THE PRISMATIC REALIZATION FUNCTOR FOR SHIMURA VARIETIES OF ABELIAN TYPE

NAOKI IMAI[(1)], HIROKI KATO[(2)], AND ALEX YOUCIS[(3)]

ABSTRACT. For the integral canonical model $\mathscr{S}_{\mathsf{K}^p}$ of a Shimura variety $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}(\mathbf{G},\mathbf{X})$ of abelian type at hyperspecial level $K_0=\mathcal{G}(\mathbb{Z}_p)$, we construct a prismatic model for the 'universal' $\mathcal{G}(\mathbb{Z}_p)$-local system on $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}(\mathbf{G},\mathbf{X})$. We use this to obtain new p-adic Hodge theoretic information about these Shimura varieties, and to provide a prismatic characterization of these models. To do this, we make several advances in integral p-adic Hodge theory, notably the development of an integral analogue of the functor D_{crys}.

CONTENTS

(1,3) GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO, 153-8914, JAPAN
(2) MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY
E-mail addresses: (1) naoki@ms.u-tokyo.ac.jp, (2) hkato@mpim-bonn.mpg.de, (3) ayoucis@ms.u-tokyo.ac.jp.
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Introduction

Shimura varieties are a class of arithmetic-geometric objects associated to a reductive $\mathbb{Q}$-group $\mathbf{G}$, and a piece of ancillary Hodge-theoretic data $\mathbf{X}$, which sit at the intersection of differential geometry, algebraic geometry, and number theory. Central to their importance in the Langlands program are certain p-adic $\mathbf{G}$-local systems $\omega_{\text{ét}}$,[1] whose cohomology is expected (and known in many cases) to encode the global Langlands correspondence for $\mathbf{G}$ (see [Kot90]).

A guiding principle concerning Shimura varieties is that they should be moduli spaces of $\mathbf{G}$-motives of type $\mathbf{X}$ with level structure. Via this principle there should be a universal $\mathbf{G}$-motive ω_{mot} whose p-adic étale realization is $\omega_{\text{ét}}$. It has long been an area of active research to apply relative p-adic Hodge theory to $\omega_{\text{ét}}$ (over a p-adic completion of the ground field) to derive what should be the realization of ω_{mot} in other categories, and to study the implications of these constructions for $\omega_{\text{ét}}$. Notable examples of this are [LZ17] and [DLLZ23] which study the de Rham aspects of this question, and [Lov17a] which studies the crystalline aspects (with abelian type and hyperspecial level assumptions), thus arriving at realizations ω_{dR} and ω_{crys}.

In [BS22], Bhatt and Scholze crystallized and explicated a large portion of recent work in integral p-adic Hodge theory, developing the notion of the prismatic cohomology of a p-adic formal scheme. This cohomology specializes to many classical cohomology theories (e.g. étale, de Rham, and crystalline), and so is conceptually a closer approximation to the true motive of that formal scheme. In this article we construct, in the case of Shimura varieties of abelian type and hyperspecial level, what should be a *prismatic realization* ω_{Δ} of ω_{mot} on (the completion of) its integral canonical model, and show it specializes to $\omega_{\text{ét}}$, ω_{dR}, and ω_{crys}. Using this, we obtain new results concerning such Shimura varieties, and more streamlined proofs of old results.

To achieve this we exploit the relationship between crystalline local systems on a smooth rigid space and prismatic F-crystals on a smooth model of that space as established in [BS23], [DLMS22], and [GR22]. In particular, we develop the Tannakian theory of such results. We further relate our prismatic realization to the prevous crystalline construction in [Lov17a], which is an important ingredient in many of our applications. To do this we develop an integral analogue of the functor D_{crys} and show it has good properties, especially in the Fontaine–Laffaille range.

Finally, we use our prismatic realization functor to give a prismatic characterization of these integral models. This characterization is modeled on ideas from [Pap23] and [PR22], but has several noted improvements and simplifications coming both from our use of prismatic theory and formal/rigid geometry. Chief amongst these is that our characterization works for individual levels, unlike those in [Kis10], [Pap23], and [PR22] which can only be formulated for the entire system as the level varies.

$\mathcal{G}$-objects in crystalline local systems and (analytic) prismatic F-crystals. Fix a complete discrete valuation ring $\mathcal{O}_K$ with perfect residue field k, and a smooth formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ with generic fiber X. In [BS23], Bhatt and Scholze define a category $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ of prismatic F-crystals on $\mathfrak{X}$, which presents the correct notion of a 'deformation' of an F-crystal on $\mathfrak{X}_k$ to $\mathfrak{X}$. They further construct a functor $T_{\text{ét}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}_{\mathbb{Z}_p}(X)$, which is a prismatic analogue of the Riemann–Hilbert correspondence. When $\mathfrak{X} = \operatorname{Spf}(\mathcal{O}_K)$, they show that $T_{\text{ét}}$ induces an

[1] For general $\mathbf{G}$ this is not quite correct, and one needs to instead consider $\mathbf{G}^c$-local systems for a certain modified group $\mathbf{G}^c$. See [LS18, §3] and §4.3 for details. For the sake of simplicity, we ignore this subtlety in the introduction and assume $\mathbf{G} = \mathbf{G}^c$, but do not make this assumption in the main body of the article.

equivalence between $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ and the category $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$ of crystalline lattices on X (i.e., $\mathbb{Z}_p$-lattices in crystalline $\mathbb{Q}_p$-local systems).

In general, $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ is not sufficient to recover every crystalline lattice on X (see [DLMS22, Example 3.35]), and Guo and Reinecke in [GR22] consider an enlargement $\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ consisting of so-called analytic prismatic F-crystals. The functor $T_{\text{ét}}$ extends to this larger category, and [GR22] showed that $T_{\text{ét}}$ forms an equivalence between $\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ and $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$ (cf. the results of [DLMS22]). The difference between $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ and $\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ suggests a strengthening of the notion of crystalline. Namely, let $\mathbf{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(X)$ be the category of *prismatically good reduction lattices*: those crystalline lattices $\mathbb{L}$ such that $T_{\text{ét}}^{-1}(\mathbb{L})$ is a prismatic F-crystal.

Our applications to Shimura varieties are benefited by studying the Tannakian aspects of this theory. To this end, fix a reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$. For an exact $\mathbb{Z}_p$-linear $\otimes$-category $\mathcal{C}$, denote by $\mathcal{G}$-$\mathcal{C}$ the category of $\mathcal{G}$-objects in $\mathcal{C}$, i.e., exact $\mathbb{Z}_p$-linear $\otimes$-functors $\omega\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathcal{C}$.

The functor $T_{\text{ét}}\colon \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$ is an exact $\mathbb{Z}_p$-linear $\otimes$-equivalence with exact quasi-inverse (see Proposition 2.21), and so evidently induces an equivalence of categories of $\mathcal{G}$-objects. But, while $T_{\text{ét}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(X)$ is an exact $\mathbb{Z}_p$-linear $\otimes$-equivalence, its quasi-inverse is not exact, as it involves the extension of a vector bundle on an open subset of some space over its non-trivial closed complement.

Our first observation is that despite this, it still induces an equivalence on the categories of $\mathcal{G}$-objects.

Theorem 1 (see Theorem 2.27)**.** *The functor*

$$T_{\text{ét}}\colon \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathcal{G}\text{-}\mathbf{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(X), \qquad \omega \mapsto T_{\text{ét}} \circ \omega,$$

is an equivalence, i.e., $T_{\text{ét}}^{-1} \circ \nu$ is exact for any ν in $\mathcal{G}$-$\mathbf{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(X)$.

We further remark that both $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ and $\mathbf{Loc}_{\mathbb{Z}_p}(X)$ satisfy reasonable Tannakian properties, and so may be interpreted in terms of torsors (with extra structure) (see §1.2 and §2.1.2). In the former case, we refer to such objects as *prismatic $\mathcal{G}$-torsors with F-structure.*

Remark 1. The main technical result needed to prove Theorem 1 may be proven independently of many of the main results of [GR22] and [DLMS22], and in greater generality, using an adaptation of an idea of Kisin. See §2.5 for details.

Shimura varieties of abelian type. Let $(\mathbf{G},\mathbf{X})$ be a Shimura datum of abelian type with reflex field $\mathbf{E}$. Fix a prime p and let E be the completion of $\mathbf{E}$ at a place above p. Set $G = \mathbf{G}_{\mathbb{Q}_p}$, and fix a reductive $\mathbb{Z}_p$-model $\mathcal{G}$ of G, letting $\mathsf{K}_0 = \mathcal{G}(\mathbb{Z}_p)$ be the associated hyperspecial subgroup. For $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f)$ a (neat) compact open subgroup, write Sh_{K} for $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G},\mathbf{X})_E$. Then, the map

$$\varprojlim_{\mathsf{K}_p \subseteq \mathsf{K}_0} \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p} \to \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p},$$

is a $\underline{\mathsf{K}_0}$-torsor on the pro-étale site of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$, and we let

$$\omega_{\mathsf{K}^p,\text{ét}}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}),$$

be the associated exact $\mathbb{Z}_p$-linear $\otimes$-functor, an object of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})$. We are interested in better understanding the integral p-adic Hodge theory of this $\mathcal{G}(\mathbb{Z}_p)$-local system and, in particular, its relationship to the theory of prismatic F-crystals.

To discuss such a relationship one must first fix a formal scheme. There is a natural choice, as associated to $(\mathbf{G},\mathbf{X})$ and $\mathcal{G}$ is the integral canonical model $\mathscr{S}_{\mathsf{K}^p}$ over $\mathcal{O}_E$ as in [Kis10] and its p-adic completion $\widehat{\mathscr{S}}_{\mathsf{K}^p}$. One may then consider the open subspace $(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta \subseteq \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_0\mathsf{K}^p}$, and define

$$\omega_{\mathsf{K}^p,\mathrm{an}}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathbf{Loc}_{\mathbb{Z}_p}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta), \qquad \xi \mapsto \omega_{\mathsf{K}^p,\mathrm{an}}(\xi) := \omega_{\mathsf{K}^p,\text{ét}}(\xi)^{\mathrm{an}}|_{(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta},$$

which is an exact $\mathbb{Z}_p$-linear $\otimes$-functor, i.e., an object of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}_p}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta)$.

Theorem 2 (see Theorem 4.12)**.** *The $\mathbb{Z}_p$-local system $\omega_{\mathsf{K}^p,\mathrm{an}}(\xi)$ has prismatically good reduction for all ξ.*

Combining Theorem 1 and Theorem 2, we deduce the existence of a *prismatic realization functor* $\omega_{\mathsf{K}^p,\Delta}$ in $\mathcal{G}$-$\mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_{\Delta})$ with the property that $T_{\text{ét}} \circ \omega_{\mathsf{K}^p,\Delta} = \omega_{\mathsf{K}^p,\mathrm{an}}$. In the case when $(\mathbf{G},\mathbf{X})$ is of Hodge type, this can be more concretely described in terms of the prismatic cohomology of the 'universal' abelian scheme over $\mathscr{S}_{\mathsf{K}^p}$ (see Theorem 4.14).

Of course, while the decision to use $\mathscr{S}_{\mathsf{K}^p}$ is natural, it presents a question: is there another open subspace U of $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_0\mathsf{K}^p}$ and a model $\mathfrak{U}$ of U over which $\omega^{\mathrm{an}}_{\mathsf{K}^p,\text{ét}}|_U$ has a natural prismatic model? We begin to address this question by proving the following folkloric result, showing that $(\widehat{\mathscr{S}_{\mathsf{K}^p}})_\eta$ is the maximal open subspace of $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_0\mathsf{K}^p}$ where such a prismatic theory could be found.

Proposition 1 (see Proposition 4.16)**.** *The potentially crystalline locus of $\omega_{\mathsf{K}^p,\text{ét}}$ is $(\widehat{\mathscr{S}_{\mathsf{K}^p}})_\eta$, and all the classical points there are crystalline.*

While there are ways of proving Proposition 1 without Theorem 2, it certainly streamlines the proof. Namely, it shows that each point of $(\widehat{\mathscr{S}_{\mathsf{K}^p}})_\eta$ is crystalline for $\omega_{\mathsf{K}^p,\text{ét}}$.

Additionally, from Theorem 2 we may obtain immediate cohomological consequences using the results of the recent paper [GL23]. For the prismatic notation used in the following statement see [GL23, Definitions 2.10, 2.14, and 2.17].

Proposition 2 (see Proposition 4.22)**.** *Suppose $\mathbf{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic. Let $f\colon \widehat{\mathscr{S}_{\mathsf{K}^p}} \to \mathrm{Spec}(\mathcal{O}_E)$ be the structure map and d its relative dimension. Fix an object ξ of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ and let a and b denote the maximum and minimum weight of the Hodge cocharacter μ_h-height of $\xi[1/p]$ respectively. Then, for each $i \geqslant 0$, the following statements are true.*

(1) One has that $R^i f_{\Delta,*}\omega_{\mathsf{K}^p,\Delta}$ is a coherent prismatic F-crystal on $\mathcal{O}_E$ whose $\mathcal{I}_\Delta$-torsion-free quotient has Frobenius height in $[a+\max\{0,i-d\}, b+\min\{i,d\}]$.

(2) There exists a map (a 'Verschiebung operator')

$$\psi_i\colon \mathcal{I}_\Delta^{(i+b)\otimes} \otimes_{\mathcal{O}_\Delta} R^i f_{\Delta,*}\omega_{\mathsf{K}^p,\Delta}(\xi) \to \phi^* R^i f_{\Delta,*}\omega_{\mathsf{K}^p,\Delta}(\xi)$$

which is inverse to the Frobenius operator on $R^i f_{\Delta,*}$ up to multiplication by $\mathcal{I}_\Delta^{(i+b)\otimes}$.

Remark 2. It is natural to ask whether or not the prismatic realization functor $\omega_{\mathsf{K}^p,\Delta}$ admits an upgrade to *prismatic F-gauges* in the sense of [Bha23] (see also [GL23, §2.2]). In other words, one might wonder about the existence of an *F-gauge realization functor* $\omega_{\mathsf{K}^p,\mathrm{Gauge}}$ with the property that its image under the natural forgetful map (using notation from [GL23, §2.2])

$$\mathcal{G}\text{-}F\text{-}\mathbf{Gauge}^{\mathrm{vect}}(\widehat{\mathscr{S}_{\mathsf{K}^p}}) \to \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_\Delta),$$

is $\omega_{\mathsf{K}^p,\Delta}$. This question has an affirmative answer, and follows already from material established in this article.

More precisely, combining [GL23, Theorem 1.2] with [GL23, Corollary 2.53], one sees that this forgetful functor is fully faithful with essential image precisely $\mathcal{G}$-$\mathbf{Vect}^{\varphi,\mathrm{lff}}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_\Delta)$. Let Π denote the natural quasi-inverse to this functor on its essential image as described in [GL23, Theorem 1.2], which can be seen to be a $\mathbb{Z}_p$-linear $\otimes$-functor. Then, using Corollary 5.20 we see that it makes sense to define $\omega_{\mathsf{K}^p,\mathrm{Gauge}} := \Pi^{-1} \circ \omega_{\mathsf{K}^p,\Delta}$. The only remaining question is whether $\omega_{\mathsf{K}^p,\mathrm{Gauge}}$ is exact. To show this it suffices to show that the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*\omega_{\mathsf{K}^p,\Delta}(\Lambda))$ is an exact functor in Λ. To see this observe that for a short exact sequence of objects of $\mathbf{Vect}^{\varphi,\mathrm{lff}}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_\Delta)$ one may check exactness of the resulting sequence of filtrations on a cover by Breuil–Kisin prisms $\{(\mathfrak{S}_R,(E))\}$ (cf. Proposition 1.15), and modulo (E), from where we may appeal to Proposition 3.36 to conclude.

Prismatic characterization of integral canonical models. Suppose that $(\mathbf{G},\mathbf{X})$ is a Shimura datum of abelian type as above. The integral canonical models $\{\mathscr{S}_{\mathsf{K}^p}\}$ of Kisin are uniquely characterized *as a system* by a *strong extension* property:

(StExt) for every regular formally smooth $\mathcal{O}_E$-algebra R, one has

$$\varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}^p}(R) = \varprojlim_{\mathsf{K}^p} \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}(R[1/p]).$$

While this characterization is sufficient for many applications, it is incapable of being adapted to characterize the models $\mathscr{S}_{\mathsf{K}^p}$ for individual levels K^p, and is quite far from a moduli-theoretic characterization.

Given our guiding principle for $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$, it is natural to expect that a canonical model $\mathscr{S}_{\mathsf{K}^p}$ of $\mathrm{Sh}_{\mathsf{K}^p\mathsf{K}_0}$ should be a moduli space of $\mathcal{G}$-motives in some sense. If one thinks of prismatic F-crystals as being a reasonably good approximation to a theory of motives, then together with our guiding principle the following definition is reasonably motivated.

Definition 1 (see Definition 5.30)**.** Let K^p be a neat compact open subgroup of $\mathbf{G}(\mathbb{A}_f^p)$. Then a *prismatic integral canonical model* of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ is a smooth and separated $\mathcal{O}_E$-model $\mathscr{X}_{\mathsf{K}^p}$ of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ such that

(1) $(\widehat{\mathscr{X}_{\mathsf{K}^p}})_\eta$ is the potentially crystalline locus of $\omega_{\mathsf{K}^p,\text{ét}}$,
(2) there exists a prismatic model ζ_{K^p} of $\omega_{\mathsf{K}^p,\text{ét}}$ such that for every point x of $\mathscr{X}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ the restriction of ζ_{K^p} to $\widehat{\mathcal{O}}_{\mathscr{X}_{\mathsf{K}^p},x}$ is universal.

By *universal* in this definition, we mean that it forms a *universal deformation* of $(\zeta_{\mathsf{K}^p})_x$ (the restriction of ζ_{K^p} to the closed point x) as a prismatic $\mathcal{G}$-torsor with F-structure bounded by μ_h (the Hodge cocharacter of $(\mathbf{G},\mathbf{X})$), in the sense of §5.2 and Definition 5.8.[2]

Theorem 3 (see Theorem 5.19 and Theorem 5.34)**.** *The unique prismatic integral canonical model of* $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ *is* $\mathscr{S}_{\mathsf{K}^p}$.

To further conceptualize Definition 1 and this result, it is useful to observe that for a separated E-scheme X, every separated and flat finite type $\mathcal{O}_E$-model $\mathscr{X}$ is obtained by 'gluing' a formal scheme $\mathfrak{X}$ to X along an open subspace $\mathfrak{X}_\eta \cong U \subseteq X^{\mathrm{an}}$ (see Remark 5.3). One might intuitively imagine this idea as being represented by the equation

$$\mathscr{X} = X \sqcup_U \mathfrak{X}$$

although the precise meaning of this equality is slightly subtle (again see Remark 5.3).

Thinking from this perspective, and using our guiding intuition, to obtain an integral canonical model there is only one reasonable guess for what open subspace of $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_0\mathsf{K}^p}$ we should be gluing along: the (potentially) crystalline locus of $\omega_{\mathsf{K}^p,\text{ét}}$ (i.e., the 'good reduction locus' of $\omega_{\mathsf{K}^p,\text{ét}}$), which we denote by U_{K^p}. The first condition in Definition 1 precisely declares this, saying that a prismatic integral canonical model $\mathscr{X}_{\mathsf{K}^p}$ of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ is of the form

$$\mathscr{X}_{\mathsf{K}^p} = \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p} \sqcup_{U_{\mathsf{K}^p}} \mathfrak{X}_{\mathsf{K}^p},$$

for some smooth formal model $\mathfrak{X}_{\mathsf{K}^p}$ of U_{K^p}.

Thus, we see that the only thing left to do is to characterize $\mathfrak{X}_{\mathsf{K}^p}$, and this is where the second condition of Definition 1 comes into play. Our guiding intuition says that $\mathfrak{X}_{\mathsf{K}^p}$ should be a fine moduli space of $\mathcal{G}$-motives. While we cannot make such a fine moduli problem precise, the second condition to be a prismatic integral canonical model says that there is a global object which is universal *formally locally at every closed point of* $\mathfrak{X}_{\mathsf{K}^p}$, at least in a prismatic sense.

Of course, this usefulness of these ideas hinges on the following non-evident claims:

(a) this condition on $\mathfrak{X}_{\mathsf{K}^p}$ is enough to uniquely specify it,
(b) that the formal model $\widehat{\mathscr{S}}_{\mathsf{K}^p}$ of U_{K^p} (see Proposition 1) with its prismatic realization functor $\omega_{\mathsf{K}^p,\Delta}$ satisfies this formal-local universality.

[2]Such universal deformations were previously studied by Ito (from a different perspective) in [Ito23a].

The content of Theorem 3 (cf. Theorem 5.27), is then precisely that both (a) and (b) hold true.

Relationship to the work of Pappas–Rapoport. Suppose that $(\mathbf{G},\mathbf{X})$ is a Shimura datum of abelian type, but relax the reductivity condition on $\mathcal{G}$, allowing $\mathcal{G}$ to be parahoric.

In [PR22], Pappas and Rapoport define the notion of a $\mathcal{G}$-shtuka on any pre-adic space over $\mathbb{Z}_p$, and explain how to associate to $\omega_{\mathsf{K}^p,\text{ét}}$ a $\mathcal{G}$-shtuka $\mathscr{P}_{\mathsf{K}^p,E}$ on $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ (see [PR22, §4.1]). In [PR22, Conjecture 4.2.2], building off previous ideas from [Pap23], they conjecture the existence of a unique system $\{\mathscr{S}_{\mathsf{K}^p}\}$ of normal flat $\mathcal{O}_E$-models of $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$, with an action of $\mathbf{G}(\mathbb{A}_f^p)$ by finite étale morphisms, satisfying:

(WeExt) for every discrete valuation ring R over $\mathcal{O}_E$ of mixed characteristic $(0,p)$, one has

$$\varprojlim_{\mathsf{K}^p}\mathscr{S}_{\mathsf{K}^p}(R)=\varprojlim_{\mathsf{K}^p}\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}(R[1/p]),$$

(Univ) for each K^p, the $\mathcal{G}$-shtuka $\mathscr{P}_{\mathsf{K}^p,E}$ admits a (necessarily unique) model $\mathscr{P}_{\mathsf{K}^p}$ over $\mathscr{S}_{\mathsf{K}^p}$ such that for each x in $\mathscr{S}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ there exists an isomorphism

$$\Theta_x\colon \widehat{\mathcal{M}}^{\mathrm{int}}_{(\mathcal{G},b_x,\mu_h)/x_0}\xrightarrow{\sim}\mathrm{Spd}(\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}),$$

such that $\Theta_x^*(\mathscr{P}_{\mathsf{K}^p})$ agrees with the universal $\mathcal{G}$-shtuka on $\widehat{\mathcal{M}}^{\mathrm{int}}_{(\mathcal{G},b_x,\mu_h)/x_0}$.

Here μ_h is the Hodge cocharacter of $(\mathbf{G},\mathbf{X})$, b_x is an element of $C(\mathcal{G})$ (an integral version of $B(G)$) associated to $(\mathscr{P}_{\mathsf{K}^p})_x$, $\mathcal{M}^{\mathrm{int}}_{(\mathcal{G},b_x,\mu_h)}$ is the integral moduli space of shtukas as in [SW20, Definition 25.1], and $\widehat{\mathcal{M}}^{\mathrm{int}}_{(\mathcal{G},b_x,\mu_h)/x_0}$ is the completion at the neutral point x_0 in the sense of [Gle22]. This conjecture has been established in most Hodge-type cases (see [PR22, §4.5]) and all special-type cases (see [Dan22, Theorem A]).

To relate our work to that of Pappas–Rapoport, let us now again assume that $\mathcal{G}$ is reductive. We then call a system $\{\mathscr{X}_{\mathsf{K}^p}\}$ of normal flat (equiv. smooth) $\mathcal{O}_E$-models of $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$, with an action of $\mathbf{G}(\mathbb{A}_f^p)$ by finite étale morphisms a *prismatic integral canonical model* if each $\mathscr{X}_{\mathsf{K}^p}$ is a prismatic integral canonical model of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$.

It is a trivial consequence of Theorem 3 that $\{\mathscr{S}_{\mathsf{K}^p}\}$ is the unique prismatic integral canonical model of $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$. That said, we have the following result which is independent of Theorem 3, and provides an alternative proof of the unicity of a prismatic integral canonical model of $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$ (although *not* of individual levels) using [PR22, Theorem 4.2.4].

Proposition 3 (see Proposition 5.35)**.** *Suppose $\{\mathscr{X}_{\mathsf{K}^p}\}$ is a prismatic integral canonical model of $\{\mathrm{Sh}_{\mathsf{K}^p\mathsf{K}_0}\}$. Then, $\{\mathscr{X}_{\mathsf{K}^p}\}$ satisfies the conditions of the Pappas–Rapoport conjecture.*

Combining this with Theorem 3 allows us to prove the remaining cases of the Pappas–Rapoport conjecture at hyperspecial level not addressed in [PR22] and [Dan22] (i.e., abelian type but not of toral or Hodge type).

Theorem 4. *The Pappas–Rapoport conjecture holds in all cases of hyperspecial level.*

Remark 3. While our characterization as in Definition 1 and Theorem 3 ultimately did not rely on [Pap23] and [PR22], it is certainly inspired by these papers. In fact, this paper began as an attempt to understand the conjecture made in [PR22, §4.4].

An integral version of D_{crys} and comparison to work of Lovering. Finally, to obtain further applications of the prismatic realization functor $\omega_{\mathsf{K}^p,\Delta}$, we need to compare them to a construction of Lovering. Namely, in [Lov17a], Lovering constructs a crystalline realization of $\omega_{\mathsf{K}^p,\text{ét}}$, i.e., an exact $\mathbb{Z}_p$-linear $\otimes$-functor

$$\omega_{\mathsf{K}^p,\mathrm{crys}}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})\to\mathbf{VectF}^{\varphi,\mathrm{div}}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\mathrm{crys}}),$$

where the target is the category of strongly divisible filtered F-crystals on $\widehat{\mathscr{S}}_{\mathsf{K}^p}$ (see §3.2.1). This has the property that there exists a canonical identification

$$D_{\mathrm{crys}}\circ\omega_{\mathsf{K}^p,\text{ét}}[1/p]\xrightarrow{\sim}\omega_{\mathsf{K}^p,\mathrm{crys}}[1/p].$$

Moreover, he shows that the lattices $\omega_{\mathsf{K}^p,\text{ét}}(\xi)$ and $\omega_{\mathsf{K}^p,\text{crys}}(\xi)$ are matched by Fontaine–Laffaille theory when it applies (i.e., when $\omega_{\mathsf{K}^p,\text{ét}}(\xi)$ has Hodge–Tate weights in $[0,p-2]$).

To compare $\omega_{\mathsf{K}^p,\Delta}$ to $\omega_{\mathsf{K}^p,\text{crys}}$ we develop an integral analogue of the functor D_{crys}, which we expect to be of independent interest. Namely, assume that k is a perfect extension of $\mathbb{F}_p$, and $\mathfrak{X}$ is a smooth formal scheme over $W := W(k)$ with generic fiber X.[3] In §3.2 we construct a functor

$$\mathbb{D}_{\text{crys}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{VectWF}^{\varphi,\text{div}}(\mathfrak{X}_{\text{crys}}).$$

Here $\mathbf{VectWF}^{\varphi,\text{div}}(\mathfrak{X}_{\text{crys}})$ is a certain enlargement of $\mathbf{VectF}^{\varphi,\text{div}}(\mathfrak{X}_{\text{crys}})$ (see §3.2.1).

This is achieved in a surprisingly pleasant way, by showing that for an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ there is a canonical *crystalline-de Rham comparison* isomorphism of vector bundles.

Proposition 4 (see Proposition 3.13)**.** *Let $\mathfrak{X}$ be a smooth formal W-scheme, and $\mathcal{E}$ a prismatic F-crystal on $\mathfrak{X}$. Then there exists a canonical isomorphism*

$$\iota_{\mathfrak{X}}\colon \underline{\mathbb{D}}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}})_{\mathfrak{X}} \xrightarrow{\sim} \mathbb{D}_{\text{dR}}(\mathcal{E},\varphi_{\mathcal{E}}) \tag{0.0.1}$$

of vector bundles on $\mathfrak{X}$.

Here we are using the following notation:

- $\underline{\mathbb{D}}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$, the *crystalline realization*, denotes the F-crystal $\mathcal{E}^{\text{crys}}$ associated to $\mathcal{E}|_{\mathfrak{X}_k}$,
- and $\mathbb{D}_{\text{dR}}(\mathcal{E},\varphi_{\mathcal{E}})$, the *de Rham realization*, denotes the vector bundle on $\mathfrak{X}$ given by $\nu_*(\phi^*(\mathcal{E})\otimes_{\mathcal{O}_{\Delta}}\mathcal{O}_{\Delta}/\mathcal{I}_{\Delta})$, where $\nu_*\colon \mathbf{Shv}(\mathfrak{X}_{\Delta}) \to \mathbf{Shv}(\mathfrak{X}_{\text{ét}})$ is the natural pushforward.

Then, $\mathbb{D}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ has underlying F-crystal $\underline{\mathbb{D}}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ and obtains a filtration on its restriction to $\mathfrak{X}_{\text{Zar}}$ via the crystalline-de Rham comparison, and the Nygaard filtration on $\phi^*(\mathcal{E})$.

Remark 4. It is interesting to note that the crystalline-de Rham comparison isomorphism is new even in the absolute case. Indeed, there one may view it as an integral refinement of the isomorphism constructed in [Kis06, §1.2.7] (see Remark 3.12).

As indicated by our choice of terminology, $\mathbb{D}_{\text{crys}}$ may be seen as an integral analogue of the classical functor D_{crys} in the sense that there are canonical identifications

$$\mathbb{D}_{\text{crys}}[1/p] \xrightarrow{\sim} D_{\text{crys}} \circ T_{\text{ét}},$$

(see Proposition 3.26). Lying deeper, is an integral comparison between $\mathbb{D}_{\text{crys}}$ and the functor

$$T^*_{\text{crys}}\colon \mathbf{VectF}^{\varphi,\text{div}}_{[0,p-2]}(\mathfrak{X}_{\text{crys}}) \to \mathbf{Loc}_{\mathbb{Z}_p}(X),$$

constructed by Fontaine and Laffaille in [FL82] when $\mathfrak{X} = \operatorname{Spf}(W)$ and by Faltings in [Fal89] for general smooth $\mathfrak{X}$, where the subscript denotes those strongly divisible filtered F-crystals with filtration level in $[0,p-2]$. Namely, if $\mathbf{Vect}^{\varphi,\text{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta})$ denotes the category of effective locally filtered free prismatic F-crystals of height at most $p-2$ (this locally filtered free condition being of a technical nature, see §3.2.1), then we have the following.

Theorem 5 (see Proposition 3.38)**.** *For $(\mathcal{E},\varphi_{\mathcal{E}})$ in $\mathbf{Vect}^{\varphi,\text{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta})$, the strongly divisible filtered F-crystal $\mathbb{D}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ lies in $\mathbf{VectF}^{\varphi,\text{div}}_{[0,p-2]}(\mathfrak{X}_{\text{crys}})$ and there is a canonical identification*

$$T^*_{\text{crys}}(\mathbb{D}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}})) \xrightarrow{\sim} T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})^{\vee}.$$

Remark 5. The proof of Theorem 5 heavily utilizes the work of Tsuji in [Tsu20]. Similar, but independent results, were obtained in [Wü23] and forthcoming work of Christian Hokaj. See Remark 3.46 for more details.

