \cdot \mathrm{RZ}_{(k,\mathrm{id})} = \mathrm{RZ}_{(k,\gamma)}.$$

If $k = n/2$ then we use (6.3.2) and Theorem 6.3.4 to see that

$$s \in \gamma \cdot \mathrm{RZ}_{\Lambda'}^{\heartsuit,\mathrm{red}} = \mathrm{RZ}_{(n/2,\gamma')}$$

for some scalar-self-dual Λ' and some $\gamma' \in G(\mathbb{Q}_p)$. □

References

[1] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine Grassmannian. *Invent. Math.*, 209(2):329–423, 2017.

[2] Cédric Bonnafé and Raphaël Rouquier. On the irreducibility of Deligne-Lusztig varieties. *C. R. Math. Acad. Sci. Paris*, 343(1):37–39, 2006.

[3] Maria Fox. The GL_4 Rapoport-Zink space. *Int. Math. Res. Not. IMRN*, (3):1825–1892, 2022.

[4] Maria Fox and Naoki Imai. The supersingular locus of the Shimura variety of $\mathrm{GU}(2, n-2)$. *Preprint. arXiv:2108.03584*, 2021.

[5] Ulrich Görtz and Xuhua He. Basic loci of Coxeter type in Shimura varieties. *Camb. J. Math.*, 3(3):323–353, 2015.

[6] Ulrich Görtz, Xuhua He, and Sian Nie. Fully Hodge-Newton decomposable Shimura varieties. *Peking Math. J.*, 2(2):99–154, 2019.

[7] Paul Hamacher and Eva Viehmann. Irreducible components of minuscule affine Deligne-Lusztig varieties. *Algebra Number Theory*, 12(7):1611–1634, 2018.

[8] Xuhua He, Rong Zhou, and Yihang Zhu. Stabilizers of irreducible components of affine Deligne-Lusztig varieties. *Preprint. arXiv:2109.02594*, 2021.

[9] Benjamin Howard and Georgios Pappas. On the supersingular locus of the $\mathrm{GU}(2,2)$ Shimura variety. *Algebra Number Theory*, 8(7):1659–1699, 2014.

[10] Benjamin Howard and Georgios Pappas. Rapoport-Zink spaces for spinor groups. *Compos. Math.*, 153(5):1050–1118, 2017.

[11] Stephen Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties I. Unramified local theory. *Invent. Math.*, 184(3):629–682, 2011.

[12] Chao Li and Yihang Zhu. Remarks on the arithmetic fundamental lemma. *Algebra Number Theory*, 11(10):2425–2445, 2017.

[13] Sian Nie. Irreducible components of affine Deligne-Lusztig varieties. *Camb. J. Math.*, 10(2):433–510, 2022.

[14] Michael Rapoport, Ulrich Terstiege, and Sean Wilson. The supersingular locus of the Shimura variety for GU$(1, n-1)$ over a ramified prime. *Math. Z.*, 276(3-4):1165–1188, 2014.

[15] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[16] Inken Vollaard. The supersingular locus of the Shimura variety for GU$(1, s)$. *Canad. J. Math.*, 62(3):668–720, 2010.

[17] Inken Vollaard and Torsten Wedhorn. The supersingular locus of the Shimura variety of GU$(1, n-1)$ II. *Invent. Math.*, 184(3):591–627, 2011.

[18] Haining Wang. On the Bruhat-Tits stratification of a quaternionic unitary Rapoport-Zink space. *Math. Ann.*, 376(3-4):1107–1144, 2020.

[19] Liang Xiao and Xinwen Zhu. Cycles on Shimura varieties via geometric Satake. *Preprint. arXiv:1707.05700*, 2017.

[20] Rong Zhou and Yihang Zhu. Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties. *Camb. J. Math.*, 8(1):149–241, 2020.

[21] Xinwen Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. of Math. (2)*, 185(2):403–492, 2017.

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

Email address: `maria.fox@okstate.edu`

Department of Mathematics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA 02467, USA

Email address: `howardbe@bc.edu`

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan

Email address: `naoki@ms.u-tokyo.ac.jp`