[3] For many of these results we actually allow for more general p-adic formal schemes (e.g. $\operatorname{Spf}(W[\![t_1,\ldots,t_n]\!])$), and this generality is actually used in the proof of Theorem 3.

In addition, we are also able to show that $\mathbb{D}_{\mathrm{crys}}$ allows one to establish a connection between the prismatic Dieudonné crystal of a p-divisible group H as in [ALB23], and the more classical filtered Dieudonne crystal (see Proposition 3.50). In turn, this enables us to compare the prismatic Dieudonné crystal of H and the Kisin module associated to H in [Kim15] (see Proposition 3.53), which further allows one to bootstrap from the work in loc. cit to give a more a concrete description of prismatic Deudonné crystals in many cases (see Corollary 3.55).

Returning to Shimura varieties, we are able to, using our functor $\mathbb{D}_{\mathrm{crys}}$, establish the expected relationship between $\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ and $\omega_{\mathsf{K}^p,\mathrm{crys}}$, which allows us to compare the lattices $\omega_{\mathsf{K}^p,\mathbb{\Delta}}(\xi)$ and $\omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)$ even outside of the Fontaine–Laffaille range.

Theorem 6 (see Theorem 4.18)**.** *There is a canonical identification*

$$\mathbb{D}_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathbb{\Delta}} \xrightarrow{\sim} \omega_{\mathsf{K}^p,\mathrm{crys}}.$$

If $(\mathbf{G},\mathbf{X})$ is of Hodge type, then this identification can be made more explicit as a comparison between prismatic and crystalline cohomology of the 'universal' abelian variety, matching tensors (see Proposition 4.19).

One consequence of these results is a new proof that the cohomology of automorphic étale sheaves is crystalline, and of the matching of lattices for low Hodge–Tate weights.

Proposition 5 (see [Lov17a, Theorem 3.6.1], Proposition 4.21)**.** *Suppose that $\mathbf{G}^{\mathrm{der}}$ is $\mathbb{Q}$-anisotropic. Then, for any representation ξ of $\mathcal{G}$, the representation $H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi)[1/p])$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$ is crystalline and there exists an isomorphism of filtered F-isocrystals*

$$D_{\mathrm{crys}}\left(H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi)[1/p])\right) \xrightarrow{\sim} H^i_{\mathrm{crys}}\left(((\mathscr{S}_{\mathsf{K}^p})_{\breve{\mathbb{Z}}_p}/\breve{\mathbb{Z}}_p)_{\mathrm{crys}}, \omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)[1/p]\right).$$

If the Hodge weights of $\xi[1/p]$ are at most $p-2-i$, then this isomorphism sends the lattice $H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi))$ onto the lattice $H^i_{\mathrm{crys}}\left(((\mathscr{S}_{\mathsf{K}^p})_{\breve{\mathbb{Z}}_p}/\breve{\mathbb{Z}}_p)_{\mathrm{crys}}, \omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)\right)$.

Finally, another consequence is the ability to relate via specialization certain important functions on the generic and special fiber of $\widehat{\mathscr{S}}_{\mathsf{K}^p}$. Namely, let $C(\mathcal{G})$ denote the quotient of $G(\breve{\mathbb{Q}}_p)$ by σ-conjugation under $\mathcal{G}(\breve{\mathbb{Z}}_p)$, and fix a neat compact open subgroup $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}^p_f)$. We then have

$$\Upsilon^\circ_{\mathsf{K}^p} \colon \mathscr{S}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p) \to C(\mathcal{G}), \qquad \Sigma^\circ_{\mathsf{K}^p} \colon |(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta|^{\mathrm{cl}} \to C(\mathcal{G}),$$

where $|(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta|^{\mathrm{cl}}$ denotes the classical points of $(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta$. The first of these functions is that defining the central leaf decomposition (see [SZ22, §4.1] and the references therein), and the latter is given by associating to x the F-crystal with $\mathcal{G}$-structure given by $\underline{\mathbb{D}}_{\mathrm{crys}} \circ (\omega_{\mathsf{K}^p,\mathrm{\acute{e}t}})_x$ (see Example 3.5). Denote by $\mathrm{sp}\colon |(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta|^{\mathrm{cl}} \to \mathscr{S}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ the specialization map.

Proposition 6 (see Corollary 4.23)**.** *The function $\Sigma^\circ_{\mathsf{K}^p}$ factorizes through* sp, *and if $(\mathbf{G},\mathbf{X})$ is of Hodge type, or $Z(\mathbf{G})$ is connected, then $\Upsilon^\circ_{\mathsf{K}^p} = \Sigma^\circ_{\mathsf{K}^p} \circ \mathrm{sp}$.*

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Notation and conventions.

- The symbol p will always denote a (rational) prime.
- All rings are assumed commutative and unital unless stated otherwise.
- Formal schemes are assumed to have a locally finitely generated ideal sheaf of definition.
- For a property P of morphisms of schemes, an adic morphism of formal schemes $\mathfrak{X} \to \mathfrak{Y}$, where $\mathfrak{Y}$ has an ideal sheaf of definition $\mathcal{I}$, is *adically P (or $\mathcal{I}$-adically P)* if the reduction modulo $\mathcal{I}^n$ is P for all n. If $A \to B$ is an adic morphism of rings with the I-adic topology, for $I \subseteq A$ an ideal, then we make a similar definition.
- For a morphism of (formal) schemes/adic spaces $f\colon X \to Y$ we denote $R^i f_* \underline{\mathbb{Z}_p}$ and $R^i f_* \underline{\mathbb{Q}_p}$ by $\mathcal{H}^i_{\mathbb{Z}_p}(X/Y)$ and $\mathcal{H}^i_{\mathbb{Q}_p}(X/Y)$, respectively. Similar notation will be applied for de Rham and crystalline cohomology.
- For a non-archimedean field K, a *rigid K-space* X is an adic space locally of finite type over K. We denote the set of *classical points* by $|X|^{\mathrm{cl}} := \{x \in X : [k(x) : K] < \infty\}$.
- For two categories $\mathscr{C}$ and $\mathscr{D}$, the notation $(F, G)\colon \mathscr{C} \to \mathscr{D}$ means a pair functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$, with F being right adjoint to G.
- For an R-module M and an ideal $I \subseteq R$ we write M/I as shorthand for M/IM.
- A filtration always means a decreasing, separated, and exhaustive $\mathbb{Z}$-filtration.
- For a ring A which is a-adically separated for a in A, denote by $\mathrm{Fil}^\bullet_a$ the filtration with $\mathrm{Fil}^r_a = a^r A$ for $r > 0$, and $\mathrm{Fil}^r_a = A$ for $r \leqslant 0$. Define $\mathrm{Fil}^\bullet_{\mathrm{triv}} := \mathrm{Fil}^\bullet_0$.
- A filtration of modules (of sheaves) is *locally split* if its graded pieces are locally free.
- For an $\mathbb{F}_p$-algebra R (resp. $\mathbb{F}_p$-scheme X), we denote by F_R (resp. F_X) the absolute Frobenius of R (resp. X).

1. The Tannakian framework for (analytic) prismatic F-crystals

In this section we discuss the Tannakian theory of (analytic) prismatic F-crystal on a quasi-syntomic p-adic formal scheme $\mathfrak{X}$. See Appendix A for our conventions concerning, topoi, formal schemes, and the Tannakian formalism.

Notation. Throughout this section, unless stated otherwise, we fix the following notation:

- k is a perfect extension of $\mathbb{F}_p$, $W := W(k)$, and $K_0 := \mathrm{Frac}(W)$,
- K is a finite totally ramified extension of K_0, with ring of integers $\mathcal{O}_K$, residue field k, and ramification index e,
- π is a uniformizer of K, which we take to be p if $K = K_0$, and $E \in W[u]$ is the minimal polynomial for π over K_0,
- $\overline{K}$ is an algebraic closure of K and C is its p-adic completion,
- $\pi^\flat$ and $p^\flat$ in $C^\flat$ are as in [SW20, Lemma 6.2.2],
- $\varepsilon = (1, \zeta_p, \ldots)$ in $C^\flat$ is a compatible system of p^{th}-power roots of 1,
- $q = [\varepsilon]$ in $\mathrm{A_{inf}}(\mathcal{O}_C)$, and $t = \log(q) = -\sum_{n \geqslant 1} \frac{(1-q)^n}{n}$,
- $\xi_0 = p - [p^\flat]$ and $\tilde{\xi}_0 = p - [p^\flat]^p$, elements of $\mathrm{A_{inf}}(\mathcal{O}_C)$,
- $(\Lambda_0, \mathbb{T}_0)$ is a tensor package over $\mathbb{Z}_p$, with $\mathcal{G} := \mathrm{Fix}(\mathbb{T}_0)$ smooth over $\mathbb{Z}_p$ (see §A.5),
- G is defined to be $\mathcal{G}_{\mathbb{Q}_p}$.

1.1. **The absolute prismatic and quasi-syntomic sites.** We now record notation and basic results about the prismatic and quasi-syntomic sites developed in [BMS19], [BS22], and [BS23].

1.1.1. *Prisms.* For a $\mathbb{Z}_{(p)}$-algebra A, a *δ-structure* is a map $\delta\colon A \to A$ with $\delta(0) = \delta(1) = 0$ and

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y), \qquad \delta(x+y) = \delta(x) + \delta(y) + \tfrac{1}{p}(x^p + y^p - (x+y)^p).$$

Associated to δ is a Frobenius lift $\phi\colon A \to A$ given by $\phi(x) = x^p + p\delta(x)$, which we call the *Frobenius*. If A is p-torsion-free then any Frobenius lift ϕ on A defines a δ structure by $\frac{1}{p}(\phi(x) - x^p)$, establishing a bijection between the two types of structures, and we conflate the two notions. We call the pair (A, δ) a *δ-ring*. We often suppress δ from the notation, writing δ_A (or ϕ_A) when we want to be clear. A morphism of δ-rings is a ring map that intertwines the δ-structures.

A *prism* is a pair (A, I) where A is a δ-ring and $I \subseteq A$ is an invertible ideal with A derived (p, I)-adically complete (see [BMS18, §6.2]), and $p \in I + \phi(I)$. Thus, I is finitely generated and $\mathrm{Spec}(A) - V(I)$ is affine (see [SP, Tag 07ZT]), and we denote by $A[1/I]$ its global sections. Let $j_{(A,I)}$ denote the natural inclusion of $U(A, I) := \mathrm{Spec}(A) - V(p, I)$ into $\mathrm{Spec}(A)$. For a prism (A, I), unless stated otherwise, we view A as being equipped with the (p, I)-adic topology.

A prism (A, I) is *bounded* if A/I has bounded p^∞-torsion. The following two results will be used without comment in sequel.

Lemma 1.1. *Let (A, I) be a bounded prism. Then, A is (p, I)-adically complete, and A/I is p-adically complete.*

Proof. The first claim is precisely [BS22, Lemma 3.7 (1)]. For the second claim, let $I = (d_1, \ldots, d_n)$. Then, $A/I = \mathrm{coker}(f)$, where $f\colon A^n \to A$ is given by $f(a_1, \ldots, a_n) = \sum_{i=1}^n d_i a_i$. As A^n and A are p-adically complete, we know by [BS15, Lemma 3.4.14] that A/I is derived p-adically complete. But, it is then p-adically complete by [BMS19, Lemma 4.7]. □

A *morphism* $(A, I) \to (B, J)$ is a morphism $A \to B$ of δ-rings mapping I into J. By the *rigidity property of morphisms* of prisms (see [BS22, Proposition 3.5]), if $(A, I) \to (B, J)$ is a morphism of prisms, then $I \otimes_A B$ maps isomorphically onto J and, in particular, $J = IB$. A morphism $(A, I) \to (B, IB)$ is *I-completely (faithfully) flat (resp. étale, smooth)* when $B \otimes^L_A (A/I)$ is concentrated in degree 0 and $A/I \to B \otimes^L_A (A/I)$ is (faithfully) flat (resp. étale, smooth).

Lemma 1.2. *Let (A, I) be a bounded prism. Let B be a (p, I)-adically complete A-algebra. Then $A \to B$ is (p, I)-completely (faithfully) flat (resp. étale, smooth) if and only if $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is adically (faithfully) flat (resp. étale, smooth).*

Proof. We put $J = IB$. If $(A, I) \to (B, J)$ is (p, I)-completely (faithfully) flat then [Yek18, Theorem 4.3] implies the map $A/(p, I)^n \to B/(p, J)^n$ is (faithfully) flat for all n, and so $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is adically (faithfully) flat. Suppose that $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is adically (faithfully) flat. Then [Yek21, Theorem 7.3] implies the ideal $(p, I) \subseteq A$ is weakly proregular in the sense of op. cit. Therefore, we deduce that $A \to B$ is (p, I)-completely (faithfully) flat by completeness of these modules and [Yek18, Theorem 6.9]. The second claim follows from this as $A \to B$ is (p, I)-completely étale (resp. smooth) if and only if it is (p, I)-completely flat and $A/(p, I) \to B/(p, J)$ is étale (resp. smooth), and $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is adically étale if and only if it is adically flat and $A/(p, I) \to B/(p, J)$ is étale (resp. smooth). □

Proposition 1.3. *Let (A, I) be a bounded prism and $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ an adically étale map, where B is (p, I)-adically complete. Then, there exists a unique δ-structure on B such that $(A, I) \to (B, IB)$ is a morphism of bounded prisms.*

Proof. By [BS22, Lemma 2.18 and Lemma 3.7 (3)], we obtain a morphism $(A, I) \to (B, IB)$ of prisms. Then by [BS15, Lemma 3.4.14], B/IB is derived p-adically complete. Hence (B, IB) is a bounded prism by [BMS19, Corollary 4.8 (1)]. □

A prism (A, I) is *perfect* if ϕ_A is an isomorphism, in which case it is bounded (see [BS22, Lemma 2.34]). For a perfectoid ring R, we have the perfect prism $(\mathrm{A_{inf}}(R), \ker(\theta_R))$ where $\mathrm{A_{inf}}(R) = W(R^\flat)$ is Fontaine's ring, which comes equipped with a natural Frobenius ϕ_R, and $\theta_R\colon \mathrm{A_{inf}}(R) \to R$ is Fontaine's map. We also have the perfect prism $(\mathrm{A_{inf}}(R), \ker(\widetilde{\theta}_R))$, where $\widetilde{\theta}_R := \theta_R \circ \phi_R^{-1}$, which is isomorphic to $(\mathrm{A_{inf}}(R), \ker(\theta_R))$ via ϕ_R.[4] We often fix a generator ξ_R of $\ker(\theta_R)$ and set $\tilde{\xi}_R := \phi_R(\xi_R)$ so that $\ker(\widetilde{\theta}_R) = (\tilde{\xi}_R)$. When R is clear from context we shall omit the decoration of R at all places. When R is an $\mathcal{O}_C$-algebra, we may take $\xi = \xi_0$ and $\tilde{\xi} = \tilde{\xi}_0$ which we often implicitly do.

[4]See Remark 1.9 to see why we introduce these two choices.

1.1.2. *The absolute prismatic site.* Let $\mathfrak{X}$ be a p-adic formal scheme. Consider the category $\mathfrak{X}_{\Delta}^{\mathrm{op}}$ of triples (A, I, s) where (A, I) is a bounded prism, and $s\colon \mathrm{Spf}(A/I) \to \mathfrak{X}$ is a morphism, and where morphisms are maps of prisms commuting with the maps to $\mathfrak{X}$. We often omit s from the notation. The *absolute prismatic site* $\mathfrak{X}_{\Delta}$ of $\mathfrak{X}$ (see [BS23, Definition 2.3]) is the opposite category of $\mathfrak{X}_{\Delta}^{\mathrm{op}}$, endowed with the topology where $\{\alpha_i\colon (A, I) \to (B_i, J_i)\}$ in $\mathfrak{X}_{\Delta}^{\mathrm{op}}$ corresponds to a cover if $\{\mathrm{Spf}(\alpha_i)\colon \mathrm{Spf}(B_i) \to \mathrm{Spf}(A)\}$ is a cover in $\mathrm{Spf}(A)_{\mathrm{fl}}$ (see §A.4). That this is a site follows from the argument in [BS22, Corollary 3.12], which also shows that for a diagram in $\mathfrak{X}_{\Delta}^{\mathrm{op}}$

$$(B, IB) \leftarrow (A, I) \to (C, IC),$$

with one of the maps adically faithfully flat, then its cofibered product is $(B\widehat{\otimes}_A C, I(B\widehat{\otimes}_A C))$ with the obvious δ-structure and map to $\mathfrak{X}$. We often abuse notation when working in $\mathfrak{X}_{\Delta}$, writing objects and morphisms as in $\mathfrak{X}_{\Delta}^{\mathrm{op}}$. We also shorten $\mathrm{Spf}(R)_{\Delta}$ to R_{Δ}, in which case we often write $s\colon \mathrm{Spf}(A/I) \to \mathrm{Spf}(R)$ as $s\colon R \to A/I$.

By [BS22, Corollary 3.12], the presheaves $\mathcal{O}_{\mathfrak{X}_{\Delta}}(A, I) := A$ and $\overline{\mathcal{O}}_{\mathfrak{X}_{\Delta}}(A, I) := A/I$ are sheaves. By the rigidity property of morphisms of prisms, $\mathcal{I}_{\mathfrak{X}_{\Delta}}(A, I) := I$ is a quasi-coherent ideal sheaf of $\mathcal{O}_{\mathfrak{X}_{\Delta}}$. The association $\mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\Delta}](A, I) := A[1/I]$ is a sheaf of rings as $A \to A[1/I]$ is flat. Define

$$\mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\Delta}]_p^{\wedge} := \varprojlim_n \mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\Delta}]/p^n\mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\Delta}] = R\varprojlim_n \mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\Delta}]/p^n\mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\Delta}],$$

(see [BS23, Remark 2.4] and [BS15, Proposition 3.1.10] for the second equality). The morphisms ϕ_A induce a morphism of sheaves of rings $\phi_{\mathfrak{X}_{\Delta}}\colon \mathcal{O}_{\mathfrak{X}_{\Delta}} \to \mathcal{O}_{\mathfrak{X}_{\Delta}}$. We often omit the $\mathfrak{X}$ from the above notation when it is clear from context, writing $\mathcal{O}_{\Delta}$, $\overline{\mathcal{O}}_{\Delta}$, $\mathcal{I}_{\Delta}$, $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]$, $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}$, and ϕ.

For a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$, the association $((A, I), s) \mapsto ((A, I), f \circ s)$ is cocontinuous, so gives a morphism of topoi $(f_{\Delta *}, f_{\Delta}^*)\colon \mathbf{Shv}(\mathfrak{X}_{\Delta}) \to \mathbf{Shv}(\mathfrak{Y}_{\Delta})$. We shorten $R^i f_{\Delta *}\mathcal{O}_{\mathfrak{X}_{\Delta}}$ to $\mathcal{H}^i_{\Delta}(\mathfrak{X}/\mathfrak{Y})$.

1.1.3. *The quasi-syntomic site.* As in [BMS19], call a p-adically complete ring R *quasi-syntomic* if it has bounded p^∞-torsion and the cotangent complex $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$ (see [BMS19, Definition 4.1]), which we consider as having the p-adic topology. A map $R \to S$ of p-adically complete rings with bounded p^∞-torsion is called a *quasi-syntomic morphism (resp. cover)* if it is adically flat (resp. adically faithfully flat)[5] and $L_{S/R}$ has p-complete Tor-amplitude in $[-1, 0]$. By [BMS19, Proposition 4.19] a perfectoid ring R is quasi-syntomic.

One extends these definitions to (maps of) p-adic formal schemes by working affine locally. For a quasi-syntomic p-adic formal scheme $\mathfrak{X}$ the *big (resp. small) quasi-syntomic site* of $\mathfrak{X}$, denoted $\mathfrak{X}_{\mathrm{QSYN}}$ (resp. $\mathfrak{X}_{\mathrm{qsyn}}$), has objects maps (resp. quasi-syntomic maps) $\mathrm{Spf}(R) \to \mathfrak{X}$ for R a p-adically complete ring with bounded p^∞-torsion, morphisms given by $\mathfrak{X}$-morphisms, and covers given by quasi-syntomic covers (see [BMS19, Lemma 4.17]).

The functor

$$u\colon \mathfrak{X}_{\Delta} \to \mathfrak{X}_{\mathrm{QSYN}}, \qquad ((A, I), s) \mapsto s,$$

is cocontinuous (see [ALB23, Corollary 3.24]), and therefore gives rise to a morphism of topoi $(u_*, u^{-1})\colon \mathbf{Shv}(\mathfrak{X}_{\Delta}) \to \mathbf{Shv}(\mathfrak{X}_{\mathrm{QSYN}})$. The inclusion $\mathfrak{X}_{\mathrm{qsyn}} \to \mathfrak{X}_{\mathrm{QSYN}}$, while continuous, may not induce a morphism of sites (see [ALB23, §4.1]), but we may still consider the functor

$$v_*\colon \mathbf{Shv}(\mathfrak{X}_{\Delta}) \to \mathbf{Shv}(\mathfrak{X}_{\mathrm{qsyn}}), \qquad \mathcal{F} \mapsto u_*(\mathcal{F})|_{\mathfrak{X}_{\mathrm{qsyn}}}.$$

Following [ALB23, Definition 4.1], we use $\mathcal{O}_{\mathfrak{X}}^{\mathrm{pris}}$ to denote $v_*(\mathcal{O}_{\mathfrak{X}_{\Delta}})$ and $\mathcal{I}_{\mathfrak{X}}^{\mathrm{pris}}$ for $v_*(\mathcal{I}_{\mathfrak{X}_{\Delta}})$. Define

$$\mathcal{O}_{\mathfrak{X}}^{\mathrm{pris}}[1/\mathcal{I}_{\mathfrak{X}}^{\mathrm{pris}}](R) := \mathcal{O}_{\mathfrak{X}}^{\mathrm{pris}}(R)[1/\mathcal{I}_{\mathfrak{X}}^{\mathrm{pris}}(R)] = v_*(\mathcal{O}_{\mathfrak{X}_{\Delta}}[1/\mathcal{I}_{\mathfrak{X}_{\Delta}}]).$$

There is a morphism of sheaves of rings $\phi_{\mathfrak{X}}^{\mathrm{pris}} := v_*(\phi_{\mathfrak{X}_{\Delta}})\colon \mathcal{O}_{\mathfrak{X}}^{\mathrm{pris}} \to \mathcal{O}_{\mathfrak{X}}^{\mathrm{pris}}$. When no confusion will arise, we omit $\mathfrak{X}$ from notation, writing $\mathcal{O}^{\mathrm{pris}}$, $\mathcal{I}^{\mathrm{pris}}$, $\mathcal{O}_{\mathrm{pris}}[1/\mathcal{I}^{\mathrm{pris}}]$, and ϕ. There are also obvious analogues of these objects using u_* in place of v_*, which we denote by $\mathcal{O}^{\mathrm{PRIS}}$, etc.

A quasi-syntomic ring R is called *quasi-regular semi-perfectoid* (see [BMS19, Definition 4.20 and Remark 4.22]), abbreviated *qrsp*, if there exists a surjection $S \to R$ with S a perfectoid ring.

[5] By [BMS19, Corollary 4.8], $R \to S$ is adically (faithfully) flat if and only if it is p-completely (faithfully) flat.

Lemma 1.4. *Let $R \to R'$ is a p-adically flat morphism of p-adically complete rings. If $N \geqslant 1$ is such that $R[p^N] = R[p^\infty]$, then $R'[p^N] = R'[p^\infty]$.*

Proof. Let r' in $R'[p^\infty]$ and $n \geqslant N$ such that $p^n r' = 0$. As R' is p-adically separated, it suffices to show that $p^N r' = 0 \mod p^m$ for all $m \geqslant 0$. Since the image of $(R/p^{m+n-N})[p^n]$ in $(R/p^m)[p^n]$ is contained in $(R/p^m)[p^N]$, as $R[p^n] = R[p^N]$. But, $(R'/p^\ell)[p^n] = (R'/p^\ell) \otimes_{R/p^\ell} (R/p^\ell)[p^n]$ for any $\ell \geqslant 0$ by p-adic flatness, so the image of r' in $(R'/p^m)[p^n]$ is contained in $(R'/p^m)[p^N]$. $\square$

Lemma 1.5. *If R is qrsp and $\mathrm{Spf}(R') \to \mathrm{Spf}(R)$ is an adically étale map, then R' is qrsp.*

Proof. By [BMS19, Remark 4.22] it is sufficient to show that R' is quasi-syntomic, R'/p is semi-perfect, and that there exists a perfectoid ring S and a morphism $S \to R'$. The last of these is clear. To prove the first claim, we observe that as R' has bounded p^∞-torsion by Lemma 1.4, that $R \to R'$ is p-completely flat by [BMS19, Corollary 4.8] and so $R' \otimes^L_R (R/p) = R'/pR'$. Thus, by [SP, Tag 08QQ] we have that $L_{R'/R} \otimes^L_{R'} (R'/p) = L_{(R'/p)/(R/p)} = 0$. For semi-perfectness, observe that as $\mathrm{Frob}_{R/p} \colon R/p \to R/p$ is surjective, thus is the induced map $R'/p \to R'/p \otimes_{R/p, \mathrm{Frob}_{R/p}} R/p$. But, this map is identified with $\mathrm{Frob}_{R'/p} \colon R'/p \to R'/p$ as $R/p \to R'/p$ is étale. $\square$

Denote by $\mathfrak{X}_{\mathrm{qrsp}}$ the full subcategory of $\mathfrak{X}_{\mathrm{qsyn}}$ consisting of qrsp objects, with the induced topology. By [BMS19, Lemma 4.28 and Proposition 4.31], $\mathfrak{X}_{\mathrm{qrsp}}$ is a basis for $\mathfrak{X}_{\mathrm{qsyn}}$. By a *qrsp cover* $\{\mathrm{Spf}(R_i) \to \mathfrak{X}\}$, we mean a quasi-syntomic cover where each R_i is qrsp.

1.1.4. *Initial prisms.* When R is a qrsp ring, the category R_{Δ} has an initial object (Δ_R, I_R), necessarily unique up to unique isomorphism (see [BS22, Proposition 7.2]).

Example 1.6. If R is perfectoid, then by [BS22, Lemma 4.8], $(\mathrm{A_{inf}}(R), (\xi), \mathrm{nat.})$, or the isomorphic $(\mathrm{A_{inf}}(R), (\tilde{\xi}), \widetilde{\mathrm{nat.}})$, are initial objects. Here we denote by $\mathrm{nat.} \colon R \xrightarrow{\sim} \mathrm{A_{inf}}(R)/(\xi)$ (resp. $\widetilde{\mathrm{nat.}} \colon R \xrightarrow{\sim} \mathrm{A_{inf}}(R)/(\tilde{\xi})$) the natural isomorphism induced by θ (resp. $\tilde{\theta}$).

Example 1.7. Let R be a qrsp k-algebra. If $R^\flat := \varprojlim_{F_R} R$ and J denotes the kernel of the composition $W(R^\flat) \to R^\flat \to R$, set $\mathrm{A_{crys}}(R) := W(R^\flat)[\{\frac{x^n}{n!} : x \in J\}]^\wedge_p$ to be the p-completed divided power envelope of $(W(R^\flat), K)$, which constitutes the universal pro-(PD thickening) of R over W (see [Fon94, Théorème 2.2.1]). Let $\phi_R \colon \mathrm{A_{crys}}(R) \to \mathrm{A_{crys}}(R)$ denote the morphism induced from F_R by this universality. Then, by [BMS19, Theorem 8.14], $\mathrm{A_{crys}}(R)$ is p-torsion-free and ϕ_R is a Frobenius lift on $\mathrm{A_{crys}}(R)$ and so $(\mathrm{A_{crys}}(R), (p))$ is a prism.

Under the natural Frobenius-equivariant morphism $W(R^\flat) \to \mathrm{A_{crys}}(R)$, the ideal $\phi_R(J)$ maps to (p).[6] We then obtain a morphism $\widetilde{\mathrm{nat.}} \colon R \to \mathrm{A_{crys}}(R)/p$ obtained as the composition

$$R \xrightarrow{\sim} W(R^\flat)/J \xrightarrow{\phi_R} W(R^\flat)/\phi_R(J) \to \mathrm{A_{crys}}(R)/p.$$

So, $(\mathrm{A_{crys}}(R), (p), \widetilde{\mathrm{nat.}})$ constitutes an element of R_{Δ}, and is initial by [ALB23, Lemma 3.27].

Example 1.8. Let R be a perfectoid ring, and set $\mathrm{A_{crys}}(R) := \mathrm{A_{crys}}(R/p)$. Then, the universal property of $\mathrm{A_{crys}}(R)$ implies that the map $\theta \colon \mathrm{A_{inf}}(R) \to R$ extends to a map $\theta \colon \mathrm{A_{crys}}(R) \to R$ which is an initial object of $(R/W)_{\mathrm{crys}}$ (see §2.3.1 for this notation). We write ϕ_R instead of $\phi_{R/p}$. From Example 1.7, we have a morphism of prisms $(\mathrm{A_{inf}}(R), (\tilde{\xi}), \widetilde{\mathrm{nat.}}) \to (\mathrm{A_{crys}}(R), (p), \widetilde{\mathrm{nat.}})$. This is injective by [SW13, Lemma 4.1.7], as R/p is f-adic by [SW13, Proposition 4.1.2] as $\ker(R^\flat \to R/p)$ is generated by the image of ξ under $\mathrm{A_{inf}}(R) = W(R^\flat) \to R^\flat$.

Remark 1.9. The morphism $(\mathrm{A_{inf}}(R), (\tilde{\xi}), \widetilde{\mathrm{nat.}}) \to (\mathrm{A_{crys}}(R), (p), \widetilde{\mathrm{nat.}})$ in Example 1.8 justifies the appearance of the element $\tilde{\xi}$. One cannot generally undo these Frobenius twists as while ϕ_R is an isomorphism on $\mathrm{A_{inf}}(R)$, it is not on $\mathrm{A_{crys}}(R)$.

[6] Indeed, as $\phi_R(J)$ contains p it suffices to show that if x is in J, then p divides $\phi_R(x)$ in $\mathrm{A_{crys}}(R)$. But, observe that $x^p = p(p-1)!\frac{x^p}{p!}$ and so $\phi_R(x) = x^p = 0 \mod p\mathrm{A_{crys}}(R)$.

If $\mathrm{Spf}(R)\to\mathfrak{X}$ is an object of $\mathfrak{X}_{\mathrm{qrsp}}$ then $u^{-1}(h_{\mathrm{Spf}(R)})$ is equal to $h_{(\Delta_R,I_R)}$. Using this and the cocontinuity of u (see [ALB23, Corollary 3.24]), one deduces the following (cf. Lemma A.8).

Proposition 1.10. *If $\{\mathrm{Spf}(R_i)\to\mathfrak{X}\}$ is a qrsp cover, then $\{(\Delta_{R_i},I_{R_i})\}$ covers $*$ in* $\mathbf{Shv}(\mathfrak{X}_{\Delta})$.

1.1.5. *Small and base $\mathcal{O}_K$-algebras.* We now discuss the rings of main interest in this article.

First definitions. Let R be a p-adically complete $\mathcal{O}_K$-algebra with $\mathrm{Spec}(R)$ connected. Call R a *base $\mathcal{O}_K$-algebra* if $R=R_0\otimes_W\mathcal{O}_K$ where R_0 is a J_{R_0}-adically complete ring for which the pair $(R_0,J_{R_0})=(A_n,I_n)$, is obtained from the following iterative procedure. For some $d\geqslant 0$, let A_0 be $T_d:=W\langle t_1^{\pm1},\ldots,t_d^{\pm1}\rangle$, and set $I_0=(p)$. For each $i=0,\ldots,n-1$ iteratively form the pair (A_{i+1},I_{i+1}) by one of the following operations:

- A_{i+1} is the p-adic completion of an étale A_i-algebra,[7] and $I_{i+1}=I_iA_{i+1}$,
- A_{i+1} is the p-adic completion of a localization $A_i\to(A_i)_{\mathfrak{p}}$ at a prime $\mathfrak{p}$ containing p, and $I_{i+1}=I_iA_{i+1}$,
- A_{i+1} is the I-adic completion of A_i with respect to an ideal $I\subseteq A_i$ containing p, and $I_{i+1}=(I_i,I)A_{i+1}$.

While the discussion of base $\mathcal{O}_K$-algebras implicitly entails other topologies, we always think of a base $\mathcal{O}_K$-algebra as being equipped with the p-adic topology.

A map $t\colon T_d\to R_0$ (where we implicitly have $R=R_0\otimes_W\mathcal{O}_K$) of the form constructed above is called a *presentation*. In the following we use terminology from [Kim15, §2.2]. Moreover, for a map of rings $R\to S$ and an ideal $J\subseteq S$, we say that $R\to S$ is *J-formally étale* if the condition in [SP, Tag 00UQ] holds whenever the ideal generated by the image of J in A/I (with notation as in loc. cit.) is nilpotent.

Proposition 1.11. *A base $\mathcal{O}_K$-algebra R is excellent and regular, and R/π has a finite p-basis. For a presentation $t\colon T_d\to R_0$, the k-algebra R_0/J_{R_0} is finite type, and t is J_{R_0}-formally étale.*

Proof. Fix a presentation $t\colon T_d\to R_0$. We first prove that R is excellent. As $R_0\to R$ is finite type, it suffices to prove that R_0 is excellent (see [SP, Tag 07QU]). We prove that A_i is excellent by induction on i. As T_d is the p-adic completion of $W[t_1^{\pm1},\ldots,t_d^{\pm1}]$, which is excellent by loc. cit., we know that T_d is excellent by [KS21, Main Theorem 2]. If A_i is excellent, then A_{i+1} is excellent regardless of which of the three constructions is applied to A_i by combining [SP, Tag 07QU] and [KS21, Main Theorem 2]. To prove that t is J_{R_0}-formally étale, it suffices by induction to prove that $A_i\to A_{i+1}$ is I_{i+1}-formally étale for each $i=0,\ldots,n-1$, but this is clear. Thus, $T_d\otimes_W\mathcal{O}_K\to R$ is J_{R_0}-formally étale, and thus it follows from [MR10, Theorem 6.4.2] that $T_d\otimes_W\mathcal{O}_K\to R$ is regular and thus R is regular.[8] The final claims concerning $R/\pi=R_0/p$ and R_0/J_{R_0} may be checked iteratively, the latter being obvious. The former being preserved by our three operations requires [dJ95a, Lemma 1.1.3], and the observation that if $\{x_\alpha\}$ is a p-basis for an $\mathbb{F}_p$-algebra A, and $A\to B$ is étale then $\{x_\alpha\}$ is a p-basis for B as $F_A\otimes 1$ is identified with F_B via the isomorphism $A\otimes_{F_A,A}B\xrightarrow{\sim}B$. $\square$

We call a decomposition $R=R_0\otimes_W\mathcal{O}_K$ and a J_{R_0}-formally étale map $t\colon T_d\to R_0$, where J_{R_0} is any ideal coming from a presentation, a *formal framing*. For a formal framing t, the ring R_0 carries a unique Frobenius lift ϕ_t such that $\phi_t\circ t=t\circ\phi_0$, where ϕ_0 is the Frobenius lift on T_d acting as usual on W and with $\phi_0(t_i)=t_i^p$.

Perfectoid type prisms. Let R be a base $\mathcal{O}_K$-algebra. Write $\mathcal{K}$ for $\mathrm{Frac}(R)$. Fix an algebraic closure $\overline{\mathcal{K}}$ containing $\overline{K}$, and denote by $\mathcal{K}^{\mathrm{ur}}$ the maximal subfield of $\overline{\mathcal{K}}$ unramified along $R[1/p]$ (cf. [SP, Tag 0BQJ]). Denote $\mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K})$ by Γ_R, which agrees with $\pi_1^{\text{ét}}(\mathrm{Spec}(R[1/p]),\overline{x})$ where $\overline{x}$ is the geometric point determined by $\overline{\mathcal{K}}$ (see [SP, Tag 0BQM]). Denote by $\overline{R}$ (resp. R^{ur}) the

[7] By Elkik's theorem (cf. [SP, Tag 0AKA]) we may replace this with: 'a p-adically étale map A_i-algebra'.

[8] For a map $f\colon(R,\mathfrak{m})\to(S,\mathfrak{n})$ of local rings, the phrase 'formally smooth' in [MR10, Theorem 6.4.2] means $\mathfrak{n}$-formally smooth in our terminology.

integral closure of R in $\overline{\mathcal{K}}$ (resp. $\mathcal{K}^{\mathrm{ur}}$), and set $\widetilde{R}$ (resp. $\check{R}$) to be its p-adic completion.[9] The $\mathcal{O}_C$-algebras $\check{R}$ and $\widetilde{R}$ are perfectoid by the following lemma.

Lemma 1.12 ([CS21, Proposition 2.1.8])**.** *Let A be a p-torsion-free $\mathcal{O}_C$-algebra which is p-integrally closed in $A[1/p]$ (i.e., if x^p is in A for x in $A[1/p]$, then x is in A). Then, the p-adic completion $\widehat{A}$ is perfectoid if and only if A/p is semi-perfect.*

Thus, we have the objects $(\mathrm{A}_{\mathrm{inf}}(\check{R}), (\tilde{\xi}), \widetilde{\mathrm{nat.}})$ and $(\mathrm{A}_{\mathrm{crys}}(\check{R}), (p), \widetilde{\mathrm{nat.}})$ of R_{Δ}, and their $\widetilde{R}$ counterparts.

Breuil–Kisin type prisms. Let $R = R_0 \otimes_W \mathcal{O}_K$ be a base $\mathcal{O}_K$-algebra and $t\colon T_d \to R_0$ a formal framing. We consider the following objects of R_{Δ}.

(1) The object $(R_0^{(\phi_t)}, (p), q)$. Here $R_0^{(\phi_t)}$ is the ring R_0 equipped with the δ-strucure corresponding to the Frobenius lift ϕ_t, and q is the quotient map $R \to R/\pi \xrightarrow{\sim} R_0/p$.

(2) **(Breuil–Kisin prism)** The object $(\mathfrak{S}_R^{(\phi_t)}, (E), \mathrm{nat.})$. Here $\mathfrak{S}_R^{(\phi_t)} := \mathfrak{S}_R := R_0[\![u]\!]$ is equipped with the δ-structure corresponding to the Frobenius $\phi_t\colon \mathfrak{S}_R \to \mathfrak{S}_R$ extending ϕ_t on R_0 and satisfying $\phi_t(u) = u^p$. The map $\mathrm{nat.}\colon R \xrightarrow{\sim} \mathfrak{S}_R/(E)$ is the natural one.

(3) **(Breuil prism)** Consider the Breuil ring S_R, defined to be the p-adic completion of the PD-envelope of $\mathfrak{S}_R \twoheadrightarrow R$, which can be explicitly described as follows:

$$S_R = \left\{ \sum_{m=0}^{\infty} a_m \frac{u^m}{\lfloor m/e \rfloor !} \in R_0[1/p][\![u]\!] : a_m \text{ converge to 0 } p\text{-adically} \right\}. \tag{1.1.1}$$

The ring $S_R^{(\phi_t)} := S_R$ has a unique Frobenius ϕ_t extending that on $\mathfrak{S}_R$, and thus an associated δ-structure. We then have the triple $(S_R^{(\phi_t)}, (p), \overline{i} \circ \overline{\phi}_t \circ \mathrm{nat.})$, where $i\colon \mathfrak{S}_R \to S_R$ is the natural inclusion, and $\overline{\phi}_t\colon \mathfrak{S}_R/(E) \to \mathfrak{S}_R/(\phi_t(E))$ is the map induced by ϕ_t, and this composition makes sense as a map $R \to S_R/p$ as $\frac{\phi_t(E)}{p}$ is a unit in S_R.

We often omit the decoration $(-)^{(\phi_t)}$ when the choice of a particular formal framing is clear or unimportant, in which case we just write ϕ for ϕ_t.

Various morphisms of prisms. Let $R = R_0 \otimes_W \mathcal{O}_K$ be a base $\mathcal{O}_K$-algebra and choose a formal framing $t\colon T_d \to R_0$. Denote by $t^\flat$ the choice of p^{th}-power roots $t_i^\flat$ of t_i in $\check{T}_d^\flat$.

Define $R_0 \to \mathrm{A}_{\mathrm{inf}}(\check{R})$, using the J_{R_0}-formal étaleness of t, as the unique extension of the map $T_d \to \mathrm{A}_{\mathrm{inf}}(\check{R})$ sending t_i to $[t_i^\flat]$ and inducing the natural map $R_0 \to \check{R}$ after composition with $\widetilde{\theta}$. We further define $\alpha_{\mathrm{inf}} = \alpha_{\mathrm{inf},t^\flat}\colon \mathfrak{S}_R \to \mathrm{A}_{\mathrm{inf}}(\check{R})$ to be the unique extension of this map such that $\alpha_{\mathrm{inf},t^\flat}(u) = [\pi^\flat]$. We then have the following diagram of prisms, where each inclusion arrow is the obvious inclusion, and all other not previously defined arrows are determined uniquely by commutativity.

[9] Note that our notation here differs from some references (e.g. [DLMS22]) where the notation $\overline{R}$ is used for what we denote by $\check{R}$.

$$
\begin{tikzcd}
 & (R_0^{(\phi_t)},(p),q) \arrow[r, "\phi_t"] & (R_0^{(\phi_t)},(p),F_{R_0/p}\circ q) \arrow[d, hook, "(*)"] \arrow[dd, bend left, "\beta_{t^\flat}"] \\
(\mathfrak{S}_R^{(\phi_t)},(E),\mathrm{nat.}) \arrow[r, "\phi_t"] \arrow[dr, "\widetilde{\alpha}_{\mathrm{inf},t^\flat}"'] & (\mathfrak{S}_R^{(\phi_t)},(\phi_t(E)),\overline{\phi}_t\circ\mathrm{nat.}) \arrow[r, hook] \arrow[d, "\alpha_{\mathrm{inf},t^\flat}"'] \arrow[dr, "\alpha_{\mathrm{crys},t^\flat}"] & (S_R^{(\phi_t)},(p),\bar{i}\circ\overline{\phi}_t\circ\mathrm{nat.}) \arrow[d, "\gamma_{t^\flat}"] \\
 & (\mathrm{A}_{\mathrm{inf}}(\check{R}),(\tilde{\xi}),\widetilde{\mathrm{nat.}}) \arrow[r, hook] \arrow[d, hook] & (\mathrm{A}_{\mathrm{crys}}(\check{R}),(p),\widetilde{\mathrm{nat.}}) \arrow[d, hook] \\
 & (\mathrm{A}_{\mathrm{inf}}(\widetilde{R}),(\tilde{\xi}),\widetilde{\mathrm{nat.}}) \arrow[r, hook] & (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}),(p),\widetilde{\mathrm{nat.}})
\end{tikzcd}
\tag{1.1.2}
$$

Note that for all of these triples, save $(R_0^{(\phi_t)},(p),q)$ and $(R_0^{(\phi_t)},(p),F_{R_0/p}\circ q)$, the structure morphism is unambiguous given the first two entries, so we often omit it. If we write $(R_0^{(\phi_t)},(p))$ then the reader should assume we mean $(R_0^{(\phi_t)},(p),q)$.

While every morphism in Diagram (1.1.2) is a morphism of prisms, the arrow labeled $(*)$ is the only map which may not be a morphism in $R_{\mathbbl{\Delta}}$. In fact, this happens precisely when $\mathcal{O}_K = W$.

Lemma 1.13. *For an integer $a \geqslant 0$, let s_{a+1}, denote the following composition:*

$$R \xrightarrow{\mathrm{nat.}} \mathfrak{S}_R/(E) \xrightarrow{\bar{\phi}_t} \mathfrak{S}_R/(\phi_t(E)) \xrightarrow{\bar{i}} S_R/p \xrightarrow{F^a_{S_R/p}} S_R/p.$$

Then $(R_0,(p),F^{a+1}_{R_0/p}\circ q) \hookrightarrow (S_R,(p),s_{a+1})$ is a morphism in $R_{\mathbbl{\Delta}}$ if and only if $p^a \geqslant e$.

Proof. It suffices to observe that the following diagram commutes if and only if $p^a \geqslant e$:

$$
\begin{tikzcd}
\mathfrak{S}_R/(\phi_t(E)) \arrow[r, "\bar{i}"] & S_R/p \arrow[r, "F^a_{S_R/p}"] & S_R/p & R_0/p \arrow[l] & R_0/p \arrow[l, "F^{a+1}_{R_0/p}"'] \\
\mathfrak{S}_R/(E) \arrow[u, "\bar{\phi}_t"] & & R \arrow[ll, "\mathrm{nat.}"] \arrow[u] \arrow[rr] & & R/(\pi) \arrow[u, "\wr"']
\end{tikzcd}
$$

Note that $R = R_0[\pi]$ and that the diagram always commutes on R_0. Then chasing where π is sent, one sees this commutativity holds if and only if $u^{p^{a+1}}$ is divisible by p in S_R. Using Equation (1.1.1), one easily sees that this happens if and only if $\frac{p^{a+1}}{e} \geqslant p$ or, equivalently, that $p^a \geqslant e$. ☐

Miscellanea. Following [DLMS22], we call a Zariski connected p-adically complete $\mathcal{O}_K$-algebra R *small* if there exists a p-adically étale morphism $t\colon \mathcal{O}_K\langle t_1^{\pm 1},\ldots,t_d^{\pm 1}\rangle \to R$ for some $d \geqslant 0$, called a *framing*. By the topological invariance of the étale site of a formal scheme, there is a unique p-adically étale morphism $T_d \to R_0$ with $R = R_0 \otimes_W \mathcal{O}_K$ such that the composition $T_d \to R_0 \to R$ is equal to t. We denote the map $T_d \to R_0$ also by t. Thus, R is a base $\mathcal{O}_K$-algebra.

By a *base formal $\mathcal{O}_K$-scheme* we mean a morphism of formal schemes $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ such that there exists an open cover $\{\mathrm{Spf}(R_i)\}$ of $\mathfrak{X}$ with each R_i a base $\mathcal{O}_K$-algebra. By the discussion in the last paragraph, this includes smooth formal schemes $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$.

Lemma 1.14 (cf. [Bha20, Theorem 5.16])**.** *Let R be a base $\mathcal{O}_K$-algebra. Then, $R \to \widetilde{R}$ is faithfully flat and quasi-syntomic.*

Proof. The faithful flatness follows from [Bha20, Theorem 5.16]. To prove quasi-syntomicness, first observe that $L_{\widetilde{R}/\mathcal{O}_K} \otimes^L_{\mathcal{O}_K} (\mathcal{O}_K/p)$ is concentrated in degree -1 (cf. the proof of [BMS19, Proposition 4.19], using $\mathcal{O}_K \to \mathrm{A}_{\mathrm{inf}}(\widetilde{R}) \otimes_W \mathcal{O}_K \xrightarrow{\theta\otimes 1} \widetilde{R}$). On the other hand, $\mathcal{O}_K \to R$ is p-completely flat, and so $R \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/p = R/p$. Thus, by [SP, Tag 08QQ] we have that $L_{R/\mathcal{O}_K} \otimes^L_{\mathcal{O}_K} (\mathcal{O}_K/p)$ is equal to $L_{(R/p)/\mathcal{O}_K/p}$. But, as $\mathcal{O}_K/p \to R/p$ is formally smooth, $L_{(R/p)/(\mathcal{O}_K/p)}$ is concentrated in degree 0. So, the claim follows by the triangle property for the cotangent complex. □

Proposition 1.15. *Let $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be a base formal $\mathcal{O}_K$-scheme, and $\{\mathrm{Spf}(R_i)\}$ an open cover with each R_i a base $\mathcal{O}_K$-algebra. Then, $\{(\mathfrak{S}_{R_i}, (E))\}$ is a cover of $*$ in $\mathbf{Shv}(\mathfrak{X}_{\Delta})$.*

Proof. The maps $\widetilde{\alpha}_{i,\mathrm{inf}} \colon (\mathfrak{S}_{R_i}, (E)) \to (\mathrm{A}_{\mathrm{inf}}(\widetilde{R}_i), (\tilde{\xi}))$ in $\mathfrak{X}_{\Delta}$, shows that $\{(\mathfrak{S}_{R_i}, (E)) \to *\}$ refines $\{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}_i), (\tilde{\xi})) \to *\}$. But, combining Proposition 1.10 with Lemma 1.14, we see that $\{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}_i), (\tilde{\xi})) \to *\}$ is a cover, and thus so is $\{(\mathfrak{S}_{R_i}, (E)) \to *\}$ (see Lemma A.1). □

1.2. **(Analytic) prismatic torsors with F-structure.** We now discuss the Tannakian aspects of the theory of (analytic) prismatic F-crystals, as in [BS22] and [GR22]. Fix a quasi-syntomic formal scheme $\mathfrak{X}$, and an object T of $\mathbf{Shv}(\mathfrak{X}_{\Delta})$, which we omit from the notation when $T = *$.

1.2.1. *Prismatic F-crystals.* Define the category of *prismatic crystals in vector bundles (resp. perfect complexes)* over T as follows:

$$\operatorname*{2-lim}_{(A,I) \in \mathfrak{X}_{\Delta}/T} \mathbf{Vect}(A) \qquad \left(\text{resp.} \lim_{(A,I)\in \mathfrak{X}_{\Delta}/T} \mathbf{D}_{\mathrm{perf}}(A)\right).$$

Concretely, a prismatic crystal in vector bundles (resp. perfect complexes) is a collection of finite projective A-modules $M_{(A,I)}$ (resp. perfect complexes $K^\bullet_{(A,I)}$), indexed by objects (A, I) of $\mathfrak{X}_{\Delta}/T$, together with (quasi-)isomorphisms $M_{(A,I)} \otimes_A B \xrightarrow{\sim} M_{(B,J)}$ (resp. $K^\bullet_{(A,I)} \otimes^L_A B \xrightarrow{\sim} K^\bullet_{(B,J)}$) for any morphism $(A, I) \to (B, J)$ in $\mathfrak{X}_{\Delta}$ (the *crystal property*), satisfying the obvious compatibility conditions. The category of prismatic crystals in vector bundles carries the structure of an exact $\mathbb{Z}_p$-linear $\otimes$-category where exactness and tensor products are defined term-by-term.

Proposition 1.16 (cf. [BS23, Proposition 2.7])**.** *The global sections functor*

$$\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta}) \to \operatorname*{2-lim}_{(A,I)\in\mathfrak{X}_{\Delta}/T} \mathbf{Vect}(A)$$

is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence. Moreover, the derived global sections functor

$$\mathbf{D}_{\mathrm{perf}}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta}) \to \lim_{(A,I)\in\mathfrak{X}_{\Delta}/T} \mathbf{D}_{\mathrm{perf}}(A)$$

is an equivalence of ∞-categories.

Proof. By [BS23, Proposition 2.7], it remains only to verify that the first functor is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence. By [BS23, Proposition 2.7], if $\mathcal{F}$ and $\mathcal{G}$ are objects of $\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$ then the presheaf $(A, I) \mapsto \mathcal{F}(A, I) \otimes_A \mathcal{G}(A, I)$ is a sheaf, and so the global sections functor preserves tensor products. Indeed, as $(\mathcal{F}(A, I) \otimes_A \mathcal{G}(A, I))$ forms a prismatic crystal in vector bundles by the crystal property for $\mathcal{F}$ and $\mathcal{G}$, there exists by [BS23, Proposition 2.7] a prismatic crystal $\mathcal{H}$ with $\mathcal{H}(A, I) = \mathcal{F}(A, I) \otimes_A \mathcal{G}(A, I)$, and as $\mathcal{H}$ is a sheaf the claim follows. The bi-exactness claim follows easily from Lemma 1.17. □

Lemma 1.17 (cf. [BS22, Corollary 3.12])**.** *Let $\mathcal{E}$ be an object of $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$. Then for any object (A, I) of $\mathfrak{X}_{\Delta}$, we have that $H^i((A, I), \mathcal{E}) = 0$ for any $i > 0$.*

Proof. The proof of [BS22, Corollary 3.12] applies as the Čech complex for $\mathcal{E}$ is the result of tensoring the Čech complex for $\mathcal{O}_{\Delta}$ with the flat module $M = \mathcal{E}(A, I)$, and so still exact. □

As in [BS23, Definition 4.1], define the category $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}/T)$ of *prismatic F-crystals over T* to have objects $(\mathcal{E}, \varphi_{\mathcal{E}})$ where $\mathcal{E}$ is an object of $\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$ and

$$\varphi_{\mathcal{E}} \colon (\phi^*\mathcal{E})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}],$$

is an isomorphism in $\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}])$, called the *Frobenius*, and morphisms are morphisms in $\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$ commuting with the Frobenii. Likewise, define the category $\mathbf{D}^{\varphi}_{\mathrm{perf}}(\mathfrak{X}_{\Delta}/T)$ of *prismatic F-crystals in perfect complexes* to be the category of pairs $(\mathcal{E}^\bullet, \varphi_{\mathcal{E}^\bullet})$ where $\mathcal{E}^\bullet$ is an object of $\mathbf{D}_{\mathrm{perf}}(\mathfrak{X}_{\mathbb{\Delta}}/T, \mathcal{O}_{\mathbb{\Delta}})$ together with a *Frobenius isomorphism* in $\mathbf{D}_{\mathrm{perf}}(\mathfrak{X}_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}])$

$$\varphi_{\mathcal{E}^\bullet} \colon L\phi^*\mathcal{E}^\bullet[1/\mathcal{I}_{\mathbb{\Delta}}] \xrightarrow{\sim} \mathcal{E}^\bullet[1/\mathcal{I}_{\mathbb{\Delta}}],$$

and with morphisms being those in $\mathbf{D}_{\mathrm{perf}}(\mathfrak{X}_{\mathbb{\Delta}}/T, \mathcal{O}_{\mathbb{\Delta}})$ commuting with Frobenii. By Proposition 1.16, when $\mathbf{Vect}^\varphi(A, I)$ and $\mathbf{D}^\varphi_{\mathrm{perf}}(A, I)$ are given the obvious meanings, then

$$\mathbf{Vect}^\varphi(\mathfrak{X}_{\mathbb{\Delta}}/T) = \underset{(A,I)\in\mathfrak{X}_{\mathbb{\Delta}}/T}{2\text{-}\lim} \mathbf{Vect}^\varphi(A, I), \qquad \mathbf{D}^\varphi_{\mathrm{perf}}(\mathfrak{X}_{\mathbb{\Delta}}/T) = \lim_{(A,I)\in\mathfrak{X}_{\mathbb{\Delta}}/T} \mathbf{D}^\varphi_{\mathrm{perf}}(A, I).$$

Observe that $\mathbf{Vect}^\varphi(\mathfrak{X}_{\mathbb{\Delta}}/T)$ inherits from $\mathbf{Vect}(\mathfrak{X}_{\mathbb{\Delta}}/T, \mathcal{O}_{\mathbb{\Delta}})$ the structure of an exact $\mathbb{Z}_p$-linear $\otimes$-category. We say $(\mathcal{E}, \varphi_{\mathcal{E}})$ is *effective* if $\varphi_{\mathcal{E}}$ is induced from a morphism $\varphi_{\mathcal{E},0} \colon \phi^*\mathcal{E} \to \mathcal{E}$, which is automatically injective, and the minimal r such $\mathcal{I}^r_{\mathbb{\Delta}}$ kills $\mathrm{coker}(\varphi_{\mathcal{E},0})$ is the *height* of $(\mathcal{E}, \varphi_{\mathcal{E}})$.

Remark 1.18. Suppose that (A, I, s) is an object of $\mathfrak{X}_{\mathbb{\Delta}}$ with the property that $\phi_A(I)$ is an invertible ideal (e.g., if ϕ_A is flat, see [SP, Tag 02OO], or $I = (p)$). In this case $(A, \phi_A(I), s \circ \mathrm{Spf}(\overline{\phi}_A))$ is an object of $\mathfrak{X}_{\mathbb{\Delta}}$, where $\overline{\phi}_A \colon A/I \to A/\phi_A(I)$ is the natural map. Indeed, if $p = i + \phi_A(i')$ with $i, i' \in I$ then evidently $p = \phi_A(i) + \phi_A(\phi_A(i'))$ so that $p \in \phi_A(I) + \phi_A(\phi_A(I))$, and A is $(p, \phi_A(I))$-adically complete as $(p, I)^p \subseteq (p, \phi_A(I)) \subseteq (p, I)$. Moreover, we observe that $\phi_A \colon (A, I, s) \to (A, \phi_A(I), s \circ \mathrm{Spf}(\overline{\phi}_A))$ is a morphism in $\mathfrak{X}_{\mathbb{\Delta}}$ so that, by the crystal property, we have an identification of A-modules

$$\phi^*\mathcal{E}(A, I, s) = \phi_A^*(\mathcal{E}(A, I, s)) \xrightarrow{\sim} \mathcal{E}(A, \phi_A(I), s \circ \mathrm{Spf}(\overline{\phi}_A)).$$

Thus, we, in particular, see that via this identification there is an isomorphism

$$\varphi_{\mathcal{E}} \colon (\phi^2)^*\mathcal{E}(A, I, s)[1/\phi_A(I)] \xrightarrow{\sim} \phi^*\mathcal{E}(A, I, s)[1/\phi_A(I)],$$

which provides $\phi^*\mathcal{E}(A, I, s)$ with a Frobenius-like structure.

1.2.2. *Analytic prismatic F-crystals.* Following [GR22, Definition 3.1], define the category of *analytic prismatic crystals* over T as follows (where we recall that $U(A, I) := \mathrm{Spec}(A) - V(p, I)$):

$$\mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\mathbb{\Delta}}/T) := \underset{(A,I)\in\mathfrak{X}_{\mathbb{\Delta}}/T}{2\text{-}\lim} \mathbf{Vect}(U(A, I)).$$

We denote an object of $\mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\mathbb{\Delta}}/T)$ as $\mathcal{V} = (\mathcal{V}_{(A,I)})$. By the argument given in [GR22, Proposition 3.7], the following restriction is fully faithful:

$$\mathbf{Vect}(\mathfrak{X}_{\mathbb{\Delta}}/T, \mathcal{O}_{\mathbb{\Delta}}) \xrightarrow{\sim} \underset{(A,I)\in\mathfrak{X}_{\mathbb{\Delta}}/T}{2\text{-}\lim} \mathbf{Vect}(A) \to \underset{(A,I)\in\mathfrak{X}_{\mathbb{\Delta}}/T}{2\text{-}\lim} \mathbf{Vect}(U(A, I)) = \mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\mathbb{\Delta}}/T).$$

Endow $\mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\mathbb{\Delta}}/T)$ with the structure of a $\mathbb{Z}_p$-linear $\otimes$-category defined term-by-term in the two-limit then the above restriction is a $\mathbb{Z}_p$-linear $\otimes$-functor. Denote the object $(\mathcal{O}_{U(A,I)})$ by $\mathcal{O}^{\mathrm{an}}_{\mathbb{\Delta}}$.

We say that a sequence of analytic prismatic crystals

$$0 \to \mathcal{V}^1 \to \mathcal{V}^2 \to \mathcal{V}^3 \to 0$$

is *exact* if the sequence of vector bundles

$$0 \to \mathcal{V}^1_{(A,I)} \to \mathcal{V}^2_{(A,I)} \to \mathcal{V}^3_{(A,I)} \to 0$$

on $\mathrm{Spec}(A) - V(p, I)$ is exact for all (A, I) in $\mathfrak{X}_{\mathbb{\Delta}}$.

Proposition 1.19. *A sequence*

$$0 \to \mathcal{V}^1 \to \mathcal{V}^2 \to \mathcal{V}^3 \to 0$$

in $\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\mathbb{\Delta}})$ is exact if and only if there exists a cover $\{(A_i, I_i)\}$ of $*$ such that

$$0 \to \mathcal{V}^1_{(A_i,I_i)} \to \mathcal{V}^2_{(A_i,I_i)} \to \mathcal{V}^3_{(A_i,I_i)} \to 0$$

is exact for all i.

Using Lemma A.1, and the fact that an exact sequence of vector bundles is universally exact (see [SP, Tag 058H]), the desired result is a special case of the following.

Proposition 1.20. *Let $f\colon A \to B$ be a map of rings, and suppose that the finitely generated ideal $J \subseteq A$ is contained in the Jacobson radical of A. If $f\colon A/J^n \to B/J^nB$ is faithfully flat for all n then, a sequence of vector bundles*

$$P\colon \quad 0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0,$$

on $\mathrm{Spec}(A) - V(J)$ *is exact if and only if the sequence*

$$f^*P\colon \quad 0 \to f^*\mathcal{M}_1 \to f^*\mathcal{M}_2 \to f^*\mathcal{M}_3 \to 0,$$

of vector bundles on $\mathrm{Spec}(B) - V(JB)$ *is exact.*

But, the proof of Proposition 1.20 is quickly obtained by combining the following two lemmas, and the fact that any point of $\mathrm{Spec}(B) - V(JB)$ contains a closed point in its closure (since J is finitely generated, so $\mathrm{Spec}(B) - V(JB)$ is quasi-compact).

Lemma 1.21. *Let $(Y, \mathcal{O}_Y)$ be a locally ringed space and S a subset of Y such that every point of Y admits an element of S as a specialization. Then, a sequence*

$$Q\colon \quad 0 \to V_1 \to V_2 \to V_3 \to 0,$$

of $\mathcal{O}_Y$-modules, where each V_i is a vector bundle on $(Y, \mathcal{O}_Y)$, is exact if and only if for all y in S the induced sequence

$$0 \to (V_1)_{k(y)} \to (V_2)_{k(y)} \to (V_3)_{k(y)} \to 0,$$

of vector spaces over the residue field $k(y)$ is exact. In particular, if $f\colon (Y', \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ is a map of locally ringed spaces containing S in its image, then Q is exact if and only if

$$f^*Q\colon \quad 0 \to f^*V_1 \to f^*V_2 \to f^*V_3 \to 0,$$

is exact.

Proof. The second claim follows easily from the first, and only the if condition of the first statement requires proof. To prove this, it suffices to show that for each y in S, the sequence of projective $\mathcal{O}_{Y,y}$-modules

$$Q_y\colon \quad 0 \to (V_1)_y \to (V_2)_y \to (V_3)_y \to 0,$$

is exact. Indeed, if y' is an arbitrary point of Y and y in S is a specialization, then the sequence $Q_{y'}$ is obtained by base changing the sequence Q_y along the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y'}$, from where the claim follows as $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y'}$ is a localization, and thus exact.

So, to show that the sequence Q_y is exact, let n_i, for $i = 1, 2, 3$, be the rank of the finite free $\mathcal{O}_{Y,y}$-module $(V_i)_y$. That $(V_2)_y \to (V_3)_y$ is surjective follows from Nakayama's lemma. But, we then see that the sequence

$$0 \to \ker((V_2)_y \to (V_3)_y) \to (V_2)_y \to (V_3)_y \to 0 \tag{1.2.1}$$

is exact. As V_3 is a vector bundle, this implies that this sequence splits, and thus $\ker((V_2)_y \to (V_3)_y)$ is a vector bundle over $\mathcal{O}_{Y,y}$ of rank equal to $n_2 - n_3$. Observe that as $(V_1)_{k(y)}$ has rank equal to n_1 but, by exactness over $k(y)$, has rank $n_2 - n_3$. Thus, $(V_1)_y \to \ker((V_2)_y \to (V_3)_y)$ is a map of finite free $\mathcal{O}_{Y,y}$-modules of the same rank. Thus, it is an isomorphism if and only if it is surjective.[10] But, by Nakayama's lemma, it suffices to check this claim after base change to $k(y)$. But, as (1.2.1) is exact, and $(V_3)_y$ flat, this is equvivalent to checking that $(V_1)_{k(y)} \to \ker((V_2)_{k(y)} \to (V_3)_{k(y)})$ is surjective, which is true by assumption. $\square$

Lemma 1.22. *Let $f\colon A \to B$ be a map of rings, and suppose that the finitely generated ideal $J \subseteq A$ is contained in the Jacobson radical. If $f\colon A/J^n \to B/J^nB$ is faithfully flat for all n, then the map* $\mathrm{Spec}(B) - V(JB) \to \mathrm{Spec}(A) - V(J)$ *is surjective on closed points.*

[10] See [Orz71] for a generalization of this fact.

Proof. Let $\mathfrak{p}$ be a prime of A constituting a closed point of $\mathrm{Spec}(A) - V(J)$. We aim to show that $\mathfrak{p}$ is in the image of f. Suppose first that $\mathfrak{p} = (0)$. In this case we see that it suffices to show that $\mathrm{Spec}(B) - V(JB)$ is non-empty. If it were empty, then JB would be contained in the nilradical of B and as J is finitely generated, this implies that $J^nB = 0$ for some n. This implies that $J^nB/J^{n+1}B$ is zero, and as $A/J^{n+1} \to B/J^{n+1}B$ is faithfully flat, this implies that J^n/J^{n+1} is zero. Since J is contained in the Jacobson radical of A, we deduce from Nakayama's lemma (see [SP, Tag 00DV]) that J^n is zero, but this implies that $\mathrm{Spec}(A) - V(J)$ is empty, which is a contradiction.

In general, let $\overline{A} := A/\mathfrak{p}$, $\overline{B} := B/\mathfrak{p}B$, and $\overline{J} := J\overline{A}$. Then, $\overline{J}$ is in the Jacobson radical of $\overline{A}$, and $\overline{A}/\overline{J}^n \to \overline{B}/\overline{J}^n\overline{B}$ is equal to the base change of $A/J^n \to B/J^nB$ along $A \to \overline{A}$, and so faithfully flat. In particular, from the argument in the previous paragraph there exists some $\overline{\mathfrak{q}}$ in $\mathrm{Spec}(\overline{B}) - V(\overline{JB})$. Let $\mathfrak{q}$ be an prime of B lying over $\overline{\mathfrak{q}}$. Observe that $\mathfrak{q}$ belongs to $\mathrm{Spec}(B) - V(JB)$, and by construction $f(\mathfrak{q})$ lies in $V(\mathfrak{p}) \cap (\mathrm{Spec}(A) - V(J))$. But, as $\mathfrak{p}$ is closed in $\mathrm{Spec}(A) - V(J)$, one checks that $V(\mathfrak{p}) \cap (\mathrm{Spec}(A) - V(J)) = \{\mathfrak{p}\}$ from where the claim follows. $\square$

While not strictly necessary, we include the following beautiful observation of Ofer Gabber showing that the reduction to closed points in the above arguments was not strictly necessary in the case when the rings involved are complete.

Proposition 1.23 (Gabber)**.** *Let $J \subseteq A$ be a finitely generated ideal, and $f\colon A \to B$ a ring map. Suppose that A (resp. B) is complete with respect to J (resp. JB), and that $A/J^n \to B/J^nB$ is faithfully flat for all n. Then, the map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective.*

Proof. For an ideal I of A, let us denote by I^{ec} the ideal $f^{-1}(IB)$, and similarly for the ring maps $f_n\colon A/J^n \to B/J^nB$. Observe that by assumption that each f_n is faithfully flat, we have that $I^{\mathrm{ec}} = I$ for any ideal $I \subseteq A/J^n$. Thus, if I is an ideal of A closed in the J-adic topology, then by passing to the limit we see that $I^{\mathrm{ec}} = I$.

Claim 1: For any ideal finitely generated ideal I of A, we have that $\overline{I}^2 \subseteq I$, where $\overline{I}$ here denotes the closure of I in A.

Proof. More generally we show that for finitely generated ideals I_1 and I_2 of A, the product of their closures is contained in $I_1 + I_2$. If x is in the closure of I_1 we can write $x = \sum_n x_n$ with x_n in $I_1 \cap J^n$. Similarly if y is in the closure of I_2 then $y = \sum_n y_n$, with y_n in $I_2 \cap J^n$. Thus,

$$xy = \sum_{n \leqslant m} x_n y_m + \sum_{n > m} x_n y_m. \tag{1.2.2}$$

Let f_k be a finite system of generators of I_1 and write $x_n = \sum_k x_{nk} f_k$. Then the first term on the right-hand side of (1.2.2) is $\sum_k (\sum_m (\sum_{n \leqslant m} x_{nk}) y_m) f_k$, which is in I_1, and similarly the second term on the right-hand side of (1.2.2) is in I_2. $\square$

Claim 2: For every ideal I of A, one has $(I^{\mathrm{ec}})^2 \subseteq I$.

Proof. One quickly reduces to the case when I is finitely generated, in which case we observe that

$$(I^{\mathrm{ec}})^2 \subseteq (\overline{I}^{\mathrm{ec}})^2 = \overline{I}^2 \subseteq I,$$

the second equality by our initial observations, and the second containment by Claim 1. $\square$

So, if $\mathfrak{p}$ is a prime ideal of A, then from Claim 2 we have that $(\mathfrak{p}^{\mathrm{ec}})^2 \subseteq \mathfrak{p}$. As $\mathfrak{p}$ is radical this implies that $\mathfrak{p}^{\mathrm{ec}} \subseteq \mathfrak{p}$ and thus $\mathfrak{p}^{\mathrm{ec}} = \mathfrak{p}$, and thus $\mathfrak{p}$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Indeed, it suffices to show that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is non-zero, but $A_{\mathfrak{p}}/\mathfrak{p} = A_{\mathfrak{p}}/f^{-1}(\mathfrak{p}B)A_{\mathfrak{p}}$ is non-zero and embeds into this ring via f. $\square$

We end by observing a simple criterion to check when an analytic prismatic (F-)crystal comes from a prismatic (F-)crystal when $\mathfrak{X}$ is a base formal $\mathcal{O}_K$-scheme.

Proposition 1.24. *Let $\mathfrak{X}$ be a base formal $\mathcal{O}_K$-scheme and let $\{\mathrm{Spf}(R_i)\}$ be an open cover of $\mathfrak{X}$ where R_i is a (formally framed) base $\mathcal{O}_K$-algebra. Then, the essential image of*

$$\mathbf{Vect}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\Delta}), \qquad \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}),$$

consist of those $\mathcal{W}$ and those $(\mathcal{V}, \varphi_{\mathcal{V}})$, such that $(j_{(\mathfrak{S}_{R_i},(E))})_*\mathcal{W}_{(\mathfrak{S}_{R_i},(E))}$ and $(j_{(\mathfrak{S}_{R_i},(E))})_*\mathcal{V}_{(\mathfrak{S}_{R_i},(E))}$ is a vector bundle on $\mathfrak{S}_{R_i}$ for all i, respectively.

Proof. It suffices to prove the first claim. Moreover, we are clearly reduced to the case when $\mathfrak{X} = \mathrm{Spf}(R)$. For $i = 1, 2, 3$, let $\mathfrak{S}_R^{(i)}$ be the i-fold self-product of $\mathfrak{S}_R$ in the topos $\mathbf{Shv}(\mathfrak{X}_{\Delta})$ (see [DLMS22, Example 3.4]), and let $p_k^{(i)}$ for $k = 1, \ldots, i$ denote the projection maps, which are flat by [DLMS22, Lemma 3.5]. Let $j^{(i)}$, for $i = 1, 2, 3$ denote the inclusion of $U(\mathfrak{S}_R^{(i)}, (E))$ into $\mathrm{Spec}(\mathfrak{S}_R^{(i)})$. Observe that we have the 2-commutative diagram of categories.

$$\begin{tikzcd}
\mathbf{Vect}(\mathfrak{X}_{\Delta}) \arrow[r] \arrow[d] & \mathbf{Vect}(\mathfrak{S}_R) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(j^{(1)})^*"] & \mathbf{Vect}(\mathfrak{S}_R^{(2)}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "(j^{(2)})^*"] & \mathbf{Vect}(\mathfrak{S}_R^{(3)}) \arrow[d, "(j^{(3)})^*"] \\
\mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\Delta}) \arrow[r] & \mathbf{Vect}(U(\mathfrak{S}_R, (E))) \arrow[r, shift left] \arrow[r, shift right] & \mathbf{Vect}(U(\mathfrak{S}_R^{(2)}, (E))) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{Vect}(U(\mathfrak{S}_R^{(3)}, (E))).
\end{tikzcd}$$

The top row is obviously exact, and the bottom row is exact by [Mat22] (cf. [BS23, Theorem 2.2]). Observe that, by inspection, $(p, (E))$ has height 2 in $\mathfrak{S}_R$, and thus by the flatness of $p_k^{(i)}$, the same holds for $(p, (E))$ in $\mathfrak{S}^{(i)}$ for $i = 1, 2, 3$. Thus, we observe that vertical maps are faithfully flat by Proposition A.21. Suppose now that $j_*^{(1)}\mathcal{W}_{(\mathfrak{S}_R,(E))}$ is a vector bundle. Let us observe that

$$(j^{(i)})^*(p_k^{(i)})^* j_*^{(1)}\mathcal{W}_{(\mathfrak{S}_R,(E))} \cong (p_k^{(i)})^*\mathcal{W}_{(\mathfrak{S}_R,(E))},$$

for each $i = 1, 2, 3$ and $k = 1, \ldots, i$, again by Proposition A.21. Thus, by the fully faithfulness of the vertical arrows $(j^{(i)})^*$, we may pullback the descent data on $\mathcal{W}_{(\mathfrak{S}_R,(E))}$ relative to $\{p_k^{(2)} \colon U(\mathfrak{S}_R^{(2)}, (E)) \to U(\mathfrak{S}_R, (E))\}_{k=1,2}$ to descent data on $j_*^{(1)}\mathcal{W}_{(\mathfrak{S}_R,(E))}$ relative to $\{p_k^{(2)} \colon \mathrm{Spec}(\mathfrak{S}_R^{(2)}) \to \mathrm{Spec}(\mathfrak{S}_R)\}_{k=1,2}$. This gives an object $\mathcal{F}$ of $\mathbf{Vect}(\mathfrak{X}_{\Delta})$ whose image in $\mathbf{Vect}^{\mathrm{an}}(\mathfrak{X}_{\Delta})$ is isomorphic to $\mathcal{W}$.

Thus, we have shown that anything satisfying this condition on the pushforward is in the essential image. Conversely, if $\mathcal{W}$ is in the essential image, then the fact that the desired pushforward condition holds follows easily again by applying Proposition A.21. □

Remark 1.25. A more succinct (but potentially opaque) way of phrasing the argument for Proposition 1.24 is the following, where we use the obvious extension of the notation from the proof of Proposition 1.24. Consider the topos theoretic Čech nerve, $\mathfrak{S}_R^{(\bullet)}$, then we have that $\mathbf{Vect}(\mathfrak{X}_{\Delta}) = 2\text{-}\lim \mathbf{Vect}(\mathfrak{S}_R^{(\bullet)})$. If $j_*^{(1)}\mathcal{W}_{(\mathfrak{S}_R,(E))}$ is a vector bundle, then the object $j_*\mathcal{W}$ of $\mathbf{Vect}(\mathfrak{X}_{\Delta})$ may be defined to be the object $(j_*^{(i)}\mathcal{W}_{(\mathfrak{S}_R^{(i)},(E))})$ of $2\text{-}\lim \mathbf{Vect}(\mathfrak{S}_R^{(\bullet)})$. Implicitly here we are claiming that the functors $j_*^{(i)}$ and $(p_k^{(i)})^*$ commute, which follows by flat base change (see [SP, Tag 02KH]) as each $p_k^{(i)}$ is flat.

That said, we observe that we also have that $\mathbf{Vect}(\mathfrak{X}_{\Delta}) = 2\text{-}\lim_{(A,I)\in\mathfrak{X}_{\Delta}} \mathbf{Vect}(A)$, but the system $((j_{(A,I)})_*\mathcal{W}_{(A,I)})$ is not an object of this two-limit category, as the terms will not necessarily satisfy compatibility in (A, I), as the transition maps $(A, I) \to (B, J)$ need not be flat or, more importantly, need not satisfy the above-mentioned pushforward-pullback commtuativity. In particular, we observe that $(j_*\mathcal{W})_{(A,I)}$ does not necessarily equal $(j_{(A,I)})_*\mathcal{W}_{(A,I)}$.

1.2.3. *Tannakian theory.* Denote by $\mathfrak{X}_{\Delta,\text{ét}}$ the subcategory of $\mathfrak{X}_{\Delta}$ with the same objects as $\mathfrak{X}_{\Delta}$ but with only those morphisms $(A, I) \to (B, J)$ with $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ adically étale, with the induced topology. By Proposition 1.3 (and [SP, Tag 00X5]), the functor $\mathfrak{X}_{\Delta,\text{ét}} \to \mathfrak{X}_{\Delta}$ induces a morphism of sites, and there are equivalences of sites $\mathfrak{X}_{\Delta,\text{ét}}/(A, I) \xrightarrow{\sim} \mathrm{Spf}(A)_{\text{ét}}$.

We shorten the notation $\mathcal{G}_{\mathcal{O}_{\Delta}}$ from §A.5 to $\mathcal{G}_{\Delta}$. We further abbreviate $\mathbf{Tors}_{\mathcal{G}_{\Delta}}(\mathfrak{X}_{\Delta}/T)$ to $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$, and likewise for similar categories. By combining Proposition 1.16 and Theorem A.18, we see that $\mathcal{G}_{\Delta}$ is reconstructable in $(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$, and every object of $\mathcal{G}$-$\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$ is locally trivial.

Proposition 1.26. *The following restriction functors are equivalences of categories*

$$\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T) \to \mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta,\text{ét}}/T) \to \underset{(A,I)\in\mathfrak{X}_{\Delta}/T}{2\text{-lim}} \mathbf{Tors}_{\mathcal{G}}(A).$$

Proof. Fix an object (A, I) of $\mathfrak{X}_{\Delta}$ and set $\mathfrak{X}'_{\Delta}/(A, I)$ to be the full subcategory of $\mathfrak{X}_{\Delta}/(A, I)$ consisting of covers $(A, I) \to (B, J)$, with the induced topology. We use this notation in a similar way for other sites. The naturally defined functor $\mu_1\colon \mathfrak{X}'_{\Delta}/(A, I) \to \mathrm{Spf}(A)'_{\mathrm{fl}}$ is continuous functor and has exact pullback by [SP, Tag 00X6] and [DLMS22, Lemma 3.3], and we have the equality $\mathcal{O}_{\mathfrak{X}_{\Delta}/(A,I)} = \mu_{1*}(\mathcal{O}_{\mathrm{Spf}(A)})$. The same properties hold mutatis mutandis for $\mu_3\colon \mathfrak{X}'_{\Delta,\text{ét}} \to \mathrm{Spf}(A)'_{\text{ét}}$. From the following commutative diagram

$$\begin{tikzcd}
\mathfrak{X}'_{\Delta}/(A,I) \arrow[r, "\mu_1"] & \mathrm{Spf}(A)'_{\mathrm{fl}} \arrow[r, "\mu_2"] & \mathrm{Spf}(A)_{\mathrm{fl}} \\
\mathfrak{X}'_{\Delta,\text{ét}}/(A,I) \arrow[u, "\nu_1"] \arrow[r, "\mu_3"'] & \mathrm{Spf}(A)'_{\text{ét}} \arrow[u, "\nu_2"] \arrow[r, "\mu_4"'] & \mathrm{Spf}(A)_{\text{ét}}, \arrow[u, "\nu_3"]
\end{tikzcd}$$

(where all other functors are the natural inclusions), we obtain the diagram

$$\begin{tikzcd}
\mathbf{Tors}_{\mathcal{G}_{\Delta}}(\mathfrak{X}'_{\Delta}/(A,I)) \arrow[r, "\mu_1^*"] & \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(A)'_{\mathrm{fl}}) \arrow[r, "\mu_2^*"] & \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(A)_{\mathrm{fl}}) \\
\mathbf{Tors}_{\mathcal{G}_{\Delta}}(\mathfrak{X}'_{\Delta,\text{ét}}/(A,I)) \arrow[u, "\nu_1^*"] \arrow[r, "\mu_3^*"'] & \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(A)'_{\text{ét}}) \arrow[u, "\nu_2^*"] \arrow[r, "\mu_4^*"'] & \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(A)_{\text{ét}}). \arrow[u, "\nu_3^*"]
\end{tikzcd}$$

Note that μ_4^* and μ_2^* are evidently equivalences, ν_2^* and ν_3^* are equivalences by Corollary A.11, and μ_3^* is an equivalence by above discussion. By a diagram chase we see that μ_1^* is essentially surjective and so it is an equivalence by Corollary A.6. Thus, ν_1^* is an equivalence. We deduce that for any object of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$, its restriction to (A, I) can be trivialized on an étale cover, and thus the restriction $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T) \to \mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta,\text{ét}}/T)$ is an equivalence by Corollary A.6.

To prove that $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta,\text{ét}}/T) \to 2\text{-lim}\,\mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(A)_{\text{ét}})$ is an equivalence we exhibit a quasi-inverse. For an object $(\mathcal{A}_{(A,I)})$ of the 2-limit category define the object $\mathcal{A}$ of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta,\text{ét}})$ to be the one sending (A, I) to $\mathcal{A}_{(A,I)}(A)$. This is a presheaf carrying an action of $\mathcal{G}_{\Delta}$ and for which the action on (A, I)-points is simply transitive whenever non-empty. Thus, we are done as it is simple to see that this presheaf is locally isomorphic to $\mathcal{G}_{\Delta}$. □

Set $\phi\colon \mathcal{G}_{\Delta} \to \mathcal{G}_{\Delta}$ to be the morphism of group sheaves associating to every (A, I) the map $\mathcal{G}(A) \to \mathcal{G}(A)$ obtained from $\phi_A\colon A \to A$. For an object $\mathcal{A}$ of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$, denote by $\phi^*\mathcal{A}$ the $\mathcal{G}_{\Delta}$-torsor $\phi_*\mathcal{A}$, in the notation of §A.1. Denote by $\mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]$ the group sheaf $\mathcal{G}_{\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]}$ and let $\iota\colon \mathcal{G}_{\Delta} \to \mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]$ be the obvious monomorphism. Finally, set $\mathcal{A}[1/\mathcal{I}_{\Delta}]$ to be $\iota_*\mathcal{A}$.

Definition 1.27. The category $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$ of *prismatic $\mathcal{G}$-torsors with F-structure* over T has objects $(\mathcal{A}, \varphi_{\mathcal{A}})$ where $\mathcal{A}$ is an object of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$ and φ is a *Frobenius isomorphism*

$$\varphi_{\mathcal{A}}\colon (\phi^*\mathcal{A})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{A}[1/\mathcal{I}_{\Delta}],$$

and morphisms are morphisms in $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$ commuting with the Frobenii.

The objects of $\mathcal{G}$-$\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}/T)$ may be written as pairs (ω, φ_ω) where ω is an object of $\mathcal{G}$-$\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$ and $\varphi_\omega\colon (\phi^*\omega)[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \omega[1/\mathcal{I}_{\Delta}]$ is an isomorphism in $\mathcal{G}$-$\mathbf{Vect}^{\mathrm{lt}}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}])$. For an object $\mathcal{A}$ in $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$ there is a natural identification between $\phi^*\omega_{\mathcal{A}}$ and $\omega_{\phi^*\mathcal{A}}$ as objects of $\mathcal{G}$-$\mathbf{Vect}(\mathfrak{X}_{\Delta}/T, \mathcal{O}_{\Delta})$. So, if $(\mathcal{A}, \varphi_{\mathcal{A}})$ is an object of $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta}/T)$ then $\omega_{\mathcal{A}}$ has a natural Frobenius $\varphi_{\omega_{\mathcal{A}}}$, and thus $(\omega_{\mathcal{A}}, \varphi_{\omega_{\mathcal{A}}})$ defines an object of $\mathcal{G}$-$\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}/T)$.

Define $\mathbf{Twist}^{\varphi}_{\mathcal{O}_{\mathbbl{\Delta}}|_T}(\Lambda_0, \mathbb{T}_0)$ to be the groupoid consisting of pairs $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T})$ where $(\mathcal{E}, \varphi_{\mathcal{E}})$ is an object of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbbl{\Delta}}/T)$, the pair $(\mathcal{E}, \mathbb{T})$ is an object of $\mathbf{Twist}_{\mathcal{O}_{\mathbbl{\Delta}}|_T}(\Lambda_0, \mathbb{T}_0)$, and $\mathbb{T}$ constitutes a set of tensors on $(\mathcal{E}, \varphi_{\mathcal{E}})$ or, equivalently, each tensor in $\mathbb{T}$ is *Frobenius invariant*: fixed under the composition $\mathcal{E} \to \phi^*\mathcal{E} \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$, where the first map sends x to $x \otimes 1$.

Observe that there is a natural identification

$$\phi^*\underline{\mathrm{Isom}}(\mathcal{O}^n_{\mathbbl{\Delta}}, \mathcal{E}) \xrightarrow{\sim} \underline{\mathrm{Isom}}(\mathcal{O}^n_{\mathbbl{\Delta}}, \phi^*\mathcal{E}),$$

for an object $\mathcal{E}$ of $\mathbf{Vect}(\mathfrak{X}_{\mathbbl{\Delta}}/T, \mathcal{O}_{\mathbbl{\Delta}})$. On the other hand, there is an identification between $\underline{\mathrm{Isom}}(\mathcal{O}^n_{\mathbbl{\Delta}}, \mathcal{E})[1/\mathcal{I}_{\mathbbl{\Delta}}]$ and $\underline{\mathrm{Isom}}(\mathcal{O}_{\mathbbl{\Delta}}[1/\mathcal{I}_{\mathbbl{\Delta}}]^n, \mathcal{E}[1/\mathcal{I}_{\mathbbl{\Delta}}])$. These observations show that if $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T})$ is an object of $\mathbf{Twist}^{\varphi}_{\mathcal{O}_{\mathbbl{\Delta}}|_T}(\Lambda_0, \mathbb{T}_0)$, then the $\mathcal{G}_{\mathbbl{\Delta}}$-torsor $\underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbbl{\Delta}}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}))$ carries a natural Frobenius isomorphism denoted also by $\varphi_{\mathcal{E}}$.

Combining Proposition 1.26 with results from Appendix A (notably Proposition A.17 applied to both $\mathcal{G}_{\mathbbl{\Delta}}$ and $\mathcal{G}_{\mathbbl{\Delta}}[1/\mathcal{I}_{\mathbbl{\Delta}}]$), we deduce the following, where $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(A, I)$ has the obvious meaning.

Proposition 1.28. *The natural functor*

$$\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\mathbbl{\Delta}}/T) \to \underset{(A,I) \in \mathfrak{X}_{\mathbbl{\Delta}}/T}{2\text{-}\lim} \mathbf{Tors}^{\varphi}_{\mathcal{G}}(A, I),$$

is an equivalence. Moreover, we have a commuting triangle of equivalences

$$\begin{tikzcd}
\mathbf{Twist}^{\varphi}_{\mathcal{O}_{\mathbbl{\Delta}}|_T}(\Lambda_0, \mathbb{T}_0) \arrow[rr, "((\mathcal{E},\varphi_{\mathcal{E}}),\mathbb{T}) \mapsto (\underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbbl{\Delta}}, \mathbb{T}_0 \otimes 1),(\mathcal{E},\mathbb{T})),\varphi_{\mathcal{E}})"] & & \mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\mathbbl{\Delta}}/T) \arrow[d, "(\mathcal{A},\varphi_{\mathcal{A}}) \mapsto (\omega_{\mathcal{A}},\varphi_{\omega_{\mathcal{A}}})"] \\
& & \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbbl{\Delta}}/T). \arrow[ull, "(\omega,\varphi_{\omega}) \mapsto ((\omega(\Lambda_0),\varphi_{\omega}),\omega(\mathbb{T}_0))"]
\end{tikzcd}$$

By Theorem A.18, and some of the ideas applied in the proof of Proposition 1.28, one has

$$\mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\mathbbl{\Delta}}/T) = \underset{(A,I) \in \mathfrak{X}_{\mathbbl{\Delta}}/T}{2\text{-}\lim} \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(U(A, I)) = \underset{(A,I) \in \mathfrak{X}_{\mathbbl{\Delta}}/T}{2\text{-}\lim} \mathbf{Tors}^{\varphi}_{\mathcal{G}}(U(A, I)),$$

and the exact restriction map

$$\mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbbl{\Delta}}/T) \xrightarrow{\sim} \underset{(A,I) \in \mathfrak{X}_{\mathbbl{\Delta}}}{2\text{-}\lim} \mathbf{Tors}^{\varphi}_{\mathcal{G}}(A) \to \mathbf{Tors}^{\varphi,\mathrm{an}}_{\mathcal{G}}(\mathfrak{X}_{\mathbbl{\Delta}}) := \underset{(A,I) \in \mathfrak{X}_{\mathbbl{\Delta}}}{2\text{-}\lim} \mathbf{Tors}^{\varphi}_{\mathcal{G}}(U(A, I)) \xrightarrow{\sim} \mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\mathbbl{\Delta}}/T)$$

is fully faithful by [GR22, Proposition 3.7], where $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(U(A, I))$ has the obvious meaning.

1.3. Quasi-syntomic torsors with F-structure.

For the sake of completeness, we compare the material above to the analogous theory in the quasi-syntomic setting.

Abbreviate $\mathcal{G}_{\mathcal{O}^{\mathrm{pris}}}$ to $\mathcal{G}^{\mathrm{pris}}$ and $\mathcal{G}_{\mathcal{O}^{\mathrm{pris}}[1/\mathcal{I}^{\mathrm{pris}}]}$ to $\mathcal{G}^{\mathrm{pris}}[1/\mathcal{I}^{\mathrm{pris}}]$. Write $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}})$ instead of $\mathbf{Tors}_{\mathcal{G}^{\mathrm{pris}}}(\mathfrak{X}_{\mathrm{qsyn}})$. We make similar notational conventions concerning $(\mathfrak{X}_{\mathrm{QSYN}}, \mathcal{O}^{\mathrm{PRIS}})$.

Proposition 1.29. *The following are well-defined equivalences of categories functorial in $\mathcal{G}$:*

$$\begin{tikzcd}
& \mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathbbl{\Delta}}) \arrow[dl, "u_*"'] \arrow[d, "v_*"] \arrow[dr, "\mathrm{eval.}"] & \\
\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{QSYN}}) \arrow[r, "\mathrm{res.}"] & \mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}}) \arrow[r, "\mathrm{eval.}"] & \underset{\mathrm{Spf}(R) \in \mathfrak{X}_{\mathrm{qrsp}}}{2\text{-}\lim} \mathbf{Tors}_{\mathcal{G}}(\mathbbl{\Delta}_R).
\end{tikzcd}$$

Proof. It suffices to show that u_*, res., and eval.: $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}}) \to 2\text{-}\lim \mathbf{Tors}_{\mathcal{G}}(\mathbbl{\Delta}_R)$ are equivalences. To prove that eval. is an equivalence, take an open cover $\{\mathrm{Spf}(R_i)\}$ of $\mathfrak{X}$ and a qrsp cover $R_i \to S_i$. Let $S_i^{\bullet}$ denote the objects of the Čech nerve of $R_i \to S_i$, which consist of qrsp rings by [BMS19, Lemma 4.30]. Then, by descent we have that the natural evaluation functor $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}}) \to 2\text{-}\lim \mathbf{Tors}_{\mathcal{G}}(\mathbbl{\Delta}_{S_i^{\bullet}})$ is an equivalence, where we have implicitly used [ALB23, Proposition 3.30]. The fact that eval. is an equivalence easily follows.

Set $\mathfrak{X}'_{\mathrm{qsyn}}$ to have the same objects as $\mathfrak{X}_{\mathrm{qsyn}}$, but whose morphisms are required to be quasi-syntomic. From [BMS19, Lemma 4.16] and [SP, Tag 00X6], the inclusions $\mathfrak{X}'_{\mathrm{qsyn}} \to \mathfrak{X}_{\mathrm{qsyn}}$

and $\mathfrak{X}'_{\mathrm{qsyn}} \to \mathfrak{X}_{\mathrm{QSYN}}$ induce morphisms of sites. As $\mathfrak{X}'_{\mathrm{qsyn}}$ has the same set of covers of $\mathfrak{X}$ as $\mathfrak{X}_{\mathrm{QSYN}}$, from Corollary A.6 we deduce that the functors $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{QSYN}}) \to \mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}'_{\mathrm{qsyn}})$ and $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}}) \to \mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}'_{\mathrm{qsyn}})$ are equivalences, and so is their composition res.

By Corollary A.7 to show that u_* is an equivalence, it suffices to show that for an object $\mathcal{A}$ of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$, that $u_*(\mathcal{A})$ is locally non-empty. Passing to a qrsp cover $\{\mathrm{Spf}(R_i) \to \mathfrak{X}\}$ we may assume that $\mathfrak{X} = \mathrm{Spf}(R)$ where R is qrsp. By Proposition 1.26, $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\Delta}) = \mathbf{Tors}_{\mathcal{G}}(\Delta_R)$. Pulling back $\mathcal{A}$ along the closed immersion $i\colon \mathrm{Spec}(R) \to \mathrm{Spec}(\Delta_R)$ gives a $\mathcal{G}$-torsor $i^*\mathcal{A}$ on $\mathrm{Spf}(R)_{\text{ét}}$. Let $R \to S$ be a p-adically étale morphism such that $(i^*\mathcal{A})(S)$ is non-empty. By Lemma 1.5 we see S is qrsp, and so $(u_*\mathcal{A})(S) = \mathcal{A}(\Delta_S, I_S)$. But, the pair $(\Delta_S, \ker(\Delta_S \to S))$ is Henselian by [ALB23, Lemma 4.28], and so $\mathcal{A}(\Delta_S, I_S)$ is non-empty by [BČ22, Theorem 2.1.6]. $\square$

The proof of the following is obtained mutatis mutandis from the proof of Proposition 1.29.

Proposition 1.30 (cf. [ALB23, Proposition 4.4] and [BS23, Proposition 2.13 and Proposition 2.14])**.** *The following are well-defined rank-preserving bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalences:*

$$\begin{tikzcd}
 & \mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}) \arrow[dl, "u_*"'] \arrow[d, "v_*"] \arrow[dr, "\text{eval.}"] & \\
\mathbf{Vect}(\mathfrak{X}_{\mathrm{QSYN}}, \mathcal{O}^{\mathrm{PRIS}}) \arrow[r, "\text{res.}"] & \mathbf{Vect}(\mathfrak{X}_{\mathrm{qsyn}}, \mathcal{O}^{\mathrm{pris}}) \arrow[r, "\text{eval.}"] & \underset{\mathrm{Spf}(R)\in\mathfrak{X}_{\mathrm{qrsp}}}{2\text{-}\lim} \mathbf{Vect}(\Delta_R).
\end{tikzcd}$$

Set $\phi = v_*(\phi)$ on $\mathcal{G}^{\mathrm{pris}} = v_*(\mathcal{G}_{\Delta})$. For an object $\mathcal{B}$ of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}})$, define $\mathcal{B}[1/\mathcal{J}^{\mathrm{pris}}]$ as the pushforward along $\mathcal{G}^{\mathrm{pris}} \to \mathcal{G}^{\mathrm{pris}}[1/\mathcal{J}^{\mathrm{pris}}]$, and $\phi^*\mathcal{B} := \phi_*\mathcal{B}$. There are identifications $v_*\phi^*\mathcal{A} = \phi^* v_*\mathcal{A}$ and $v_*(\mathcal{A}[1/\mathcal{I}_{\Delta}]) = (v_*\mathcal{A})[1/\mathcal{J}^{\mathrm{pris}}]$. Define a *quasi-syntomic $\mathcal{G}$-torsor with F-structure* to be a pair $(\mathcal{B}, \varphi)$ where $\mathcal{B}$ is an object of $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}})$ and φ a *Frobenius isomorphism*

$$\varphi\colon (\phi^*\mathcal{B})[1/\mathcal{J}^{\mathrm{pris}}] \xrightarrow{\sim} \mathcal{B}[1/\mathcal{J}^{\mathrm{pris}}].$$

A morphism of quasi-syntomic $\mathcal{G}$-torsors with F-structure is a morphism of $\mathcal{G}^{\mathrm{pris}}$-torsors commuting with the Frobenii. Denote the category of such objects by $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}})$. One can similarly define the categories $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{qsyn}})$ and $\mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{qsyn}})$, as well as their analogues for $\mathfrak{X}_{\mathrm{QSYN}}$ or $\mathfrak{X}_{\mathrm{qrsp}}$.

Using Proposition 1.28, Proposition 1.29, and Proposition 1.30 one may deduce the following.

Proposition 1.31. *There is a commuting diagram of well-defined equivalences*

$$\begin{tikzcd}
\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta}) \arrow[r] \arrow[d, "v_*"'] & \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \arrow[d, "v_*"] \\
\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\mathrm{qsyn}}) \arrow[r] & \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{qsyn}})
\end{tikzcd}$$

which is functorial in $\mathcal{G}$. Similar assertions hold with $\mathfrak{X}_{\mathrm{qsyn}}$ replaced by $\mathfrak{X}_{\mathrm{QSYN}}$ or $\mathfrak{X}_{\mathrm{qrsp}}$.

2. $\mathcal{G}$-objects in the category of crystalline local systems

In this section we discuss various Tannakian consequences of the results of [BS23], [GR22], and [DLMS22]. Unless stated otherwise, we use the same notation from the beginning of §1.

2.1. The category of $\mathcal{G}(\mathbb{Z}_p)$-local systems.

Both in the statement of the main result of this section, and in various constructions in §4, it will be helpful to have a clear notion of a $\mathcal{G}(\mathbb{Z}_p)$-local system (on a scheme or an adic space). Fix a pro-finite group $H = \varprojlim H_n$, with each H_n finite.

2.1.1. *$\mathcal{G}(\mathbb{Z}_p)$-local systems on a scheme.* For a scheme S, recall from [BS15, Definition 4.1.1] that a morphism of schemes $f\colon S' \to S$ is *weakly étale* if f and Δ_f are flat. Moreover, we can associate to S the *proétale site* $S_{\text{proét}}$ which is the full subcategory of S_{fpqc} consisting of weakly étale morphisms $S' \to S$, with the induced topology.

We will be interested in the ringed site $(S_{\text{proét}}, \underline{\mathbb{Z}_p}_S)$, where

$$\underline{\mathbb{Z}_p}_S := R\varprojlim \underline{\mathbb{Z}/p^n}_S = \varprojlim \underline{\mathbb{Z}/p^n}_S,$$

where the second equality follows from [BS15, Proposition 3.2.3, Proposition 3.1.10, and Proposition 4.2.8]. We often use the notation $\mathbb{Z}/p^\infty := \mathbb{Z}_p$. More generally, for a topological space T, denote by $\underline{T}_S$ (or just $\underline{T}$ when S is clear from context) the sheaf (see [BS15, Lemma 4.2.12])

$$\underline{T}_S \colon S_{\text{proét}} \to \mathbf{Set}, \qquad U \mapsto \mathrm{Hom}_{\text{cont.}}(U, T).$$

If T is totally disconnected, then $\mathrm{Hom}_{\text{cont.}}(U, T)$ is equal to $\mathrm{Hom}_{\text{cont.}}(\pi_0(U), T)$, and so if $T = \varprojlim T_i$ with each T_i finite then $\underline{T} = \varprojlim \underline{T_i}$ where each $\underline{T_i}$ is the constant sheaf.

Lemma 2.1. *With notation as in §A.5, there is an identification $\mathcal{G}_{\underline{\mathbb{Z}/p^n}} \xrightarrow{\sim} \underline{\mathcal{G}(\mathbb{Z}/p^n)}$, compatible in n and functorial in $\mathcal{G}$, for $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. We handle only the case where n is finite as the case for $n = \infty$ follows by passing to the limit. We provide for U in $S_{\text{proét}}$ an isomorphism $\mathcal{G}_{\underline{\mathbb{Z}/p^n}}(U) \xrightarrow{\sim} \underline{\mathcal{G}(\mathbb{Z}/p^n)}(U)$ bi-functorial in U and $\mathcal{G}$ and compatible in n. Let $\{D_i\}$ be the set of discrete quotient spaces of U. Then, for any discrete space X there is an identification between $\mathrm{Hom}_{\text{cont.}}(U, X)$ and $\varinjlim \mathrm{Hom}(D_i, X)$, bi-functorial in U and X. As $\mathcal{G}$ is locally of finite presentation over $\mathbb{Z}_p$

$$\begin{aligned} \mathcal{G}_{\underline{\mathbb{Z}/p^n}}(U) &= \mathcal{G}(\mathrm{Hom}_{\text{cont.}}(U, \mathbb{Z}/p^n)) \\ &= \varinjlim \mathcal{G}(\mathrm{Hom}(D_i, \mathbb{Z}/p^n)) \\ &= \varinjlim \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{Z}_p}}(A_{\mathcal{G}}, \mathrm{Hom}(D_i, \mathbb{Z}/p^n)), \end{aligned}$$

(cf. [SP, Tag 01ZC]), where $A_{\mathcal{G}} := \mathcal{O}_{\mathcal{G}}(\mathcal{G})$. On the other hand we see that

$$\begin{aligned} \underline{\mathcal{G}(\mathbb{Z}_p/p^n\mathbb{Z})}(U) &= \mathrm{Hom}_{\text{cont.}}(U, \mathcal{G}(\mathbb{Z}/p^n)) \\ &= \varinjlim \mathrm{Hom}(D_i, \mathcal{G}(\mathbb{Z}/p^n)) \\ &= \varinjlim \mathrm{Hom}(D_i, \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{Z}_p}}(A_{\mathcal{G}}, \mathbb{Z}/p^n)). \end{aligned}$$

Thus, we are done via the natural isomorphism of groups

$$\mathrm{Hom}(D_i, \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{Z}_p}}(A_{\mathcal{G}}, \mathbb{Z}/p^n)) \to \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{Z}_p}}(A_{\mathcal{G}}, \mathrm{Hom}(D_i, \mathbb{Z}/p^n))$$

bi-functorial in D_i and $\mathcal{G}$, given by currying. □

Denote by $\mathbf{Loc}_{\mathbb{Z}/p^n}(S)$ the category $\mathbf{Vect}(S_{\text{proét}}, \underline{\mathbb{Z}/p^n})$ of $\mathbb{Z}/p^n$-*local systems*. By [BS15, Corollary 5.1.5 and Proposition 6.8.4], this is equivalent to $\mathbf{Loc}_{\mathbb{Z}/p^n}(S_{\text{ét}})$ (see [SGA5, Exposé VI]) as an exact $\mathbb{Z}_p$-linear $\otimes$-category. We refer to objects of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}/p^n}(S)$ as $\mathcal{G}(\mathbb{Z}/p^n)$-*local systems*.

It will also be convenient to consider the category $\mathbf{Loc}_{\mathbb{Q}_p}(S)$ of $\mathbb{Q}_p$-*local systems* on S. This has the same objects as $\mathbf{Loc}_{\mathbb{Z}_p}(S)$, denoted by $\mathbb{L}$ of $\mathbb{L}[1/p]$ when clarity is necessary, but where one defines

$$\mathbf{Hom}_{\mathbf{Loc}_{\mathbb{Q}_p}(S)}(\mathbb{L}[1/p], \mathbb{L}'[1/p]) := \Gamma(S_{\text{proét}}, \mathcal{H}om(\mathbb{L}, \mathbb{L}')[1/p]).$$

The category $\mathbf{Loc}_{\mathbb{Q}_p}(S)$ admits a fully faithful embedding into $\mathbf{Vect}(S_{\text{proét}}, \underline{\mathbb{Q}_p})$, and inherits an exact $\mathbb{Q}_p$-linear $\otimes$-structure from this embedding

Remark 2.2. Beware that, with our definition, there is a fully faithful embedding $\mathbf{Loc}_{\mathbb{Q}_p}(S) \to \mathbf{Vect}(S_{\text{proét}}, \underline{\mathbb{Q}_p})$, with essential image those vector bundles for $\underline{\mathbb{Q}_p}$ which admit a $\underline{\mathbb{Z}_p}$-lattice. This embedding is not generally essentially surjective, the reason being illustrated by the following observation. Suppose that S is connected and $\bar{s}$ is a geometric point. Then, an object $\mathbf{Vect}(S_{\text{proét}}, \underline{\mathbb{Q}_p})$ corresponds to continuous representations $\pi_1^{\text{proét}}(S, \bar{s}) \to \mathrm{GL}_n(\mathbb{Q}_p)$, where $\pi_1^{\text{proét}}(S, \bar{s})$ is the proétale fundamental group as in [BS15, §7], whereas $\mathbf{Loc}_{\mathbb{Q}_p}(S)$ corresponds to those such representations which (up to conjugation) factorize through $\mathrm{GL}_n(\mathbb{Z}_p)$ (see [BS15, Lemma 7.4.7]). But, unlike $\pi_1^{\text{ét}}(S, \bar{s})$, the group $\pi_1^{\text{proét}}(S, \bar{s})$ can be non-compact, and therefore a representation need not (up to conjugation) factorize through $\mathrm{GL}_n(\mathbb{Z}_p)$ (e.g. if S is the projective nodal curve, then $\pi_1^{\text{proét}}(S, \bar{s})$ is isomorphic to $\mathbb{Z}$ with the discrete topology). That said, if S is connected and geometrically unibranch then both agree (cf. [BS15, Lemma 7.4.10]).

Proposition 2.3. *Suppose that S is locally Noetherian. Then, for every $n \in \mathbb{N} \cup \{\infty\}$, the group $\mathcal{G}$ is reconstructable in $(S_{\text{proét}}, \underline{\mathbb{Z}/p^n})$ and every object of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}/p^n}(S)$ is locally trivial.*

Proof. For the first case, we deal only with the case $n = \infty$. Let U in $S_{\text{proét}}$ be arbitrary. One may identify the natural map $\mathcal{G}_{\underline{\mathbb{Z}_p}}(U) \to \underline{\mathrm{Aut}}(\omega_{\text{triv}})(U)$ with the map from $\mathrm{Hom}_{\text{cont.}}(\pi_0(U), \mathcal{G}(\mathbb{Z}_p))$ to the system (g_Λ) of elements of $\mathrm{Hom}_{\text{cont.}}(\pi_0(U), \mathrm{GL}(\Lambda))$ for Λ in $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, which are compatible in Λ. The projection $(g_\Lambda) \mapsto g_{\Lambda_0}$ from $\underline{\mathrm{Aut}}(\omega_{\text{triv}})(U)$ to $\mathrm{Hom}_{\text{cont.}}(\pi_0(U), \mathrm{GL}(\Lambda_0))$ is injective as every object Λ of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ is a subquotient of $\Lambda_0^{\otimes}$ (see [dS09, Proposition 12]). As the diagram

$$\begin{tikzcd} \mathcal{G}_{\underline{\mathbb{Z}_p}}(U) \arrow[r, "a"] \arrow[dr, "c"'] & \underline{\mathrm{Aut}}(\omega_{\text{triv}})(U) \arrow[d, "b"] \\ & \mathrm{Hom}_{\text{cont.}}(\pi_0(U), \mathrm{GL}(\Lambda_0)) \end{tikzcd}$$

commutes, and a and b are injective, to show that a is an isomorphism it suffices to show that $\mathrm{im}(b) \subseteq \mathrm{im}(c)$. But, by functoriality $\mathrm{im}(b)$ fixes $\mathbb{T}$ and thus lies in $\mathrm{im}(c)$.

Let ω be an object of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}_p}(S)$. For $n \in \mathbb{N} \cup \{\infty\}$ consider the sheaf $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n)$, associating to a locally Noetherian S-scheme $f\colon T \to S$ the set of isomorphisms $\omega_{\text{triv}} \to \omega_{n,T}$, where $\omega_{n,T}(\Lambda) := f^{-1}(\omega(\Lambda) \otimes_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Z}/p^n})$ is considered as an object of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}/p^n}(T)$. To prove the second claim it suffices to show that $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_\infty)$ is representable by a weakly étale cover of S. Moreover, as $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_\infty)$ is the limit of $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n)$ (cf. [BS15, Proposition 6.8.4]) it further suffices to show that $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n)$ is represented by a finite étale cover of S for every n in $\mathbb{N}$. To see this, observe that the natural map

$$\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n) \to \underline{\mathrm{Isom}}(\Lambda_0 \otimes_{\mathbb{Z}_p} \underline{\mathbb{Z}/p^n}, \omega_n(\Lambda_0))$$

is a closed embedding, cut out by the intersection of the following conditions (with terminology from [dS09, Definition 10 and Proposition 12]) on an f in $\underline{\mathrm{Isom}}(\Lambda_0 \otimes_{\mathbb{Z}_p} \underline{\mathbb{Z}/p^n}, \omega_n(\Lambda_0))$: for every tuple (a, b, W, U, q), where a, b are multi-indices, W is a special subrepresentation of $(\Lambda_0)^a_b$, and $q\colon W \to U$ is a surjection of representations, f and f^{-1} preserve W and $\ker q$. Thus, $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n) \to S$ is finite. Moreover, by the topological invariance of the pro-étale topos (see [BS15, Lemma 5.4.2]), pullback along $i\colon \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$, for a square-zero ideal I of a Noetherian ring A, defines an equivalence

$$i^{-1}\colon \mathbf{Loc}_{\mathbb{Z}/p^n}(\mathrm{Spec}(A)) \to \mathbf{Loc}_{\mathbb{Z}/p^n}(\mathrm{Spec}(A/I))$$

for $n \in \mathbb{N} \cup \{\infty\}$. From this we deduce that $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n) \to S$ is formally étale in the sense of [SP, Tag 02HG], and thus finite étale by [SP, Tag 02HM].

To show that $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n) \to S$ is surjective, it suffices to show that the pullback to each geometric point of S is a trivial torsor, and so we may assume that S is the spectrum of an algebraically closed field. In this case, there is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence between $\mathbf{Loc}_{\mathbb{Z}/p^n}(S)$ and $\mathbf{Vect}(\mathbb{Z}/p^n)$. But, by Theorem A.18, if ν_n belongs to $\mathcal{G}$-$\mathbf{Vect}(\mathbb{Z}/p^n)$ then $\underline{\mathrm{Isom}}(\omega_{\text{triv}}, \nu_n)$ is a $\mathcal{G}$-torsor. But, $H^1(\mathrm{Spec}(\mathbb{Z}/p^n), \mathcal{G})$ is trivial (e.g. by [Mil17, Corollary 17.98] and [BČ22, Theorem 2.1.6]), and so the claim follows. ☐

Given Lemma 2.1, we write $\mathbf{Tors}_{\mathcal{G}(\mathbb{Z}/p^n)}(S_{\text{proét}})$ instead of $\mathbf{Tors}_{\mathcal{G}_{\underline{\mathbb{Z}/p^n}}}(S_{\text{proét}})$, and call objects of this category $\mathcal{G}(\mathbb{Z}/p^n)$*-torsors*. By Proposition 2.3 and Proposition A.17 we know that there is a natural equivalence of categories between $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}/p^n}(S)$ and $\mathbf{Tors}_{\mathcal{G}(\mathbb{Z}/p^n)}(S_{\text{proét}})$. Explicitly, this functor associates to a $\mathcal{G}(\mathbb{Z}/p^n)$-torsor $\mathcal{A}$ the object $\omega_{\mathcal{A}}$ of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}/p^n}(S)$ given by $\omega_{\mathcal{A}}(\Lambda) := \mathcal{A} \wedge^{\mathcal{G}} \Lambda$.

Remark 2.4. The locally Noetherian hypothesis in Proposition 2.3 may be removed as follows. We may assume that n is in $\mathbb{N}$ and S is connected. Consider W in $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ with a saturated injection $W \hookrightarrow A^*_{\mathcal{G}} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{\mathcal{G}}(\mathcal{G}), \mathbb{Z}_p)$ whose image in $A^*_{\mathcal{G}}/p^n$ contains $\mathcal{G}(\mathbb{Z}/p^n)$. We may replace S by a finite étale connected cover that trivializes $\omega_n(W)$. We claim then that $\omega_n(V)$ is trivial for all V. By Proposition 2.3, for each s in S, we have an isomorphism

$$\psi\colon \underline{\mathrm{Isom}}(\omega_{\text{triv}}, \omega_n)_s \times^{\mathcal{G}(\mathbb{Z}/p^n)} W(\mathbb{Z}/p^n) \cong \omega_n(W)_s.$$

Take a trivial $\mathcal{G}(\mathbb{Z}/p^n)$-torsor $\mathcal{T}$ over S and an isomorphism $\mathcal{T} \times^{\mathcal{G}(\mathbb{Z}/p^n)} W(\mathbb{Z}/p^n) \cong \omega_n(W)$ extending ψ. For an object V of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ and v in $V(\mathbb{Z}_p)$, let $f_v \colon A^*_{\mathcal{G}} \to V$ be the morphism induced by the map $\mathcal{G} \to V$ sending g to gv. Let ev_1 in $A^*_{\mathcal{G}}/p^n$ be the unit section. Then

$$\mathcal{T} \xrightarrow{\mathrm{id} \times \mathrm{ev}_1} \mathcal{T} \times^{\mathcal{G}(\mathbb{Z}/p^n)} W(\mathbb{Z}/p^n) \cong \omega_n(W) \xrightarrow{\omega_n(f_v|_W)} \omega_n(V)$$

gives a map $\mathcal{T} \times^{\mathcal{G}(\mathbb{Z}/p^n)} V(\mathbb{Z}/p^n) \to \omega_n(V)$. One checks this is an isomorphism over s, and so is an isomorphism as the source and target are finite étale over S.

By a $G(\mathbb{Q}_p)$*-local system on* S, we mean an exact $\mathbb{Q}_p$-linear $\otimes$-functor $\mathbf{Rep}_{\mathbb{Q}_p}(G) \to \mathbf{Loc}_{\mathbb{Q}_p}(S)$, the category of which we denote by $G\text{-}\mathbf{Loc}_{\mathbb{Q}_p}(S)$. The following trivial observation allows us to define a natural functor $(-)[1/p] \colon \mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}(S) \to G\text{-}\mathbf{Loc}_{\mathbb{Q}_p}(S)$. Let us denote by $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})[1/p]$ the category which has the same objects as $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, denoted by either Λ or $\Lambda[1/p]$ for clarity, but where we set $\mathrm{Hom}_{\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})[1/p]}(\Lambda[1/p], \Lambda'[1/p])$ to be $\mathbf{Hom}_{\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})}(\Lambda, \Lambda')[1/p]$.

Lemma 2.5. *The functor*

$$\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})[1/p] \to \mathbf{Rep}_{\mathbb{Q}_p}(G), \qquad \Lambda \mapsto \Lambda[1/p],$$

is an equivalence of categories.

Proof. Let Λ and Λ' be objects of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, viewed as right comodules of $A_{\mathcal{G}} := \mathcal{O}_G(\mathcal{G})$ and view $\Lambda[1/p]$ and $\Lambda'[1/p]$ as right comodules of $A_{\mathcal{G}}[1/p] = \mathcal{O}_G(G)$. Then, it suffices to show that under the natural isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \Lambda')[1/p] \to \mathrm{Hom}_{\mathbb{Q}_p}(\Lambda[1/p], \Lambda'[1/p])$ that the subsets of comodule homomorphisms are matched. But, this is obvious as $A_{\mathcal{G}}$ is a flat $\mathbb{Z}_p$-module. To show that this functor is essentially surjective, let V be an $A_{\mathcal{G}}[1/p]$-comodule, and in particular a $A_{\mathcal{G}}$-comodule. The desired comodule lattice is then obtained using [Bro13, Lemma 3.1] applied to V, by taking a $A_{\mathcal{G}}$-comodule, locally free (and thus free) as a $\mathbb{Z}_p$-module, containing any given $\mathbb{Z}_p$-lattice of V. □

Finally, it is useful to have a more concrete description of torsors for our pro-finite group H on $S_{\text{proét}}$. To this end, consider a projective system of schemes $\{X_n\}$, each equipped with the structure of a principal homogeneous space over S for H_n (see [SP, Tag 049A]). The quotient sheaf X_n/K_n, where $K_n := \ker(H_n \to H_{n-1})$, is representable by [SGA3-1, Exposé V, §7, Théorème 7.1], and $X_n/K_n \to S$ is naturally a principal homogeneous space for H_{n-1}. We say (X_n) is an H*-covering* if each $X_n \to X_{n-1}$ is K_n-equivariant (when the target is given the trivial action) and the induced map $X_n/K_n \to X_{n-1}$ is an H_{n-1}-equivariant isomorphism. A morphism of H-coverings $(X_n) \to (Y_n)$ is a compatible family of H_n-equivariant morphisms $X_n \to Y_n$.

An S-scheme X equipped with an action of H is a *principal homogeneous space* for H if h_X with the induced action of $\underline{H}$ is an $\underline{H}$-torsor. Morphisms of principal homogeneous spaces are H-equivariant morphisms of S-schemes. If (X_n) is an H-covering then $X_\infty := \varprojlim X_n$ is a principal homogeneous space for H. Conversely, if X is a principal homogenous space for H, the system (X_n) with $X_n := X/K_n$ (a scheme by Proposition A.10) is an H-covering with $X_\infty = X$.

If S is locally topologically Noetherian then, applying [BS15, Lemma 7.3.9] to $\mathcal{T} \times^{\underline{H}} \underline{H}_n$ for every n, every object $\mathcal{T}$ of $\mathbf{Tors}_{\underline{H}}(S_{\text{proét}})$ is representable. We summarize the above as follows.

Proposition 2.6. *For locally topologically Noetherian S, the following are equivalences:*

$$\left\{\begin{matrix} H\text{-coverings} \\ \text{of } S \end{matrix}\right\} \xrightarrow{(X_n) \mapsto X_\infty} \left\{\begin{matrix} \text{Principal homogenous} \\ \text{spaces for } H \text{ on } S \end{matrix}\right\} \xrightarrow{X \mapsto h_X} \mathbf{Tors}_{\underline{H}}(S_{\text{proét}}).$$

2.1.2. $\mathcal{G}(\mathbb{Z}_p)$*-local systems on an adic space.* Let X be a locally Noetherian adic space over $\mathbb{Q}_p$ (cf. [Sch13, p. 17]), and let $X_{\text{proét}}$ be the pro-étale site as in [BMS18, §5.1]. As in [Sch13], we call an object of $X_{\text{proét}}$ *affinoid perfectoid* if it can be represented as $(\mathrm{Spa}(R_i, R_i^+))$ with finite étale surjective transition maps, and the Huber pair $(R, R^+) = ((\varinjlim R_i)^\wedge, (\varinjlim R_i^+)^\wedge)$ (endowed with the unique topology such that each $R_i \to R$ is adic) is perfectoid. In this case,

$\mathrm{Spa}(R,R^+)\sim\varprojlim\mathrm{Spa}(R_i,R_i^+)$ (see [SW13, §2.4]). The affinoid perfectoid objects of $X_{\text{proét}}$ form a basis of $X_{\text{proét}}$ (cf. [Sch13, Proposition 4.8]).[11]

We will again be interested in the ringed site $(X_{\text{proét}},\underline{\mathbb{Z}_p}_X)$, where

$$\underline{\mathbb{Z}_p}_X=R\varprojlim\underline{\mathbb{Z}/p^n}_X=\varprojlim\underline{\mathbb{Z}/p^n}_X\in\mathbf{Shv}(X_{\text{proét}}),$$

where $\underline{\mathbb{Z}/p^n}_X$ is the constant sheaf on $X_{\text{proét}}$ associated to $\mathbb{Z}/p^n$, and the second equality follows now from [Sch13, Proposition 8.2]). Again, more generally, for a topological space T, we denote by $\underline{T}_X$ (or just $\underline{T}$ when X is clear from context) the sheaf[12]

$$\underline{T}_X\colon X_{\text{proét}}\to\mathbf{Set},\qquad Y\mapsto\mathrm{Hom}_{\text{cont.}}(Y,T).$$

If T is totally disconnected, then $\mathrm{Hom}_{\text{cont.}}(Y,T)$ equals $\mathrm{Hom}_{\text{cont.}}(\pi_0(Y),T)$, and if $T=\varprojlim T_i$ with each T_i finite then $\underline{T}=\varprojlim\underline{T_i}$ where each $\underline{T_i}$ is the constant sheaf.

For $n\in\mathbb{N}\cup\{\infty\}$, write $\mathbf{Loc}_{\mathbb{Z}/p^n}(X)$ for the category $\mathbf{Vect}(X_{\text{proét}},\underline{\mathbb{Z}/p^n})$ of $\mathbb{Z}/p^n$*-local systems*. By [Sch13, Proposition 8.2], this category is equivalent to $\mathbf{Loc}_{\mathbb{Z}/p^n}(X_{\text{ét}})$ as an exact $\mathbb{Z}_p$-linear $\otimes$-category. The objects of $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}/p^n}(X)$ are called $\mathcal{G}(\mathbb{Z}/p^n)$*-local systems*. The following is proven, mutatis mutandis, as in Lemma 2.1 and Proposition 2.3.

Proposition 2.7. *For $n\in\mathbb{N}\cup\{\infty\}$, there is an identification $\mathcal{G}_{\underline{\mathbb{Z}/p^n}}\xrightarrow{\sim}\underline{\mathcal{G}(\mathbb{Z}/p^n)}$, the group $\mathcal{G}$ is reconstructable in $(X_{\text{proét}},\underline{\mathbb{Z}/p^n})$, and every object of $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}/p^n}(X)$ is locally trivial.*

It will again be convenient to consider the category $\mathbf{Loc}_{\mathbb{Q}_p}(X)$ of $\mathbb{Q}_p$*-local systems* on X. This has the same objects as $\mathbf{Loc}_{\mathbb{Z}_p}(X)$, denoted by $\mathbb{L}$ of $\mathbb{L}[1/p]$ when clarity is necessary, but where one defines

$$\mathbf{Hom}_{\mathbf{Loc}_{\mathbb{Q}_p}(X)}(\mathbb{L}[1/p],\mathbb{L}'[1/p]):=\Gamma(X_{\text{proét}},\mathcal{H}om(\mathbb{L},\mathbb{L}')[1/p]).$$

The category $\mathbf{Loc}_{\mathbb{Q}_p}(X)$ admits a fully faithful embedding into $\mathbf{Vect}(X_{\text{proét}},\underline{\mathbb{Q}_p})$, and inherits an exact $\mathbb{Q}_p$-linear $\otimes$-structure from this embedding

Remark 2.8. As in Remark 2.2, the embedding $\mathbf{Loc}_{\mathbb{Q}_p}(X)\to\mathbf{Vect}(X_{\text{proét}},\underline{\mathbb{Q}_p})$ is not generally essentially surjective. The same reasoning applies, except now the relevant non-compact group is the de Jong fundamental group $\pi_1^{\text{dJ}}(X,\overline{x})$ (see [dJ95b] and [ALY21, Corollary 4.4.2]). Here the situation is more extreme though, as even for $X=\mathbb{P}^{1,\text{an}}_{\mathbb{C}_p}$ the de Jong fundamental group is non-compact (cf. [dJ95b, Proposition 7.4]).

Given Proposition 2.7, we write $\mathbf{Tors}_{\mathcal{G}(\mathbb{Z}/p^n)}(X_{\text{proét}})$ instead of $\mathbf{Tors}_{\mathcal{G}_{\underline{\mathbb{Z}/p^n}}}(X_{\text{proét}})$, and call objects of this category $\mathcal{G}(\mathbb{Z}/p^n)$*-torsors*. Moreover, by Proposition A.17 we know that there is a natural equivalence of categories between $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}/p^n}(X)$ and $\mathbf{Tors}_{\mathcal{G}(\mathbb{Z}/p^n)}(X_{\text{proét}})$. Explicitly, this functor associates to a $\mathcal{G}(\mathbb{Z}/p^n)$-torsor $\mathcal{A}$ the object $\omega_{\mathcal{A}}$ of $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}/p^n}(X)$ given by $\omega_{\mathcal{A}}(\Lambda):=\mathcal{P}\wedge^{\mathcal{G}}\Lambda$.

We again define the category of $G(\mathbb{Q}_p)$*-local systems on* X, denoted $G\text{-}\mathbf{Loc}_{\mathbb{Q}_p}(X)$, to be the category of exact $\mathbb{Q}_p$-linear $\otimes$-functors $\mathbf{Rep}_{\mathbb{Q}_p}(G)\to\mathbf{Loc}_{\mathbb{Q}_p}(X)$. Again by Lemma 2.5, we obtain a natural functor $(-)[1/p]\colon\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}(X)\to G\text{-}\mathbf{Loc}_{\mathbb{Q}_p}(X)$.

Finally, as in §2.1.1, it is convenient to have a more down-to-earth definition of torsor for our profinite group H. The definition of an H-covering (Y_n) and principal homogenous spaces Y for H is verbatim to the case of schemes, and the proof of the next result is obtained mutatis mutandis from that of Proposition 2.6 (cf. [ALY21, Theorem 4.4.1]).

Proposition 2.9. *The following functors are equivalences functorial in H:*

$$\left\{\begin{matrix}H\text{-coverings}\\ \text{of }X\end{matrix}\right\}\xrightarrow{(Y_n)\mapsto Y_\infty}\left\{\begin{matrix}\text{Principal homogenous}\\ \text{spaces for }H\text{ on }X\end{matrix}\right\}\xrightarrow{Y\mapsto h_Y}\mathbf{Tors}_{\underline{H}}(X_{\text{proét}}).$$

[11] The site $X_{\text{proét}}$ is not subcanonical (see [ALY21, Example 4.1.7]), but h_Y and $h_Y^\#$ have the same value on affinoid perfectoid objects (see [ALY21, Proposition 4.1.8]). So, we abusively conflate the two.

[12] That this is a sheaf can be deduced from the fact that if $\{Y_i\to Y\}$ is a cover, then $\coprod_i|Y_i|\to|Y|$ is a quotient map. By [Sch22, Lemma 2.5]), to prove this it suffices to each $|Y_i|\to|Y|$ is generalizing, but this follows by combining [Hub96, Lemma 1.1.10] and [Sch22, Lemma 2.11].

2.1.3. *Comparison via analytification.* Fix a non-archimedean extension K of $\mathbb{Q}_p$ and S to be a finite type K-scheme. Consider the analytification $S^{\mathrm{an}} := S \times_{\mathrm{Spec}(K)} \mathrm{Spa}(K)$ of S (see [Hub94, Proposition 3.8]). By [Lüt93, Theorem 3.1], we have an equivalence

$$\left\{\begin{matrix} H\text{-coverings} \\ \text{of } S \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} H\text{-coverings} \\ \text{of } S^{\mathrm{an}} \end{matrix}\right\}, \qquad (Y_n) \mapsto (Y_n^{\mathrm{an}}).$$

On the other hand, using the morphism of sites $S^{\mathrm{an}}_{\text{ét}} \to S_{\text{ét}}$ as in [Hub96, §3.8], we have a functor

$$(-)^{\mathrm{an}} \colon \mathbf{Loc}_{\mathbb{Z}_p}(S) \to \mathbf{Loc}_{\mathbb{Z}_p}(S^{\mathrm{an}}),$$

which is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence by [Lüt93, Theorem 3.1] and the density of classical points. This induces an equivalence of $\mathcal{G}(\mathbb{Z}_p)$-local systems agreeing with the above equivalence of $\mathcal{G}(\mathbb{Z}_p)$-coverings under the equivalences of Proposition 2.6 and Proposition 2.9. We denote the quasi-inverse of $(-)^{\mathrm{an}}$ by $(-)^{\mathrm{alg}}$.

2.1.4. *Comparison on affinoids.* For a strongly Noetherian Huber pair (A, A^+), we have, by [Hub96, Example 1.6.6 ii)], an equivalence

$$\left\{\begin{matrix} H\text{-coverings} \\ \text{of } \mathrm{Spec}(A) \end{matrix}\right\} \to \left\{\begin{matrix} H\text{-coverings} \\ \text{of } \mathrm{Spa}(A, A^+) \end{matrix}\right\}.$$

If (A, A^+) is topologically of finite type over a non-archimedean extension K of $\mathbb{Q}_p$, this gives

$$\mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spec}(A)) \xrightarrow{\sim} \mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(A, A^+)),$$

which is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence by the density of classical points. If $\mathrm{Spec}(A)$ is connected, then the choice of a geometric point $\overline{x}$ gives a futher bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence

$$\mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spec}(A)) \to \mathbf{Rep}^{\mathrm{cont.}}_{\mathbb{Z}_p}(\pi_1^{\text{ét}}(\mathrm{Spec}(A), \overline{x})), \qquad \mathbb{L} \mapsto \mathbb{L}_{\overline{x}},$$

with the target the category of continuous representations $\pi_1^{\text{ét}}(\mathrm{Spec}(A), \overline{x}) \to \mathrm{GL}(\Lambda)$ for a finite free $\mathbb{Z}_p$-module Λ, with the obvious exact $\mathbb{Z}_p$-linear $\otimes$-structure (see [SGA5, Exposé VI, Proposition 1.2.5]). We often tacitly identify a $\mathbb{Z}_p$-local system on $\mathrm{Spa}(A, A^+)$ with such a representation.

2.1.5. *Comparison with v-sheaves.* There is a unique functor

$$(-)^{\mathrm{ad}} \colon \left\{\begin{matrix} p\text{-adic formal} \\ \text{schemes} \end{matrix}\right\} \to \left\{\begin{matrix} \text{Pre-adic spaces} \\ \text{over } \mathbb{Z}_p \end{matrix}\right\},$$

sending open covers to open covers, and with $\mathrm{Spf}(A)^{\mathrm{ad}} = \mathrm{Spa}^Y(A, A)$ (see [SW20, §3.4]). For a p-adic formal scheme $\mathfrak{Y}$, denote by $\mathfrak{Y}^\lozenge$ the v-sheaf over $\mathrm{Spd}(\mathbb{Z}_p)$ associated to $\mathfrak{Y}^{\mathrm{ad}}$ as in [SW20, §18.1]. Set $(\mathfrak{Y}^\lozenge)_\eta := \mathfrak{Y}^\lozenge \times_{\mathrm{Spd}(\mathbb{Z}_p)} \mathrm{Spd}(\mathbb{Q}_p)$. For an affine open $\mathrm{Spf}(A) \subseteq \mathfrak{Y}$, the pair $(A[1/p], \widetilde{A})$ is a Huber pair over $(\mathbb{Q}_p, \mathbb{Z}_p)$, with $\widetilde{A}$ the integral closure of A in $A[1/p]$ and $A[1/p]$ given the unique ring topology with the image of A in $A[1/p]$ open. The diamond $(\mathfrak{Y}_\eta)^\lozenge$ associated to $\mathfrak{Y}_\eta := \mathrm{colim}\, \mathrm{Spa}^Y(A[1/p], \widetilde{A})$ over $\mathbb{Q}_p$ (see [SW20, §10.1]) agrees with $(\mathfrak{Y}^\lozenge)_\eta$. Denote the common object by $\mathfrak{Y}^\lozenge_\eta$.

Consider the quasi-pro-étale site $\mathfrak{Y}^\lozenge_{\eta,\text{qproét}}$ as in [Sch22, Definition 14.1]. Consider

$$\underline{\mathbb{Z}_p}_{\mathfrak{Y}} = R\lim \underline{\mathbb{Z}/p^n}_{\mathfrak{Y}} = \lim \underline{\mathbb{Z}/p^n}_{\mathfrak{Y}}$$

where $\underline{\mathbb{Z}/p^n}_{\mathfrak{Y}}$ is the constant sheaf, and the second equality follows from [Sch22, Lemma 7.18] and [BS15, Proposition 3.1.10 and Proposition 3.2.3]. Define $\mathbf{Loc}_{\mathbb{Z}_p}(\mathfrak{Y}_\eta)$ to be $\mathbf{Vect}(\mathfrak{Y}^\lozenge_{\eta,\text{qproét}}, \underline{\mathbb{Z}_p}_{\mathfrak{Y}})$.

If $\mathfrak{Y} \to \mathrm{Spf}(\mathcal{O}_K)$ is locally of finite type, then $\mathfrak{Y}_\eta$ is the rigid K-space associated to $\mathfrak{Y}$ (see [Hub96, §1.9] or [FK18, §A.5]). Thus, in this case we have already defined an exact $\mathbb{Z}_p$-linear $\otimes$-category $\mathbf{Loc}_{\mathbb{Z}_p}(\mathfrak{Y}_\eta)$, and so there is potential for ambiguity. But, defining H-coverings in the obvious way, one may again show that there is an equivalence of categories between such H-coverings and $\mathbf{Loc}(\mathfrak{Y}_\eta)$. But, using [Sch22, Lemma 15.6] one has an equivalence

$$\left\{\begin{matrix} H\text{-coverings} \\ \text{of } \mathfrak{Y}_\eta \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} H\text{-coverings} \\ \text{of } \mathfrak{Y}_\eta^\diamondsuit \end{matrix}\right\}, \qquad (Y_n) \mapsto (Y_n^\diamondsuit).$$

Using this equivalence and [Sch22, Proposition 14.3] (for the $\mathbb{Z}/p^n\mathbb{Z}$-local system for each $n \geqslant 1$) one obtains a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence

$$\mathbf{Loc}_{\mathbb{Z}_p}(\mathfrak{Y}_\eta) \xrightarrow{\sim} \mathbf{Vect}(\mathfrak{Y}^\diamondsuit_{\eta,\text{qproét}}, \underline{\mathbb{Z}_p}_{\mathfrak{Y}}).$$

Thus, no abmiguity actually occurs if $\mathfrak{Y} \to \operatorname{Spf}(\mathcal{O}_K)$ is locally of finite type.

If $\mathfrak{Y}$ is quasi-syntomic, then there is a natural functor

$$\mathbf{Loc}_{\mathbb{Z}_p}(\mathfrak{Y}_\eta) \to \underset{\operatorname{Spf}(R)\in\mathfrak{Y}_{\text{qrsp}}}{2\text{-}\lim} \mathbf{Loc}_{\mathbb{Z}_p}(\operatorname{Spf}(R)_\eta).$$

This is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence by the next lemma and [Sch22, Proposition 9.7], which we apply by using a limit argument to reduce to the case of finite coefficients.

Lemma 2.10. *Let $\{\mathfrak{Y}_i \to \mathfrak{Y}\}$ be a faithfully flat cover of p-adic formal schemes. Then, the collection $\{\mathfrak{Y}^\diamondsuit_{i,\eta} \to \mathfrak{Y}^\diamondsuit_\eta\}$ is a cover of v-sheaves.*

Proof. By [Sch22, Lemma 12.11] it suffices to show that $\bigsqcup_i |\mathfrak{Y}^\diamondsuit_{i,\eta}| \to |\mathfrak{X}^\diamondsuit_\eta|$ is surjective and any quasi-compact open of the target is covered by a quasi-compact open of the source. By [Sch22, Lemma 15.6] this is equivalent to proving this claim for $\bigsqcup_i |\mathfrak{Y}_{i,\eta}| \to |\mathfrak{X}_\eta|$. This reduces to showing that if $\operatorname{Spf}(B) \to \operatorname{Spf}(A)$ is faithfully flat then $\operatorname{Spf}(B)_\eta \to \operatorname{Spf}(A)_\eta$ is surjective. By [SW13, Proposition 2.1.6], we must show that for an affinoid field (K, K^+) over $(\mathbb{Q}_p, \mathbb{Z}_p)$ and a morphism $\operatorname{Spf}(K^+) \to \operatorname{Spf}(A)$ there exists a surjection $\operatorname{Spf}(L^+) \to \operatorname{Spf}(K^+)$, with (L, L^+) an affinoid field over $(\mathbb{Q}_p, \mathbb{Z}_p)$, so that $\operatorname{Spf}(L^+) \to \operatorname{Spf}(A)$ lifts to $\operatorname{Spf}(B)$. This follows from the discussion in [CS21, Example (2), §2.2.1], as the argument there does not use that K^+ has rank at most 1. □

2.2. **The étale realization functor.** Let $\mathfrak{X}$ be a quasi-syntomic flat formal $\mathbb{Z}_p$-scheme, and $X = \mathfrak{X}^\diamondsuit_\eta$. We discuss the equivalence between $\mathbb{Z}_p$-local systems on X and prismatic Laurent F-crystals on $\mathfrak{X}$ given in [BS23, Corollary 3.7], explicating its bi-exactness.

Note that $\phi\colon \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}$ induces a morphism $\phi\colon \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p \to \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p$.[13] As in [BS23, Definition 3.2], define the category $\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p)$ of *prismatic Laurent F-crystals on $\mathfrak{X}$* to consist of pairs $(\mathcal{L}, \varphi)$ with $\mathcal{L}$ an object of $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p)$ and $\varphi\colon \phi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ an isomorphism in $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p)$, called the *Frobenius*, and morphisms are morphisms in $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p)$ commuting with the Frobenii.

We can endow $\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p)$ with the structure of an exact $\mathbb{Z}_p$-linear $\otimes$-category, where it inherits its exact structure from $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p)$. By [BS23, Proposition 2.7], together with an argument as in the proof of Proposition 1.16, with $\mathbf{Vect}^\varphi(A[1/I]^\wedge_p)$ having the obvious meaning, taking global sections gives a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence

$$\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p) \xrightarrow{\sim} \underset{(A,I)\in\mathfrak{X}_{\Delta}}{2\text{-}\lim} \mathbf{Vect}^\varphi(A[1/I]^\wedge_p),$$

where the right-hand side is given the term-by-term exact and $\mathbb{Z}_p$-linear $\otimes$-structure.

In [BS23, §3], Bhatt and Scholze define an *étale realization functor*

$$T_{\mathfrak{X},\text{ét}}\colon \mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p) \xrightarrow{\sim} \mathbf{Loc}_{\mathbb{Z}_p}(X).$$

which is an equivalence by [BS23, Corollary 3.8] (cf. [MW21, Theorem 3.1] and [Wu21]). When $\mathfrak{X}$ is clear from context, we shall drop $\mathfrak{X}$ from the notation. We now wish to show the following.

Proposition 2.11. *The functor $T_{\mathfrak{X},\text{ét}}$ is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence.*

[13]For (A, I), the map ϕ_A induces an endomorphism of $A[1/I]/p^n A[1/I]$ as $\phi_A(d) = d^p + p\delta(d)$ for any d in I, which as p is nilpotent in A/p^n, implies that the image of I under $\phi_A\colon A/p^n \to A/p^n[1/I]$ generates the unit ideal.

Our proof essentially recounts the construction of Bhatt–Scholze, verifying exactness at each step. Let R be qrsp ring, and define

$$\Delta_{R,\infty} := (\varinjlim_{\phi} \Delta_R)^{\wedge}_{(p,I_R)}, \quad I_{R,\infty} = I_R\Delta_{R,\infty}.$$

Then, $(\Delta_{R,\infty}, I_{R,\infty})$ is a perfect prism and the p-adically complete R-algebra

$$\Delta_{R,\infty}/I_{R,\infty} = (\varinjlim_{\phi} \Delta_R/I_R \varinjlim_{\phi} \Delta_R)^{\wedge}_p =: R_{\mathrm{perfd}}$$

is perfectoid and initial amongst maps from R to a perfectoid ring (see [BS22, Corollary 7.3]). Thus, the maps $\mathrm{Spf}(R_{\mathrm{perfd}})^{\Diamond} \to \mathrm{Spf}(R)^{\Diamond}$ and $\mathrm{Spf}(R_{\mathrm{perfd}})^{\Diamond}_{\eta} \to \mathrm{Spf}(R)^{\Diamond}_{\eta}$ are isomorphisms.[14]

Lemma 2.12 (cf. [BS23, Proposition 3.6 and Corollary 3.7])**.** *Let R be a p-torsion-free qrsp ring. Then, the base change functor*

$$\mathbf{Vect}^{\varphi}(\Delta_R[1/I_R]^{\wedge}_p) \to \mathbf{Vect}^{\varphi}(\Delta_{R,\infty}[1/I_{R,\infty}]^{\wedge}_p)$$

is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence functorial in R.

Proof. We follow the notation of [BS22, Proposition 3.6 and Corollary 3.7]. The only thing to be verified is that this equivalence is bi-exact. In turn, by the method of proof in loc. cit. we reduce to the claim that the equivalence $\mathbf{Loc}_{\mathbb{F}_p}(S) \xrightarrow{\sim} \mathbf{Vect}^{\varphi}(S)$ from [BS23, Proposition 3.4] is bi-exact. But, this is clear as the map $\mathbb{F}_p \to \mathcal{O}_S$ is faithfully flat. ☐

Proof of Proposition 2.11. The only thing to verify is bi-exactness. As the proof in [BS23, Corollary 3.7] proceeds by descent to the case when $\mathfrak{X} = \mathrm{Spf}(R)$ with R qrsp and p-torsion-free, which preserves exactness (cf. Lemma 2.10), we also reduce to this case. Further, using Lemma 2.12 and the isomorphism $\mathrm{Spf}(R_{\mathrm{perfd}})^{\Diamond}_{\eta} \to \mathrm{Spf}(R)^{\Diamond}_{\eta}$ we may reduce to the case when R is perfectoid.

By [BMS18, Lemma 3.21], $(R[1/p], R')$ is a Tate perfectoid algebra, where R' is the integral closure of R in $R[1/p]$. Thus, there is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence

$$\mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(R[1/p], R')) \xrightarrow{\sim} \mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(R[1/p], R')^{\flat}) = \mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(R^{\flat}[1/\varpi^{\flat}], (R')^{\flat}))$$

(see [Sch22, Theorem 6.3]), where ϖ is a pseudo-uniformizer of $R[1/p]$ as in [SW20, Lemma 6.2.2] which may be taken to lie in R. On the other hand, $\Delta_R = W(R^{\flat})$ and $I_R = (\tilde{\xi})$, with $\tilde{\xi} = p + [\varpi^{\flat}]\alpha$ with α in $W(R^{\flat})$ (see [SW20, Lemma 6.2.10]). But, $W(R^{\flat})[1/\tilde{\xi}]^{\wedge}_p = W(R^{\flat}[1/\varpi^{\flat}])$ as both are strict p-rings (in the sense of [KL15, Definition 3.2.1]) with the same residue ring. Thus, we will be done if the $\mathbb{Z}_p$-linear $\otimes$-equivalence between the category of φ-modules for $W(R^{\flat}[1/\varpi^{\flat}])$ and $\mathbb{Z}_p$-local systems on $\mathrm{Spec}(R^{\flat}[1/\varpi^{\flat}])$ is bi-exact, which was already discussed in the proof of Lemma 2.12. ☐

Example 2.13. Let R be a base $\mathcal{O}_K$-algebra. As $\widetilde{\theta}\colon \mathrm{A}_{\mathrm{inf}}(\check{R}) \to \check{R}$ is Γ_R-equivariant, we see that Γ_R acts on $(\mathrm{A}_{\mathrm{inf}}(\check{R}), (\tilde{\xi}))$ as an object of R_{Δ}. In this way, from a prismatic Laurent F-crystal $\mathcal{L}$ on R, we obtain a finite free $\mathbb{Z}_p$-module $\Lambda = \mathcal{L}(\mathrm{A}_{\mathrm{inf}}(\check{R}), (\tilde{\xi}))^{\varphi=1}$ with a continuous action of Γ_R, which is the Γ_R-representation associated to $T_{\text{ét}}(\mathcal{L})$.

2.3. **Crystalline local systems and analytic prismatic F-crystals.** We discuss the equivalence from [GR22] and [DLMS22] and its Tannakian consequences. Unless stated otherwise, we assume that $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth and write $X = \mathfrak{X}_{\eta}$.

2.3.1. *Filtered F-isocrystals.* In this subsection we record our notation and conventions concerning the crystalline site, F-(iso)crystals, and filtered F-isocrystals. The reader is encouraged to skip this on first reading, referring back only as is necessary.

[14] The reader may find it helpful to consult [SW13, Proposition 2.1.6] for this deduction.

PD thickenings of formal schemes. Let $\mathfrak{Z} \to \mathrm{Spf}(W)$ be an adic morphism. By a *PD thickening* of formal $\mathfrak{Z}$-schemes over W, we mean a pair $(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ where $\mathfrak{U} \to \mathfrak{Z}$ is an adic morphism of formal W-schemes, $i\colon \mathfrak{U} \to \mathfrak{T}$ is a closed immersion of adic formal W-schemes, and if $\mathcal{I} := \ker(\mathcal{O}_{\mathfrak{T}} \to i_*\mathcal{O}_{\mathfrak{U}})$ then $\gamma = (\gamma_n)\colon \mathcal{I} \to \mathcal{O}_{\mathfrak{T}}$ is a sequence of morphisms of sheaves so that for every open $\mathfrak{V} \subseteq \mathfrak{T}$ the maps $\gamma_n\colon \mathcal{I}(\mathfrak{V}) \to \mathcal{O}_{\mathfrak{T}}(\mathfrak{V})$ form a PD structure compatible with the usual one on $(p) \subseteq W$. We will often drop i and γ from the notation when they are clear from context.

A morphism $(f, g)\colon (i'\colon \mathfrak{U}' \hookrightarrow \mathfrak{T}', \gamma') \to (i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ is a morphism $f\colon \mathfrak{U}' \to \mathfrak{U}$ of formal $\mathfrak{Z}$-schemes, and $g\colon \mathfrak{T}' \to \mathfrak{T}$ a morphism of formal W-schemes, such that the diagram

$$\begin{tikzcd} \mathfrak{U}' \arrow[r, "i'"] \arrow[d, "f"'] & \mathfrak{T}' \arrow[d, "g"] \\ \mathfrak{U} \arrow[r, "i"'] & \mathfrak{T}, \end{tikzcd} \tag{2.3.1}$$

commutes, and the induced map $g\colon \mathcal{I} \to g_*\mathcal{I}'$ satisfies $g \circ \gamma = g_*(\gamma') \circ g$. A morphism (f, g) is *Cartesian* if (2.3.1) is Cartesian, and a collection $\{(f_i, g_i)\colon (i_i\colon \mathfrak{U}_i \hookrightarrow \mathfrak{T}_i, \gamma_i) \to (i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)\}$ is a *flat cover* if each (f_i, g_i) is Cartesian and $\{g_i\colon \mathfrak{T}_i \to \mathfrak{T}\}$ is a cover in $\mathrm{Spf}(W)_{\mathrm{fl}}$.

If $\mathfrak{U} = \mathrm{Spf}(B)$ and $\mathfrak{T} = \mathrm{Spf}(A)$, we will often write $(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ as $(i\colon A \twoheadrightarrow B, \gamma)$, and write morphisms of such affine objects in the direction dictated by maps of rings. If $i\colon A \to A$ is the identity map, we further shorten $(i\colon A \to A, \gamma)$ to A. Moreover, for a PD thickening of affine formal $\mathfrak{Z}$-schemes $(i\colon A \twoheadrightarrow B, \gamma)$, we have a filtration $\mathrm{Fil}^\bullet_{\mathrm{PD}}(A)$, or just $\mathrm{Fil}^\bullet_{\mathrm{PD}}$ when A is clear from context, given by $\mathrm{Fil}^r_{\mathrm{PD}}(A) := \ker(i)^{[r]}$. Here for an element a of A we sometimes abbreviate $\gamma_r(a)$ to $a^{[r]}$, and for an ideal $I \subseteq A$ by $I^{[r]}$ the ideal generated by $\gamma_{e_1}(i_1) \cdots \gamma_{e_k}(i_k)$ with $\sum e_j \geqslant r$ and i_j in I for all j.

The big crystalline site of a formal scheme. The *(big) crystalline site* $(\mathfrak{Z}/W)_{\mathrm{crys}}$ is the site consisting of PD-thickenings of formal $\mathfrak{Z}$-schemes over W, endowed with the topology whose covers are flat covers. If p is locally nilpotent on $\mathfrak{Z}$, this coincides with the crystalline site as in [SP, Tag 07I5], and in general the proof that $(\mathfrak{Z}/W)_{\mathrm{crys}}$ is a site is proven in much the same way. The site $(\mathfrak{Z}/W)_{\mathrm{crys}}$ naturally comes equipped with the sheaves $\mathcal{J}_{(\mathfrak{Z}/W)_{\mathrm{crys}}} \subseteq \mathcal{O}_{(\mathfrak{Z}/W)_{\mathrm{crys}}}$ where

$$\mathcal{O}_{(\mathfrak{Z}/W)_{\mathrm{crys}}}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) := \mathcal{O}_{\mathfrak{T}}(\mathfrak{T}), \qquad \mathcal{J}_{(\mathfrak{Z}/W)_{\mathrm{crys}}}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) := \ker\left(\mathcal{O}_{\mathfrak{T}} \to i_*\mathcal{O}_{\mathfrak{U}}\right),$$

so $\mathcal{J}_{(\mathfrak{Z}/W)_{\mathrm{crys}}}(i\colon A \twoheadrightarrow B, \gamma) = \mathrm{Fil}^1_{\mathrm{PD}}(A)$. We often shorten this notation to $\mathcal{O}_{\mathrm{crys}}$ and $\mathcal{J}_{\mathrm{crys}}$. When $\mathfrak{Z} = \mathrm{Spf}(R)$ we shall shorten $(\mathfrak{Z}/W)_{\mathrm{crys}}$ to $(R/W)_{\mathrm{crys}}$, and similarly for other notation below.

The following example will appear frequently in the sequel (compare with Example 1.7).

Example 2.14. Let S be a p-adically complete W-algebra with S/p semi-perfect. Then, if $S^\flat$ denotes the perfection of S/p, there exists a universal p-adic pro-thickening $\theta\colon W(S^\flat) \to S$ over W (see [Fon94, §1.2]). Let $K = \ker(\theta)$, and let $\mathrm{A}_{\mathrm{crys}}(S)$ denote the p-adic PD-envelope of the pair $(W(S^\flat), K)$. Then, θ extends to a p-adically continuous surjection $\theta\colon \mathrm{A}_{\mathrm{crys}}(S) \to S$. The pair $(\theta\colon \mathrm{A}_{\mathrm{crys}}(S) \to S, \gamma)$ is a final object of $(S/W)_{\mathrm{crys}}$ (see [Fon94, Théorème 2.2.1]).

There is a natural functor $u_{\mathrm{crys}}\colon (\mathfrak{Z}/W)_{\mathrm{crys}} \to \mathfrak{Z}^{\mathrm{adic}}_{\mathrm{pr}}$, where $\mathfrak{Z}^{\mathrm{adic}}_{\mathrm{pr}}$ is the category of all adic formal $\mathfrak{Z}$-schemes equipped with the pr-topology as in [Lau18b, §7.1], by $u_{\mathrm{crys}}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) := \mathfrak{U}$. This functor is cocontinuous (as one can lift pr-covers along surjective closed embeddings), and so induces a morphism of topoi

$$(u_{\mathrm{crys},*}, u^{-1}_{\mathrm{crys}})\colon \mathbf{Shv}((\mathfrak{Z}/W)_{\mathrm{crys}}) \to \mathbf{Shv}(\mathfrak{Z}^{\mathrm{adic}}_{\mathrm{pr}}).$$

If $\mathfrak{Z}$ is quasi-syntomic, then qrsp objects are a basis for $\mathfrak{Z}^{\mathrm{adic}}_{\mathrm{pr}}$ (cf. the proof of [BMS19, Lemma 4.28]). So, one may prove the following much the same way as Proposition 1.10.

Proposition 2.15. *Suppose that $\mathfrak{Z}$ is quasi-syntomic. Then, $\{(\theta_i\colon \mathrm{A}_{\mathrm{crys}}(S_i) \to S_i, \gamma_i)\}$ where $\{S_i\}$ runs over $\mathfrak{Z}_{\mathrm{qrsp}}$ forms a basis of $\mathbf{Shv}((\mathfrak{Z}/W)_{\mathrm{crys}})$.*

The topos $\mathbf{Shv}((\mathfrak{Z}/W)_{\mathrm{crys}})$ is functorial. Namely, suppose that $f\colon \mathfrak{Z}' \to \mathfrak{Z}$ is an adic morphism of formal schemes lying over a morphism $W \to W$. There is then a morphism of topoi

$$(f_*, f^{-1})\colon \mathbf{Shv}((\mathfrak{Z}'/W)_{\mathrm{crys}}) \to \mathbf{Shv}((\mathfrak{Z}/W)_{\mathrm{crys}}),$$

and f^{-1} has a very concrete description, with $f^{-1}\mathcal{O}_{(\mathfrak{Z}/W)_{\mathrm{crys}}} = \mathcal{O}_{(\mathfrak{Z}'/W)_{\mathrm{crys}}}$ (see [BBM82, 1.1.10]).

The category of (iso)crystals. Let $\mathfrak{Z} \to \mathrm{Spf}(W)$ be an adic morphism. Define the category of *finitely presented (resp. locally free) crystals* on $\mathfrak{Z}$, denoted $\mathbf{Crys}(\mathfrak{Z})$ (resp. $\mathbf{Vect}(\mathfrak{Z}_{\mathrm{crys}})$) to be the category of finitely presented (resp. locally free of finite rank) (in the sense of [SP, Tag 03DL]) $\mathcal{O}_{\mathrm{crys}}$-modules.[15] A finitely presented (resp. locally free) crystal $\mathcal{E}$ on $\mathfrak{Z}$ is equivalent to the following data: for every $(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ a finitely presented (locally free of finite rank) $\mathcal{O}_{\mathfrak{T}}$-module $\mathcal{E}_{(i\colon \mathfrak{U}\hookrightarrow\mathfrak{T},\gamma)}$ and for every morphism $(f,g)\colon (i'\colon \mathfrak{U}' \hookrightarrow \mathfrak{T}', \gamma') \to (i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ a morphism $g^{-1}\mathcal{E}_{(i\colon \mathfrak{U}\hookrightarrow\mathfrak{T},\gamma)} \to \mathcal{E}_{(i'\colon \mathfrak{U}'\hookrightarrow\mathfrak{T}',\gamma')}$ such that the induced morphism $g^*\mathcal{E}_{(i\colon \mathfrak{U}\hookrightarrow\mathfrak{T},\gamma)} \to \mathcal{E}_{(i'\colon \mathfrak{U}'\hookrightarrow\mathfrak{T}',\gamma')}$ is an isomorphism (see [SP, Tag 07IT]). The association is as in [SP, Tag 07IN].

For each $n \in \mathbb{N} \cup \{\infty\}$ set $\mathfrak{Z}_n := (|\mathfrak{Z}|, \mathcal{O}_{\mathfrak{Z}}/p^{n+1})$ (where by definition $\mathfrak{Z}_\infty = \mathfrak{Z}$). For $m \leqslant n$ in $\mathbb{N} \cup \{\infty\}$, denote by $\iota_{m,n}\colon \mathfrak{Z}_m \to \mathfrak{Z}_n$ the natural closed immersion. Then, one may check that $(\iota_{m,n})_*\mathcal{O}_{(\mathfrak{Z}_m/W)_{\mathrm{crys}}} = \mathcal{O}_{(\mathfrak{Z}_n/W)_{\mathrm{crys}}}$, and by [dJ95a, Lemma 2.1.4] there are pairs of quasi-inverse equivalences

$$\begin{aligned} ((\iota_{m,n})_*, \iota_{m,n}^*)\colon \mathbf{Crys}(\mathfrak{Z}_m) &\xrightarrow{\sim} \mathbf{Crys}(\mathfrak{Z}_n), \\ ((\iota_{m,n})_*, \iota_{m,n}^*)\colon \mathbf{Vect}((\mathfrak{Z}_m)_{\mathrm{crys}}) &\xrightarrow{\sim} \mathbf{Vect}((\mathfrak{Z}_n)_{\mathrm{crys}}). \end{aligned} \tag{2.3.2}$$

For a crystal $\mathcal{E}$ on $\mathfrak{Z}$, set $\mathcal{E}_n := \iota_{n,\infty}^*\mathcal{E}$. For an object $(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ of $(\mathfrak{Z}/W)_{\mathrm{crys}}$, we have

$$\mathcal{E}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) = \mathcal{E}_n(i_n\colon \mathfrak{U}_n \hookrightarrow \mathfrak{T}, \gamma)$$

as $\mathcal{O}_{\mathfrak{T}}(\mathfrak{T})$-modules. Here, i_n is the composition of the canonical closed embedding $\mathfrak{U}_n \hookrightarrow \mathfrak{U}$ with i which, as γ is compatible with the PD structure on W, admits a unique extension (also denoted γ) to $\ker(\mathcal{O}_{\mathfrak{T}} \to (i_n)_*\mathcal{O}_{\mathfrak{U}_n}) = (p^n, \ker(\mathcal{O}_{\mathfrak{T}} \to i_*\mathcal{O}_{\mathfrak{U}}))$.

Remark 2.16. Note that while $\mathcal{J}_{(\mathfrak{Z}/W)_{\mathrm{crys}}}$ is locally quasi-coherent (see [SP, Tag 07IS]), it does not satisfy the crystal condition. Nor is it true that $(\iota_{m,n})_*\mathcal{J}_{(\mathfrak{Z}_m/W)_{\mathrm{crys}}} = \mathcal{J}_{(\mathfrak{Z}_n/W)_{\mathrm{crys}}}$.

Finally, the category $\mathbf{Isoc}(\mathfrak{Z})$ of *isocrystals* on $\mathfrak{Z}$ has the same objects as $\mathbf{Crys}(\mathfrak{Z})$, denoted by $\mathcal{E}$ or the formal symbol $\mathcal{E}[1/p]$ when clarity is needed, but with the following morphisms

$$\mathrm{Hom}(\mathcal{E}[1/p], \mathcal{E}'[1/p]) := \Gamma((\mathfrak{Z}/W)_{\mathrm{crys}}, \mathcal{H}om(\mathcal{E}, \mathcal{E}') \otimes_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Q}_p}).$$

For any $m \leqslant n$ in $\mathbb{N} \cup \{\infty\}$ we again have an equivalence

$$((\iota_{m,n})_*, \iota_{m,n}^*)\colon \mathbf{Isoc}(\mathfrak{Z}_m) \xrightarrow{\sim} \mathbf{Isoc}(\mathfrak{Z}_n).$$

We again denote by $\mathcal{E}_n$ (or $\mathcal{E}_n[1/p]$), the pullback of $\mathcal{E}$ (or $\mathcal{E}[1/p]$) to $\mathfrak{Z}_n$.

The categories $\mathbf{Crys}(\mathfrak{Z})$, $\mathbf{Vect}(\mathfrak{Z}_{\mathrm{crys}})$, and $\mathbf{Isoc}(\mathfrak{Z})$ carry natural exact $\mathbb{Z}_p$-linear $\otimes$-structures, and for $m \leqslant n$ in $\mathbb{N} \cup \{\infty\}$ the equivalences $(\iota_{m,n})_*$ and $\iota_{m,n}^*$ are bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalences.

***F*-(iso)crystals.** Let $\mathfrak{Z} \to \mathrm{Spf}(W)$ be an adic morphism. The absolute Frobenius morphism $F_{\mathfrak{Z}_0}\colon \mathfrak{Z}_0 \to \mathfrak{Z}_0$ lies over the Frobenius $\phi\colon W \to W$, and so induces a morphism of ringed topoi

$$(\phi_*, \phi^*)\colon (\mathbf{Shv}((\mathfrak{Z}_0/W)_{\mathrm{crys}}), \mathcal{O}_{(\mathfrak{Z}_0/W)_{\mathrm{crys}}}) \to (\mathbf{Shv}((\mathfrak{Z}_0/W)_{\mathrm{crys}}), \mathcal{O}_{(\mathfrak{Z}_0/W)_{\mathrm{crys}}}).$$

The category $\mathbf{Crys}^{\varphi}(\mathfrak{Z})$ (resp. $\mathbf{Isoc}^{\varphi}(\mathfrak{Z})$) of *F-(iso)crystals* on $\mathfrak{Z}$ has as objects pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ (resp. $(\mathcal{E}[1/p], \varphi_{\mathcal{E}[1/p]})$), where $\mathcal{E}$ (resp. $\mathcal{E}[1/p]$) is an (iso)crystal on $\mathfrak{Z}$ and $\varphi_{\mathcal{E}}$ (resp. $\varphi_{\mathcal{E}[1/p]}$) is a *Frobenius isomorphism* $(\phi^*\mathcal{E}_0)[1/p] \xrightarrow{\sim} \mathcal{E}_0[1/p]$ of isocrystals, with morphisms those morphisms of (iso)crystals commuting with the Frobenii. Denote by $\mathbf{Vect}^{\varphi}(\mathfrak{Z}_{\mathrm{crys}})$ the full subcategory of $\mathbf{Crys}^{\varphi}(\mathfrak{Z})$ of F-crystal whose underlying crystal is a vector bundle. Each of the categories

[15] The forgetful functor $(\mathfrak{Z}/W(k))_{\mathrm{crys}} \to (\mathfrak{Z}/\mathbb{Z}_p)_{\mathrm{crys}}$ induces an equivalence of ringed topoi (see [BBM82, 1.1.13]). It is for this reason that we can be slightly imprecise with our notation for crystals.

$\mathbf{Crys}^\varphi(\mathfrak{Z})$, $\mathbf{Vect}^\varphi(\mathfrak{Z}_{\mathrm{crys}})$, and $\mathbf{Isoc}^\varphi(\mathfrak{Z})$ carry natural exact $\mathbb{Z}_p$-linear $\otimes$-structures. For every $m \leqslant n$ in $\mathbb{N} \cup \{\infty\}$ the morphism $(\iota_{m,n})_*$ induces quasi-inverse pairs $((\iota_{m,n})_*, \iota_{m,n}^*)$

$$\mathbf{Crys}^\varphi(\mathfrak{Z}_m) \xrightarrow{\sim} \mathbf{Crys}^\varphi(\mathfrak{Z}_n),\ \mathbf{Vect}^\varphi((\mathfrak{Z}_m)_{\mathrm{crys}}) \xrightarrow{\sim} \mathbf{Vect}^\varphi((\mathfrak{Z}_n)_{\mathrm{crys}}),\ \mathbf{Isoc}^\varphi(\mathfrak{Z}_m) \xrightarrow{\sim} \mathbf{Isoc}^\varphi(\mathfrak{Z}_n).$$

These are each bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalences.

Remark 2.17. Suppose that $(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ is an object of $(\mathfrak{Z}/W)_{\mathrm{crys}}$, and $\phi_{\mathfrak{T}}\colon \mathfrak{T} \to \mathfrak{T}$ is a Frobenius lift compatible with the Frobenius map ϕ on W. Fix a crystal $\mathcal{E}$ on $\mathfrak{Z}$. From the morphism $(F_{\mathfrak{U}_0}, \phi_{\mathfrak{T}})\colon (i_0\colon \mathfrak{U}_0' \hookrightarrow \mathfrak{T}', \gamma) \to (i_0\colon \mathfrak{U}_0 \hookrightarrow \mathfrak{T}, \gamma)$ in $(\mathfrak{Z}_0/W)_{\mathrm{crys}}$, and the crystal property, there is an identification

$$\phi^*\mathcal{E}_0(i_0\colon \mathfrak{U}_0 \hookrightarrow \mathfrak{T}, \gamma) = \mathcal{E}(i_0\colon \mathfrak{U}_0' \hookrightarrow \mathfrak{T}', \gamma) \cong \phi_{\mathfrak{T}}^*\mathcal{E}(i_0\colon \mathfrak{U}_0 \hookrightarrow \mathfrak{T}, \gamma) = \phi_{\mathfrak{T}}^*\mathcal{E}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma).$$

By $\mathfrak{T}'$ (resp. $\mathfrak{U}'$) we denote $\mathfrak{T}$ (resp. $\mathfrak{U}$) but with W-structure map (resp. $\mathfrak{Z}$-structure map) twisted by ϕ (resp. $F_{\mathfrak{Z}_0}$), so the first equality holds by definition.

Base formal schemes and modules with connection. Let us now assume that $\mathfrak{Z} \to \mathrm{Spf}(W)$ is a base formal W-scheme. Define $\mathbf{MIC}^{\mathrm{tqn}}(\mathfrak{Z})$ to be the category of pairs $(\mathcal{V}, \nabla)$, where $\mathcal{V}$ is a coherent $\mathcal{O}_{\mathfrak{Z}}$-module, and $\nabla\colon \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \Omega^1_{\mathfrak{Z}/W}$ is an integrable topologically quasi-nilpotent connection (see [dJ95a, Remark 2.2.4]). Let $\mathbf{Vect}^\nabla(\mathfrak{Z})$ denote the full subcategory of $\mathbf{MIC}^{\mathrm{tqn}}(\mathfrak{Z})$ of those pairs $(\mathcal{V}, \nabla)$ where $\mathcal{V}$ is a vector bundle. By [dJ95a, Corollary 2.2.3], there are equivalences

$$\mathbf{Crys}(\mathfrak{Z}) \xrightarrow{\sim} \mathbf{MIC}^{\mathrm{tqn}}(\mathfrak{Z}), \qquad \mathbf{Vect}(\mathfrak{Z}_{\mathrm{crys}}) \xrightarrow{\sim} \mathbf{Vect}^\nabla(\mathfrak{Z}), \qquad \mathcal{E} \mapsto (\mathcal{E}_{\mathfrak{Z}}, \nabla_{\mathcal{E}}).$$

Here, for an $\mathcal{O}_{\mathrm{crys}}$-module object $\mathcal{F}$, we denote by $\mathcal{F}_{\mathfrak{Z}}$ the $\mathcal{O}_{\mathfrak{Z}}$-module given by associating to a Zariski open $\mathfrak{U} \subseteq \mathfrak{Z}$ the value $\mathcal{F}(\mathrm{id}\colon \mathfrak{U} \hookrightarrow \mathfrak{U}, \gamma)$. These functors are bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalences, functorial in $\mathfrak{Z}$.

Filtered F-isocrystals. For a rigid K-space Y, denote by $\mathbf{Vect}^\nabla(Y)$ the category of pairs (V, ∇_V) where V is a vector bundle on Y, and $\nabla_V\colon V \to V \otimes_{\mathcal{O}_Y} \Omega^1_{Y/K}$ is an integrable connection. Denote by $\mathbf{VectF}^\nabla(Y)$ the category of triples $(V, \nabla_V, \mathrm{Fil}^\bullet_V)$ where (V, ∇_V) is an object of $\mathbf{Vect}^\nabla(Y)$ and $\mathrm{Fil}^\bullet_V$ is a locally split filtration satisfying Griffiths transversality: for all $i \geqslant 0$ the containment $\nabla_V(\mathrm{Fil}^i_V) \subseteq \mathrm{Fil}^{i-1}_V \otimes_{\mathcal{O}_Y} \Omega^1_{Y/K}$ holds. The category $\mathbf{Vect}^\nabla(Y)$ has an obvious exact $\mathbb{Z}_p$-linear $\otimes$-structure, and we endow $\mathbf{VectF}^\nabla(Y)$ with an exact $\mathbb{Z}_p$-linear $\otimes$-structure where

$$\mathrm{Fil}^k_{V_1 \otimes V_2} = \sum_{i+j=k} \mathrm{Fil}^i_{V_1} \otimes \mathrm{Fil}^j_{V_2},$$

and an exact structure where a sequence is exact if for all i the sequence of vector bundles on Y given by the i^{th}-graded pieces is exact.

Suppose now that $\mathfrak{Z} \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth with rigid generic fiber Z. In [Ogu84, Remark 2.8.1 and Theorem 2.15], Ogus constructs an exact $\mathbb{Z}_p$-linear $\otimes$-functor

$$\mathbf{Isoc}^\varphi(\mathfrak{Z}_k) \to \mathbf{Vect}^\nabla(Z), \qquad (\mathcal{E}, \varphi_{\mathcal{E}}) \mapsto (E, \nabla_E).$$

This functor possesses the following property. For any open $\mathfrak{V} \subseteq \mathfrak{Z}$, and any smooth model $\mathfrak{W}$ of $\mathfrak{V}$ over W, one has $(E, \nabla_E)|_{\mathfrak{V}_\eta}$ is isomorphic to $(\mathcal{F}_{\mathfrak{W}}, \nabla_{\mathcal{F}}) \otimes K$, where $\mathcal{F} = (\iota_{0,\infty})_*(\mathcal{E}|_{\mathfrak{W}_0})$, which is well-defined (i.e., doesn't depend on $\mathcal{E}$ within its isogeny class).

A *filtered F-isocrystal* on $\mathfrak{Z}$ is a triple $(\mathcal{E}, \varphi_{\mathcal{E}}, \mathrm{Fil}^\bullet_E)$ where $(\mathcal{E}, \varphi_{\mathcal{E}})$ is an object of $\mathbf{Isoc}^\varphi(\mathfrak{Z}_k)$ and $\mathrm{Fil}^\bullet_E$ is a locally split filtration on E satisfying Griffiths transversality. With the obvious notion of morphism, we denote by $\mathbf{IsocF}^\varphi(\mathfrak{Z})$ the category of filtered F-isocrystals on $\mathfrak{Z}$, which is seen to be identified with the fiber product $\mathbf{Isoc}^\varphi(\mathfrak{Z}_k) \times_{\mathbf{Vect}^\nabla(Z)} \mathbf{VectF}^\nabla(Z)$. We endow $\mathbf{IsocF}^\varphi(\mathfrak{Z})$ with the exact $\mathbb{Z}_p$-linear $\otimes$-structure inherited from this decomposition and those structures on $\mathbf{Isoc}^\varphi(\mathfrak{Z}_k)$ and $\mathbf{VectF}^\nabla(Z)$.

2.3.2. *de Rham local systems.* Let Y be a smooth rigid K-space. Recall the following rings for an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+) \sim \varprojlim \mathrm{Spa}(R_i, R_i^+)$ over C in $Y^{\mathrm{aff}}_{\mathrm{proét}}$:

- $\mathrm{A}_{\mathrm{inf}}(S) := \mathrm{A}_{\mathrm{inf}}(R^+)$,
- $\mathrm{B}^+_{\mathrm{dR}}(S) := \mathrm{B}^+_{\mathrm{dR}}(R^+) := \mathrm{A}_{\mathrm{inf}}(S)[1/p]^\wedge_{\xi_0}$,
- $\mathrm{B}_{\mathrm{dR}}(S) := \mathrm{B}_{\mathrm{dR}}(R^+) := \mathrm{B}^+_{\mathrm{dR}}(R^+)[1/\xi_0]$,
- $\mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(S) := \mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(R^+) := \varinjlim((R_i^+ \widehat{\otimes}_W \mathrm{A}_{\mathrm{inf}}(R^+))[1/p]^\wedge_{\ker(\theta)})$, given the $\ker(\theta)$-adic filtration, where $\theta\colon (R_i^+ \widehat{\otimes}_{W(k)} \mathrm{A}_{\mathrm{inf}}(S))[1/p] \to R$ is the base extension of $\theta\colon \mathrm{A}_{\mathrm{inf}}(S) \to R^+$,
- $\mathcal{O}\mathrm{B}_{\mathrm{dR}}(S) := \mathcal{O}\mathrm{B}_{\mathrm{dR}}(R^+) := \mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(R^+)[1/t]$ with filtration

$$\mathrm{Fil}^i\, \mathcal{O}\mathrm{B}_{\mathrm{dR}}(S) = \sum_{j \in \mathbb{Z}} t^{-j}\, \mathrm{Fil}^{i+j}\, \mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(S).$$

which extend to sheaves on $Y^{\mathrm{aff}}_{\mathrm{proét}}$. By [Sch13, Corollary 6.13], $\mathcal{O}\mathrm{B}^+_{\mathrm{dR}}$ carries a $\mathrm{B}^+_{\mathrm{dR}}$-horizontal integrable connection satisfying Griffiths transversality, extending to $\mathcal{O}\mathrm{B}_{\mathrm{dR}}$ by base change.

An object $\mathbb{L}$ of $\mathbf{Loc}_{\mathbb{Z}_p}(Y)$ is called *de Rham* if there exists an object $(V, \nabla_V, \mathrm{Fil}^\bullet_V)$ of $\mathbf{VectF}^\nabla(Y)$, such that there exists an isomorphism of sheaves of modules over $\mathrm{B}^+_{\mathrm{dR}}$:

$$c_{\mathrm{Sch}}\colon \mathbb{L} \otimes_{\mathbb{Z}_p} \mathrm{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{Fil}^0 \left(V \otimes_{\mathcal{O}_Y} \mathcal{O}\mathrm{B}_{\mathrm{dR}}\right)^{\nabla = 0},$$

where $\nabla := \nabla_V \otimes \nabla_{\mathcal{O}\mathrm{B}_{\mathrm{dR}}}$ and we endow $V \otimes_{\mathcal{O}_Y} \mathcal{O}\mathrm{B}_{\mathrm{dR}}$ with the tensor product filtration. By [Sch13, Theorem 7.6], the object $(V, \nabla_V, \mathrm{Fil}^\bullet_V)$ is functorially associated to $\mathbb{L}$, and we denote it by $D_{\mathrm{dR}}(\mathbb{L})$ (which we sometimes conflate with the underlying vector bundle). The category of *de Rham $\mathbb{Q}_p$ local systems* is the essential image of the natural functor $\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X) \to \mathbf{Loc}_{\mathbb{Q}_p}(X)$ and we set $D_{\mathrm{dR}}(\mathbb{L}[1/p]) := D_{\mathrm{dR}}(\mathbb{L})$ (which is independent of the choice of $\mathbb{L}$).

Denote by $\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(Y)$ the full subcategory of de Rham objects of $\mathbf{Loc}_{\mathbb{Z}_p}(Y)$, which is seen to be stable under tensor products and duals. The functor $D_{\mathrm{dR}}\colon \mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(Y) \to \mathbf{VectF}^\nabla(Y)$ is an exact $\mathbb{Z}_p$-linear $\otimes$-functor. If $Y = \mathrm{Spa}(K)$, these notations agree with those of Fontaine.

Fix an object ω of $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(Y)$ and $y\colon \mathrm{Spa}(K') \to Y$, for K' a finite extension of K in $\overline{K}$. Let $\mathbf{VectF}(K')$ denote the category of finite-dimensional filtered K'-vector spaces. Then, we have an exact $\mathbb{Z}_p$-linear $\otimes$-functor

$$D_{\mathrm{dR}} \circ \omega_y\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathbf{VectF}(K'), \quad \Lambda \mapsto (D_{\mathrm{dR}}(\omega(\Lambda)), \mathrm{Fil}^\bullet_{D_{\mathrm{dR}}(\omega(\Lambda))})_y.$$

Fix a conjugacy class $\boldsymbol{\mu}$ of cocharacters of $G_{\overline{K}}$. We say ω has *cocharacter $\boldsymbol{\mu}$* if for all such y the following condition holds. For every representation $\rho\colon \mathcal{G} \to \mathrm{GL}(\Lambda)$ and any (equiv. one) element μ of $\boldsymbol{\mu}$, the filtered vector space $(D_{\mathrm{dR}} \circ \omega_y)(\Lambda)_{\overline{K}}$ is isomorphic to $\Lambda_{\overline{K}}$ with filtration defined by $\mathrm{Fil}^r := \bigoplus_{i \geqslant r} \Lambda_{\overline{K}}[i]$, where $\Lambda_{\overline{K}}[i]$ is the i-weight space for the cocharacter $\rho \circ \mu$. The subcategory of $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(Y)$ of those ω which have cocharacter $\boldsymbol{\mu}$ is denoted by $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}(Y)$.

Finally, for a smooth K-scheme Y, denote by $\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(Y)$ (resp. $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}(Y)$) the full subcategory of $\mathbf{Loc}_{\mathbb{Z}_p}(Y)$ (resp. $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}(Y)$) consisting of those $\mathbb{L}$ (resp. ω) such that $\mathbb{L}^{\mathrm{an}}$ (resp. ω^{an}) belongs to $\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(Y^{\mathrm{an}})$ (resp. $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}(Y^{\mathrm{an}})$).

2.3.3. *Crystalline local systems.* As in [TT19, §2A], recall the following rings for an affinoid perfectoid C-space $S = \mathrm{Spa}(R, R^+) \sim \varprojlim \mathrm{Spa}(R_i, R_i^+)$ in $X_{\mathrm{proét}}$:

- $\mathrm{A}_{\mathrm{crys}}(S) := \mathrm{A}_{\mathrm{crys}}(R^+)$, filtered by $\mathrm{Fil}^\bullet_{\mathrm{PD}}$,
- $\mathrm{B}^+_{\mathrm{crys}}(S) := \mathrm{B}^+_{\mathrm{crys}}(R^+) := \mathrm{A}_{\mathrm{crys}}(S)[1/p]$ filtered by $\mathrm{Fil}^\bullet_{\mathrm{PD}}[1/p]$,
- $\mathrm{B}_{\mathrm{crys}}(S) := \mathrm{B}_{\mathrm{crys}}(R^+) := \mathrm{B}^+_{\mathrm{crys}}(S)[1/t]$, with filtration given by

$$\mathrm{Fil}^i\, \mathrm{B}_{\mathrm{crys}}(S) = \sum_{j \in \mathbb{Z}} t^{-j}\, \mathrm{Fil}^{i+j}\, \mathrm{B}^+_{\mathrm{crys}}(R, R^+).$$

These rings, and their filtrations, extend to sheaves on $X_{\mathrm{proét}}$. The Frobenius on $\mathrm{A}_{\mathrm{crys}}$ extends uniquely to $\mathrm{B}^+_{\mathrm{crys}}$ as p is Frobenius invariant, and further extends uniquely to $\mathrm{B}_{\mathrm{crys}}$

with $\phi(1/t) = 1/pt$ (see [TT19, §2C]). In particular, for any object $\mathbb{L}[1/p]$ of $\mathbf{Loc}_{\mathbb{Q}_p}(X)$ the sheaf $\mathrm{B}_{\mathrm{crys}} \otimes_{\underline{\mathbb{Q}_p}} \mathbb{L}[1/p]$ carries a natural Frobenius and filtration.

The Faltings formulation. For an object $(\mathcal{E}, \varphi, \mathrm{Fil}^\bullet_E)$ of $\mathbf{IsocF}^\varphi(\mathfrak{X})$ and an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over C in $X_{\text{proét}}$, which uniquely extends to a map $\mathrm{Spf}(R^+) \to \mathfrak{X}$ (e.g. those factorizing through $\mathrm{Spf}(A)_\eta$ for an affine open $\mathrm{Spf}(A) \subseteq \mathfrak{X}$), we have that $\mathrm{A}_{\mathrm{crys}}(S)$ is a pro-infinitesimal PD thickening of R^+, so we have the associated $\mathrm{B}_{\mathrm{crys}}(R, R^+)$-module

$$\mathrm{B}_{\mathrm{crys}}(\mathcal{E})(R, R^+) := \mathcal{E}(\mathrm{A}_{\mathrm{crys}}(R, R^+) \twoheadrightarrow R^+)[1/p, 1/t].$$

As in [GR22, §2.3], this extends to a sheaf on $X_{\text{proét}}$ and inherits a Frobenius morphism $\varphi\colon \phi^*\mathrm{B}_{\mathrm{crys}}(\mathcal{E}) \to \mathrm{B}_{\mathrm{crys}}(\mathcal{E})$ and a filtration $\mathrm{Fil}^\bullet_{\mathrm{B}_{\mathrm{crys}}(\mathcal{E})}$ induced from $(\mathcal{E}, \varphi, \mathrm{Fil}^\bullet_E)$.

As in [Fal89, p. 67], call an object $\mathbb{L}[1/p]$ of $\mathbf{Loc}_{\mathbb{Q}_p}(X)$ (resp. object $\mathbb{L}$ of $\mathbf{Loc}_{\mathbb{Z}_p}(X)$) *crystalline relative to* $\mathfrak{X}$ if there exists an object $(\mathcal{E}, \varphi_{\mathcal{E}}, \mathrm{Fil}^\bullet_E)$ of $\mathbf{IsocF}^\varphi(\mathfrak{X})$ and an isomorphism sheaves of $\mathrm{B}_{\mathrm{crys}}$-modules

$$c_{\mathrm{Fal}}\colon \mathrm{B}_{\mathrm{crys}} \otimes_{\underline{\mathbb{Q}_p}} \mathbb{L}[1/p] \xrightarrow{\sim} \mathrm{B}_{\mathrm{crys}}(\mathcal{E}),$$

compatible with Frobenius and filtration (resp. $\mathbb{L}$ is crystalline). Denote by $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Q}_p}(X)$ (resp. $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$), suppressing $\mathfrak{X}$ from the notation, the full subcategory of crystalline $\mathbb{Q}_p$-local (resp. $\mathbb{Z}_p$-local) systems. By the arguments in [GR22, Corollary 2.35], the filtered F-isocrystal $\mathcal{E}$ is functorial in $\mathbb{L}[1/p]$. We say that $(\mathcal{E}, \varphi, \mathrm{Fil}^\bullet_E)$ and $\mathbb{L}[1/p]$ are *associated*, and write $D_{\mathrm{crys}}(\mathbb{L})$ for $\mathcal{E}$. We futher define $D_{\mathrm{crys}}(\mathbb{L}) := D_{\mathrm{crys}}(\mathbb{L}[1/p])$ for a crystalline $\mathbb{Z}_p$-local system $\mathbb{L}$. As explained in [TT19, Proposition 3.22] and [GR22, Corollary 2.37], any crystalline local system $\mathbb{L}$ on X is de Rham, and if $D_{\mathrm{crys}}(\mathbb{L}) = (\mathcal{E}, \varphi_{\mathcal{E}}, \mathrm{Fil}^\bullet_E)$ then $D_{\mathrm{dR}}(\mathbb{L})$ is equal to $(E, \nabla_E, \mathrm{Fil}^\bullet_E)$ (cf. §2.3.1).

Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid C-space $S = \mathrm{Spa}(R, R^+)$ in $X_{\text{proét}}$, which uniquely extends to a map $\mathrm{Spf}(R^+) \to \mathfrak{X}$. We consider the isomorphism

$$\theta^+_{\mathrm{crys}}\colon D_{\mathrm{crys}}(\mathbb{L})(\mathrm{A}_{\mathrm{crys}}(S) \to R^+) \otimes_{\mathrm{A}_{\mathrm{crys}}(S)} \mathcal{O}\mathrm{B}^+_{\mathrm{crys}}(S) \xrightarrow{\sim} (D_{\mathrm{crys}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathrm{B}^+_{\mathrm{crys}})(S)$$

from the paragraph right after [TT19, Proposition 3.21]. Put $\theta_{\mathrm{crys}} := \theta^+_{\mathrm{crys}}[1/t]$. Let θ^+_{dR} denote the scalar extension of θ^+_{crys} along $\mathcal{O}\mathrm{B}^+_{\mathrm{crys}}(S) \to \mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(S)$, and

$$\theta^{+,\nabla}_{\mathrm{dR}}\colon D_{\mathrm{crys}}(\mathbb{L})(\mathrm{A}_{\mathrm{crys}}(S) \to R^+) \otimes_{\mathrm{A}_{\mathrm{crys}}(S)} \mathrm{B}^+_{\mathrm{dR}}(S) \xrightarrow{\sim} (D_{\mathrm{crys}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathrm{B}^+_{\mathrm{dR}})(S)^{\nabla=0} \qquad (2.3.3)$$

be the map induced on the spaces of horizontal sections. Put $\theta^\nabla_{\mathrm{dR}} := \theta^{+,\nabla}_{\mathrm{dR}}[1/t]$. Since, for a crystalline local system $\mathbb{L}$ on X, the isomorphism c_{Sch} is given as the scalar extension of the composition $\theta_{\mathrm{crys}} \circ c_{\mathrm{Fal}}$ as explained in the proof of [GR22, Corollary 2.37], we obtain the following lemma, which is used in the proof of Proposition 5.7.

Lemma 2.18. *Assume that $\mathbb{L}|_S$ is constant. Then the diagram*

$$\begin{tikzcd}
(\mathbb{L} \otimes_{\underline{\mathbb{Z}_p}} \mathrm{B}_{\mathrm{dR}})(S) \arrow[r, "c_{\mathrm{Fal}}\otimes 1"] \arrow[dr, "c_{\mathrm{Sch}}\otimes 1"'] & D_{\mathrm{crys}}(\mathbb{L})(\mathrm{A}_{\mathrm{crys}}(S) \to R^+) \otimes_{\mathrm{A}_{\mathrm{crys}}(S)} \mathrm{B}_{\mathrm{dR}}(S) \arrow[d, "\theta^\nabla_{\mathrm{dR}}"] \\
 & (D_{\mathrm{dR}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathrm{B}_{\mathrm{dR}})(S)^{\nabla=0}
\end{tikzcd}$$

commutes.

For a conjugacy class $\boldsymbol{\mu}$ of cocharacters of $G_{\overline{K}}$ denote by $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{crys},\boldsymbol{\mu}}_{\mathbb{Z}_p}(X)$ those $\mathcal{G}(\mathbb{Z}_p)$-local systems which lie in both $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$ and $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}(X)$.

The Brinon formulation. Fix a base $\mathcal{O}_K$-algebra $A = A_0 \otimes_W \mathcal{O}_K$, with formal framing $t\colon T_d \to A_0$. Consider the following rings:

- $\mathcal{O}\mathrm{A}_{\mathrm{crys}}(\check{A})$ is the p-adically completed PD envelope of the map $\theta\colon A_0 \otimes_W \mathrm{A}_{\mathrm{inf}}(\check{A}) \to \check{A}$, filtered by $\mathrm{Fil}^\bullet_{\mathrm{PD}}$,
- $\mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A}) := \mathcal{O}\mathrm{A}_{\mathrm{crys}}(\check{A})[1/p, 1/t]$ filtered by

$$\mathrm{Fil}^r\, \mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A}) = \sum_{n \geqslant -r} t^{-n}\, \mathrm{Fil}^{n+r}\, \mathcal{O}\mathrm{A}_{\mathrm{crys}}(\check{A}).$$

The tensor product Frobenius ϕ on $A_0 \otimes_W \mathrm{A}_{\mathrm{crys}}(\check{A})$ uniquely extends to $\mathcal{O}\mathrm{A}_{\mathrm{crys}}(\check{A})$ and we can extend the Frobenius to $\mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A})$ with $\phi(1/t) = 1/pt$. There is a natural connection

$$\nabla\colon \mathcal{O}\mathrm{A}_{\mathrm{crys}}(\check{A}) \to \mathcal{O}\mathrm{A}_{\mathrm{crys}}(\check{A}) \otimes_{A_0} \widehat{\Omega}^1_{A_0/W},$$

with $\mathrm{A}_{\mathrm{crys}}(\check{A})$ and ϕ horizontal (see [Kim15, Proposition 4.3]), which extends to $\mathrm{B}_{\mathrm{crys}}(\check{A})$ by base change. There is a natural action of Γ_A on $\mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A})$ and the tautological morphism $A_0[1/p] \to \mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A})^{\Gamma_A}$ is an isomorphism (see [Bri08, Proposition 6.2.9]).

For a finite-dimensional continuous $\mathbb{Q}_p$-representation $\rho\colon \Gamma_A \to \mathrm{GL}_{\mathbb{Q}_p}(V)$, write

$$D_{\mathrm{crys}}(V) := (\mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A}) \otimes_{\mathbb{Q}_p} V)^{\Gamma_A},$$

which is an $A_0[1/p]$-module. There is a natural morphism of $\mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A})$-modules

$$\alpha_{\mathrm{crys}}\colon \mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A}) \otimes_{A_0[1/p]} D_{\mathrm{crys}}(V) \to \mathcal{O}\mathrm{B}_{\mathrm{crys}}(\check{A}) \otimes_{\mathbb{Q}_p} V$$

and following [Bri08] we say ρ (or V) is *crystalline* if it is an isomorphism. This notion is independent of the choice of A_0 (see [Bri08, Proposition 8.3.5]). A finite-rank free $\mathbb{Z}_p$-representation Λ of Γ_A is *crystalline* if $\Lambda[1/p]$ is, in which case we write $D_{\mathrm{crys}}(\Lambda) := D_{\mathrm{crys}}(\Lambda[1/p])$.

Denote by $\mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\Gamma_A)$ (resp. $\mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_A)$) the category of crystalline $\mathbb{Q}_p$-representations (resp. $\mathbb{Z}_p$-representations) endowed with the evident structure of an exact $\mathbb{Z}_p$-linear $\otimes$-category. If V is an object of $\mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_A)$ then $D_{\mathrm{crys}}(V)$ has the structure of an object of $\mathbf{IsocF}^{\varphi}(A)$, and [Bri08, Thèoréme 8.5.1] defines a $\mathbb{Z}_p$-linear $\otimes$-functor

$$D_{\mathrm{crys}}\colon \mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\Gamma_A) \to \mathbf{IsocF}^{\varphi}(A),$$

which is a bi-exact equivalence onto its image, functorial in A.

Let Σ be either $\mathbb{Z}_p$ or $\mathbb{Q}_p$. If $\{\mathrm{Spf}(A)\}$ is a small open cover of $\mathfrak{X}$, then an object $\mathbb{L}$ of $\mathbf{Loc}_{\Sigma}(X)$ is crystalline if and only if the Σ-representation V_A associated to $\mathbb{L}|_{\mathrm{Spf}(A)_\eta}$ is crystalline for all A (see [DLMS22, Proposition A.10]). In this case we have that $D_{\mathrm{crys}}(\mathbb{L})|_{\mathrm{Spf}(A)_\eta}$ agrees with $D_{\mathrm{crys}}(V_A)$ for all A. Thus,

$$D_{\mathrm{crys}}\colon \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Q}_p}(X) \to \mathbf{IsocF}^{\varphi}(\mathfrak{X}),$$

is a $\mathbb{Z}_p$-linear $\otimes$-functor which is a bi-exact equivalent onto its image. Also, $\mathbf{Loc}^{\mathrm{crys}}_{\Sigma}(X)$ is closed under duals, tensor products, direct sums, and subquotients (see [Bri08, Théorème 8.4.2]).

Proposition 2.19. *For an object ω of $\mathcal{G}$-$\mathbf{Loc}_{\Sigma}(X)$, ω factorizes through $\mathbf{Loc}^{\mathrm{crys}}_{\Sigma}(X)$ if and only if $\omega(V_0)$ is crystalline for some faithful Σ-representation V_0.*

Proof. It suffices to prove the if condition. Moreover, as ω is crystalline if and only if $\omega[1/p]$ is, we may suppose that $\Sigma = \mathbb{Q}_p$. By [Del82, Proposition 3.1 (a)] every object V of $\mathbf{Rep}_{\mathbb{Q}_p}(G)$ occurs as a subquotient of $\bigoplus_i (V_0)^{\otimes m_i} \oplus (V_0^\vee)^{\otimes n_i}$ for some finite list of integers m_i and n_i. As $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Q}_p}(X)$ is closed under duals, tensor products, direct sums, and subquotients, the claim follows. □

2.3.4. *Crystalline $\mathcal{G}(\mathbb{Z}_p)$-local systems.* Consider the *étale realization functor*

$$T_{\text{ét}}\colon \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\mathbb{A}}) \to \mathbf{Loc}_{\mathbb{Z}_p}(X),$$

defined to be the composition

$$\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\mathbb{A}}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbb{A}}, \mathcal{O}_{\mathbb{A}}[1/\mathcal{I}_{\mathbb{A}}]^\wedge_p) \xrightarrow{T_{\text{ét}}} \mathbf{Loc}_{\mathbb{Z}_p}(X),$$

where the first functor is obtained by patching together the pullbacks $\mathrm{Spec}(A[1/I]^\wedge_p) \to U(A, I)$.

Theorem 2.20 ([GR22] (cf. [DLMS22])). *The functor $T_{\text{ét}}$ induces an equivalence*

$$T_{\text{ét}}\colon \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\mathbb{A}}) \xrightarrow{\sim} \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X).$$

We devote the rest of this subsection to proving the following claim.

Proposition 2.21. *The functor*

$$T_{\text{ét}}\colon \mathbf{Vect}^{\text{an},\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X)$$

is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence. In particular, $T_{\text{ét}}$ induces an equivalence of categories

$$\mathcal{G}\text{-}\mathbf{Vect}^{\text{an},\varphi}(\mathfrak{X}_{\Delta}) \to \mathcal{G}\text{-}\mathbf{Loc}^{\text{crys}}_{\mathbb{Z}_p}(X).$$

As the base extension functor $\mathbf{Vect}^{\text{an},\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)$ is exact, as an exact sequence of vector bundles is universally exact (in the sense of [SP, Tag 058I]), exactness of $T_{\text{ét}}$ follows from Proposition 2.11. So, we have reduced ourselves to showing that $T_{\text{ét}}^{-1}$ is exact. But, combining Proposition 1.10 and Proposition 1.19 we are reduced to showing the following.

Proposition 2.22. *If R is a small $\mathcal{O}_K$-algebra, then the following composition is exact:*

$$\mathbf{Rep}^{\text{crys}}_{\mathbb{Z}_p}(\Gamma_R) \xrightarrow{\sim} \mathbf{Loc}^{\text{crys}}_{\mathbb{Z}_p}(\operatorname{Spa}(R[1/p])) \xrightarrow{T_{\text{ét}}^{-1}} \mathbf{Vect}^{\text{an},\varphi}(R_{\Delta}) \xrightarrow{\text{eval.}} \mathbf{Vect}^{\text{an}}(\mathrm{A}_{\text{inf}}(\widetilde{R}), (\tilde{\xi})).$$

Define

$$\begin{aligned}
\mathrm{B}_{[1/p,\infty]} &:= \mathrm{A}_{\text{inf}}(\widetilde{R})[[p^\flat]^p/p]^{\wedge}_p[1/p] \\
\mathrm{B}_{[0,1/p]} &:= \mathrm{A}_{\text{inf}}(\widetilde{R})[p/[p^\flat]^p]^{\wedge}_{[p^\flat]^p}[1/[p^\flat]^p] \\
\mathrm{B}_{[1/p,1/p]} = \mathrm{B}_{1/p} &:= \mathrm{A}_{\text{inf}}(\widetilde{R})[[p^\flat]^p/p, p/[p^\flat]^p]^{\wedge}[1/p[p^\flat]^p],
\end{aligned}$$

where the final completion is either the p-adic or $[p^\flat]^p$-adic completion.[16] We denote by $\widetilde{\varphi}$ the following composition

$$\mathrm{A}_{\text{crys}}(\widetilde{R})[1/p] = \mathrm{A}_{\text{inf}}(\widetilde{R})[\tfrac{\xi^n}{n!}]^{\wedge}_p[1/p] \xrightarrow{\phi} \mathrm{A}_{\text{inf}}(\widetilde{R})[\tfrac{\tilde{\xi}^n}{n!}]^{\wedge}_p[1/p] \hookrightarrow \mathrm{A}_{\text{inf}}(\widetilde{R})[[p^\flat]^p/p]^{\wedge}_p[1/p] = B_{[1/p,\infty]},$$

which we use to view $\mathrm{B}_{[1/p,\infty]}$ as an $\mathrm{A}_{\text{crys}}(\widetilde{R})[1/p]$-algebra.

There is a natural pullback functor

$$\mathbf{Vect}\left(\operatorname{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R})) - V(p, [p^\flat]^p)\right) \to \mathbf{Vect}(\mathrm{B}_{[0,1/p]}) \times_{\mathbf{Vect}(\mathrm{B}_{1/p})} \mathbf{Vect}(\mathrm{B}_{[1/p,\infty]}),$$

as the natural maps $\operatorname{Spec}(\mathrm{B}_{[a,b]}) \to \operatorname{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R}))$ for $[a,b] \in \{[0,1/p],[1/p,\infty]\}$ factorize through $\operatorname{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R})) - V(p,[p^\flat]^p)$, and are equalized when composed with $\operatorname{Spec}(\mathrm{B}_{1/p}) \to \operatorname{Spec}(\mathrm{B}_{[a,b]})$. For a vector bundle M on $\operatorname{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R})) - V(p,[p^\flat]^p)$ we denote by $M_{[a,b]}$, for $[a,b] \in \{[0,1/p],[1/p,\infty]\}$, the induced vector bundle on $\mathrm{B}_{[a,b]}$.

Lemma 2.23 (cf. [Ked20, Theorem 3.8])**.** *The functor*

$$\mathbf{Vect}\left(\operatorname{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R}) - V(p, [p^\flat]^p))\right) \to \mathbf{Vect}(\mathrm{B}_{[0,1/p]}) \times_{\mathbf{Vect}(\mathrm{B}_{1/p})} \mathbf{Vect}(\mathrm{B}_{[1/p,\infty]})$$

is a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence.

Proof. By the proof of [Ked20, Theorem 3.8], it remains to prove this functor is bi-exact. The exactness follows as an exact sequences of vector bundles is universally exact. To prove that the quasi-inverse is exact, from acyclicity of vector bundles on sheafy affinoid adic spaces and [Ked20, Proposition 3.2] as a whole, we deduce exactness of the quasi-inverse to the functor in [Ked20, Proposition 3.2 (a)], which we apply to (defaulting to the notation in [Ked20, Definition 3.5]) $A_1 \to B_1 \oplus B_2'$ and $A_2 \to B_1' \oplus B_2$ to get the desired exactness. ☐

Remark 2.24. The proof of this lemma implies exactness of Kedlaya's equivalence between vector bundles on $\operatorname{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R})) - V\left(p, [p^\flat]^p\right)$ and those on $\operatorname{Spa}(\mathrm{A}_{\text{inf}}(\widetilde{R}))_{\text{an}}$ (cf. Footnote 16), which seems well-known. Though the adic space perspective could clarify the following proof of Lemma 2.22, we have chosen to avoid it for the sake of brevity.

[16] As in [SW20, §12.2], denote by $\mathcal{Y}$ the analytic locus $\operatorname{Spa}(\mathrm{A}_{\text{inf}}(\widetilde{R}))^{\text{an}}$. This is an analytic adic space equipped with a surjective continuous map $\kappa\colon |\mathcal{Y}| \to [0,\infty]$. For $[a,b] \subseteq [0,\infty]$ with rational (possibly infinite) endpoints, set $\mathcal{Y}_{[a,b]} := \kappa^{-1}([a,b])^{\circ}$, which is a rational open subset of $\operatorname{Spa}(\mathrm{A}_{\text{inf}}(\widetilde{R}))$. When comparing references it is useful to observe that for each $[a,b]$ in $\{[0,1/p],[1/p,\infty],[1/p,1/p]\}$ there is a natural isomorphism $\mathrm{B}_{[a,b]} \to \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}_{[a,b]})$.

We now recall some structural properties of the functor $T_{\text{ét}}^{-1}$.

Lemma 2.25. *For an object Λ of $\mathbf{Rep}_{\mathbb{Z}_p}^{\text{crys}}(\Gamma_R)$, set $\mathcal{M} := T_{\text{ét}}^{-1}(\Lambda)$ and $M := \mathcal{M}(\mathrm{A}_{\text{inf}}(\widetilde{R}), (\tilde{\xi}))$.*

(1) *There is a natural isomorphism of $\mathrm{B}_{[0,1/p]}$-modules*

$$M_{[0,1/p]} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} \mathrm{B}_{[0,1/p]}.$$

(2) *Let $\mathcal{E} = D_{\text{crys}}(\Lambda)$. Then there is a natural isomorphism*

$$M_{[1/p,\infty]}[1/\tilde{\xi}] \xrightarrow{\sim} \mathcal{E}(\mathrm{A}_{\text{crys}}(\widetilde{R}) \twoheadrightarrow \widetilde{R})[1/p] \otimes_{\mathrm{A}_{\text{crys}}(\widetilde{R})[1/p],\tilde{\varphi}} \mathrm{B}_{[1/p,\infty]}[1/\tilde{\xi}]. \tag{2.3.4}$$

Proof. In the proof of [GR22, Theorem 4.8], the prismatic crystal $\mathcal{M}$ is described as the vector bundle $\mathcal{M}_{\widetilde{R}}$ on $\mathrm{Spec}(\mathrm{A}_{\text{inf}}(\widetilde{R})) - V\left(p, [p^\flat]^p\right)$ constructed in [GR22, Theorem 4.15], together with a descent datum on $\widetilde{R}\hat{\otimes}_R\widetilde{R}$. In particular, M is naturally isomorphic to $\mathcal{M}_{\widetilde{R}}$. In the proof of [GR22, Theorem 4.15], the module $\mathcal{M}_{\widetilde{R}}$ is constructed by gluing the $\mathrm{B}_{[1/p,\infty]}$-module $\mathcal{M}_{3,\widetilde{R}}$ constructed in [GR22, Proposition 4.11] and the $\mathrm{B}_{[0,1/p]}$-module $\Lambda \otimes_{\mathbb{Z}_p} \mathrm{B}_{[0,1/p]}$ via the equivalence in Lemma 2.23. In particular, assertion (1) follows by definition. To prove assertion (2), note that $\mathcal{M}_{3,\widetilde{R}}$ is obtained by the Beauville–Laszlo gluing of the $\mathrm{B}_{[1/p,\infty]}[1/\tilde{\xi}]$-module

$$\mathcal{E}(\mathrm{A}_{\text{crys}}(\widetilde{R}) \twoheadrightarrow \widetilde{R})[1/p] \otimes_{\mathrm{A}_{\text{crys}}(\widetilde{R})[1/p],\tilde{\varphi}} \mathrm{B}_{[1/p,\infty]}[1/\tilde{\xi}]$$

and a certain $(\mathrm{B}_{[1/p,\infty]})^\wedge_{\tilde{\xi}}$-module (specifically the submodule $\widetilde{\mathrm{Fil}}^0_E$ of the the $(\mathrm{B}_{[1/p,\infty]})^\wedge_{\tilde{\xi}}[1/\tilde{\xi}]$-module $\mathcal{E}(\mathrm{A}_{\text{crys}}(\widetilde{R}) \twoheadrightarrow \widetilde{R})[1/p] \otimes_{\mathrm{A}_{\text{crys}}(\widetilde{R})[1/p],\tilde{\varphi}} (\mathrm{B}_{[1/p,\infty]})^\wedge_{\tilde{\xi}}[1/\tilde{\xi}]$ see (3.2.4)). So, (2) again follows by definition. ☐

Proof of Proposition 2.22. By Lemma 2.23 we are reduced to the assertions that the functors

$$\mathbf{Rep}_{\mathbb{Z}_p}^{\text{crys}}(\Gamma_R) \to \mathbf{Vect}(\mathrm{B}_{[0,1/p]}), \qquad \Lambda \mapsto T_{\text{ét}}^{-1}(\Lambda)(\mathrm{A}_{\text{inf}}(\widetilde{R}))_{[0,1/p]},$$
$$\mathbf{Rep}_{\mathbb{Z}_p}^{\text{crys}}(\Gamma_R) \to \mathbf{Vect}(\mathrm{B}_{[1/p,\infty]}), \qquad \Lambda \mapsto T_{\text{ét}}^{-1}(\Lambda)(\mathrm{A}_{\text{inf}}(\widetilde{R}))_{[1/p,\infty]},$$

are exact. The first functor being exact follows immediately from the first assertion of Lemma 2.25 and the fact that $\mathbb{Z}_p \to \mathrm{B}_{[0,1/p]}$ is flat. As Beauville–Laszlo gluing is exact, to prove the that the second functor is exact, it suffices to show that the functors

$$\mathbf{Rep}_{\mathbb{Z}_p}^{\text{crys}}(\Gamma_R) \to \mathbf{Vect}(\mathrm{B}_{[1/p,\infty]}), \qquad \Lambda \mapsto T_{\text{ét}}^{-1}(\Lambda)(\mathrm{A}_{\text{inf}}(\widetilde{R}))_{[1/p,\infty]}[1/\tilde{\xi}], \tag{2.3.5}$$

and

$$\mathbf{Rep}_{\mathbb{Z}_p}^{\text{crys}}(\Gamma_R) \to \mathbf{Vect}(\mathrm{B}_{[1/p,\infty]}), \qquad \Lambda \mapsto \left[T_{\text{ét}}^{-1}(\Lambda)(\mathrm{A}_{\text{inf}}(\widetilde{R}))_{[1/p,\infty]}\right]^\wedge_{\tilde{\xi}}, \tag{2.3.6}$$

are exact. The functor in (2.3.5) being exact follows from the second assertion of Lemma 2.25 as D_{crys} is exact. To see that the functor in (2.3.6) is exact, we observe that there is an identification $(\mathrm{B}_{[0,1/p]})^\wedge_{\tilde{\xi}} = (\mathrm{B}_{[1/p,\infty]})^\wedge_{\tilde{\xi}}$, and thus we are again reduced to the first assertion of Lemma 2.25. ☐

2.4. **$\mathcal{G}$-objects in the category of prismatically good reduction local systems.** We now wish to extend some of the results of the last subsection to the case of prismatic F-crystals. For the remainder of this subsection, we assume that $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth and $\mathcal{G}$ is reductive.

Definition 2.26. The category $\mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(X)$ of *prismatically good reduction* $\mathbb{Z}_p$-local systems on X (relative to $\mathfrak{X}$) is the full exact $\mathbb{Z}_p$-linear $\otimes$-subcategory of $\mathbf{Loc}_{\mathbb{Z}_p}^{\text{crys}}(X)$ consisting of those $\mathbb{L}$ with $T_{\text{ét}}^{-1}(\mathbb{L})$ a prismatic F-crystal on $\mathfrak{X}$.

If $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_K)$, then every crystalline $\mathbb{Z}_p$-local system has prismatically good reduction (cf. [GR22, Proposition 3.8]), but for higher-dimensional $\mathfrak{X}$ this ceases to be the case (cf. [DLMS22, Example 3.35]). There is a $\mathbb{Z}_p$-linear $\otimes$-equivalence

$$T_{\text{ét}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(X). \tag{2.4.1}$$

This is exact as $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ is, and so induces a functor

$$T_{\text{ét}} \colon \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(X).$$

That said, the quasi-inverse to the functor in (2.4.1) is not exact even for $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_K)$ as its evaluation at the Breuil–Kisin prism is the functor $\mathfrak{M}$ from [Kis06] (see [BS23, Remark 7.11]), which is known to not be exact (e.g. see [Liu18, Example 4.1.4]).

But, despite the functor in (2.4.1) not being bi-exact, we still have the following.

Theorem 2.27. *The functor*

$$T_{\text{ét}} \colon \mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(X)$$

is an equivalence.

Proof. From Proposition 2.21, it is clear that this functor is fully faithful. To show that it is essentially surjective, fix ω in $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}^{\mathrm{crys}}(X)$ such that $\omega(\Lambda_0)$ has prismatically good reduction. Write $(\mathcal{E}_0, \varphi_{\mathcal{E}_0})$ for the associated object of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ and let $(\mathcal{V}_0, \varphi_{\mathcal{V}_0})$ denotes its image in $\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$. As the functor $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ is fully faithful, we see that the tensors

$$T_{\text{ét}}^{-1}(\omega(\mathbb{T}_0)) \subseteq \mathrm{Hom}\left((\mathcal{O}_{\Delta}^{\mathrm{an}}, \phi), (\mathcal{V}_0, \varphi_{\mathcal{V}_0})^{\otimes}\right)$$

obtained from Proposition 2.21 uniquely lift to a set of tensors $\mathbb{T}_{\Delta} \subseteq (\mathcal{E}_0, \varphi_{\mathcal{E}_0})^{\otimes}$. Set

$$\mathcal{Q}_{\omega} := \underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), (\mathcal{E}_0, \mathbb{T}_{\Delta})\right),$$

with the Frobenius structure inherited from $(\mathcal{E}_0, \mathbb{T}_{\Delta})$. This is a pseudo-torsor for $\mathcal{G}_{\Delta}$ on $\mathfrak{X}_{\Delta}$.

Proposition 2.28. *The pseudo-torsor $\mathcal{Q}_{\omega}$ is a torsor.*

Proof. For any small affine open subset $\mathrm{Spf}(R)$ of $\mathfrak{X}$, set

$$(M_R, \mathbb{T}_R) := (\mathcal{E}_0, \mathbb{T}_{\Delta})(\mathfrak{S}_R, (E)), \quad \mathcal{Q}_{\omega,R} := \underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathfrak{S}_R, \mathbb{T}_0 \otimes 1), (M_R, \mathbb{T}_R)\right),$$

considered as a pseudo-torsor for $\mathcal{G}$ on $\mathrm{Spec}(\mathfrak{S}_R)_{\text{ét}}$. By Corollary 1.15, it suffices to show that $\mathcal{Q}_{\omega,R}$ is a torsor for all such R. But, the restriction of $\mathcal{Q}_{\omega,R}$ to $U(\mathfrak{S}_R, (E))$ is identified with

$$\underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{U(\mathfrak{S}_R,E)}, \mathbb{T}_0 \otimes 1), (\mathcal{V}_0, T_{\text{ét}}^{-1}(\omega(\mathbb{T}_0)))|_{(\mathfrak{S}_R,(E))}\right).$$

Proposition 2.21 implies that $\mathcal{Q}_{\omega,R}|_{U(\mathfrak{S}_R,E)}$ is a torsor. As the height of $(p, E) \subseteq \mathfrak{S}_R$ is 2 and M_R is a vector bundle, $\mathcal{Q}_{\omega,R}$ is a torsor by Proposition A.26 or Remark A.27. $\square$

Let ν be the object of $\mathcal{G}\text{-}\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ associated to $\mathcal{Q}_{\omega}$ by Proposition 1.28. We claim that $T_{\text{ét}} \circ \nu$ is isomorphic to ω. But, by Proposition 1.28 it suffices to observe that, by setup, both $T_{\text{ét}} \circ \nu$ and ω have the value $(\omega(\Lambda_0), \omega(\mathbb{T}_0))$ when evaluated on $(\Lambda_0, \mathbb{T}_0)$. $\square$

As a byproduct of the above proof and Theorem A.14 (which implies every faithful representation can be upgraded to a tensor package) we obtain an analogue of Proposition 2.19.

Corollary 2.29. *Let Λ be a faithful representation of $\mathcal{G}$. Then, an object ω of $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}(X)$ belongs to $\mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(\mathfrak{X})$ if and only if $\omega(\Lambda)$ is an object of $\mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(X)$.*

Proposition 1.28 and Theorem 2.27 yield an equivalence $\mathbf{Tors}_{\mathcal{G}}^{\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathcal{G}\text{-}\mathbf{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(X)$ which we also denote by $T_{\text{ét}}$ (or $T_{\mathfrak{X},\text{ét}}$), which is compatible in $\mathfrak{X}$ and $\mathcal{G}$ in the obvious way.

2.5. **Complementary results about base $\mathcal{O}_K$-algebras.** While the proof of Theorem 2.27 was built on the work of [GR22], Proposition 2.28 works more generally, using [Kis10, Proposition 1.3.4] and the results of [DLMS22]. In addition to this result being potentially useful in other contexts, it also provides an alternative method to prove our main application, Theorem 4.12.

Let R be a (formally framed) base $\mathcal{O}_K$-algebra. In [DLMS22, §4.4],[17] there is constructed a $\mathbb{Z}_p$-linear $\otimes$-functor

$$\mathfrak{M}\colon \mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_R) \to \mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{S}_R,(E)).$$

Let j denote the inclusion $U(\mathfrak{S}_R,(E)) \hookrightarrow \mathrm{Spec}(\mathfrak{S}_R)$. We say that a representation Λ in $\mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_R)$ has *prismatically good reduction* if $j_*\mathfrak{M}(\Lambda)$ is a vector bundle on $\mathrm{Spec}(\mathfrak{S}_R)$.

Let us suppose that Λ_0 carries the structure of an object of $\mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_R)$. Denote $(\mathfrak{M}(\Lambda_0),\mathfrak{M}(T_0))$ by $(M^{\mathrm{an}},\mathbb{T}^{\mathrm{an}})$, and denote the global sections of $j_*(\mathfrak{M}(\Lambda_0),\mathfrak{M}(T_0))$ by $(M,\mathbb{T})$.

Proposition 2.30. *Consider the following sheaf on $U(\mathfrak{S}_R,(E))_{\mathrm{fpqc}}$:*

$$\mathcal{Q}^{\mathrm{an}} = \underline{\mathrm{Isom}}\left((\Lambda_0\otimes_{\mathbb{Z}_p}\mathcal{O}_{U(\mathfrak{S}_R,(E))},\mathbb{T}_0\otimes 1),(M^{\mathrm{an}},\mathbb{T}^{\mathrm{an}})\right).$$

Set $\mathcal{Q} := j_*\mathcal{Q}^{\mathrm{an}}$. Then $\mathcal{Q}$ is a reflexive pseudo-torsor and if $\mathcal{G}$ is reductive, then

$$\mathcal{Q} = \underline{\mathrm{Isom}}\left((\Lambda_0\otimes_{\mathbb{Z}_p}\mathfrak{S}_R,\mathbb{T}_0\otimes 1),(M,\mathbb{T})\right),$$

and is $\mathcal{G}$=torsor on $\mathrm{Spec}(\mathfrak{S}_R)_{\mathrm{fppf}}$ if and only if Λ_0 has prismatically good reduction.

Proof. Given Proposition A.25 and Proposition A.26, the latter claims will follow if we can show that $\mathcal{Q}$ is a reflexive pseudo-torsor. As $\mathcal{Q}^{\mathrm{an}}$ is clearly affine and finite type over $U(\mathfrak{S}_R,E)$, we further deduce from Proposition A.22 that it suffices to prove $\mathcal{Q}$ is a torsor after pulled back to all codimension 1 points of $\mathrm{Spec}(\mathfrak{S}_R)$.

Let $\mathcal{O}_L$ (resp. $\mathcal{O}_{L_0}$) denote the p-adic completion of the localization $R_{(p)}$ (resp. $(R_0)_{(p)}$). Note that $\mathcal{O}_L = \mathcal{O}_{L_0}\otimes_W\mathcal{O}_K$ is a base $\mathcal{O}_K$-algebra, and write $(\mathfrak{S}_L,(E))$ for its Breuil–Kisin prism. Furthermore, let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}_R[u^{-1}]$. As the morphism $\mathrm{Spec}(\mathfrak{S}_L)\sqcup\mathrm{Spec}(\mathcal{O}_{\mathcal{E}})\to\mathrm{Spec}(\mathfrak{S}_R)$ is flat (see [SP, Tag 00MB]) and its image contains all codimension 1 points, it suffices to show that $\mathcal{Q}$ is a $\mathcal{G}$-torsor on pullback to $\mathfrak{S}_L$ and $\mathcal{O}_{\mathcal{E}}$.

To prove the first claim, note that the perfection $\varinjlim_\phi\mathcal{O}_{L_0}$ is a discrete valuation ring that is faithfully flat over $\mathcal{O}_{L_0}$. In particular, its p-adic completion $\mathcal{O}_{L_0'}$ is faithfully flat over $\mathcal{O}_{L_0}$. The ring $\mathcal{O}_{L'} := \mathcal{O}_{L_0'}\otimes_W\mathcal{O}_K$ is a base $\mathcal{O}_K$-algebra, and, with notation as above, $\mathfrak{S}_{L'}$ is faithfully flat over $\mathfrak{S}_L$. Thus, it suffices to show $\mathcal{Q}_{\mathfrak{S}_{L'}}$ is a $\mathcal{G}$-torsor. Recall though that $M\otimes_{\mathfrak{S}_R}\mathfrak{S}_{L'}$ is canonically identified with the (classical) Breuil–Kisin module associated to ρ restricted to the absolute Galois group of L' as in [Kis06] (see [DLMS22, Lemma 4.18 and Proposition 4.26]). Thus, from [Kis10, Proposition 1.3.4] we see that $\mathcal{Q}(\mathfrak{S}_{L'})$ is non-empty.

For the second claim, observe that (cf. [DLMS22, Proposition 4.26]) there is an isomorphism

$$M\otimes_{\mathfrak{S}_R}\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}}\cong\Lambda_0\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}},$$

in $\mathbf{Mod}^{\varphi}(\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}})$, functorial in Λ_0, where $\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}}$ is the p-adic completion of a colimit of finite étale extensions $\mathcal{O}^{\mathrm{ur}}_{\mathcal{E}}$ of $\mathcal{O}_{\mathcal{E}}$ with compatible extension of ϕ (see [DLMS22, §2.3]). From this functorial isomorphism, we see that $\mathcal{Q}(\widehat{\mathcal{O}}^{\mathrm{ur}}_{\mathcal{E}})$ is non-empty, and so $\mathcal{Q}|_{\mathcal{O}_{\mathcal{E}}}$ is a $\mathcal{G}$-torsor. □

3. The filtered F-crystal associated to a prismatic F-crystal

In this section, we construct an integral refinement of the classical D_{crys} functor valued in (naive) filtered F-crystals, and compare it to Fontaine–Laffaille and Dieudonné theory.

Fix k to be a perfect field of characteristic p, and set $W = W(k)$ and $K_0 = W[1/p]$. Also fix K to be a finite totally ramified extension of K_0, with uniformizer π (which we take to be p if $K = K_0$), and minimal polynomial $E\in W[u]$. We make heavy use of the notation and conventions from §1.1.5, §2.3.1, and §2.3.2.

[17] While in loc. cit. the authors only construct this functor for non-negative Hodge–Tate weights, this definition can be easily extended using Breuil–Kisin twists.

3.1. **The crystalline-de Rham comparison.** Our integral analogue of the functor D_{crys} relies on a comparison isomorphism between the crystalline and de Rham realizations of a prismatic F-crystal that we formulate and prove in this subsection.

Throughout this subsection we fix $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ to be a base formal $\mathcal{O}_K$-scheme. When $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth, we write X for its rigid generic fiber over K.

3.1.1. *The crystalline realization functor.* We now expand on the notion of the *crystalline realization functor* $\mathbb{D}_{\mathrm{crys}} \colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ as discussed in [BS23, Example 4.12].

Prismatic F-crystals on quasi-syntomic $\mathbb{F}_p$-schemes. Fix Z to be a quasi-syntomic k-scheme. As in [BS23, Example 4.7], there is an equivalence of categories

$$(-)^{\mathrm{crys}} \colon \mathbf{Vect}(Z_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathbf{Vect}(Z_{\mathrm{crys}}), \qquad \mathcal{F} \mapsto \mathcal{F}^{\mathrm{crys}},$$

defined as follows. Recall from Example 1.7 and Example 2.14 that for a qrsp k-algebra R, the categories R_{Δ} and $(R/W)_{\mathrm{crys}}$ have initial objects $(\mathrm{A}_{\mathrm{crys}}(R), (p), \widetilde{\mathrm{nat.}})$ and $\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R$, respectively. Thus, by evaluation, we have functorial equivalences

$$\mathbf{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathbf{Vect}(\mathrm{A}_{\mathrm{crys}}(R)) \xleftarrow{\sim} \mathbf{Vect}(R_{\mathrm{crys}}).$$

Passing to the limit, and using Proposition 1.30, as well as Proposition 2.15, we deduce the existence of a diagram of equivalences

$$\mathbf{Vect}(Z_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \underset{R \in Z_{\mathrm{qrsp}}}{2\text{-}\lim}\, \mathbf{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \underset{R \in Z_{\mathrm{qrsp}}}{2\text{-}\lim}\, \mathbf{Vect}(R_{\mathrm{crys}}) \xleftarrow{\sim} \mathbf{Vect}(Z_{\mathrm{crys}}).$$

We then define $(-)^{\mathrm{crys}}$ to be the obvious equivalence derived from the diagram, which is functorial in Z in the obvious way.

For an object $\mathcal{V}$ of $\mathbf{Vect}(Z_{\Delta}, \mathcal{O}_{\Delta})$, one has that $(\phi^*\mathcal{V})^{\mathrm{crys}}$ is naturally isomorphic to $\phi^*(\mathcal{V}^{\mathrm{crys}})$. Indeed, by construction it suffices to observe that for a qrsp k-algebra R over Z

$$\phi^*(\mathcal{V}^{\mathrm{crys}})(\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R) = \phi_R^*\mathcal{V}(\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R) = \phi_R^*\mathcal{V}(\mathrm{A}_{\mathrm{crys}}(R), (p)) = (\phi^*\mathcal{V})(\mathrm{A}_{\mathrm{crys}}(R), (p)),$$

where the first equality follows from Remark 2.17 and the second and third by definition. From this observation, one may upgrade $(-)^{\mathrm{crys}}$ to an equivalence

$$(-)^{\mathrm{crys}} \colon \mathbf{Vect}^{\varphi}(Z_{\Delta}) \to \mathbf{Vect}^{\varphi}(Z_{\mathrm{crys}}),$$

which is functorial in Z in the obvious way.

The crystalline realization functor. From the above discussion, we obtain a natural functor $(-)^{\mathrm{crys}} \colon \mathbf{Vect}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}((\mathfrak{X}_k)_{\mathrm{crys}})$ given by by sending $\mathcal{E}$ to $\mathcal{E}^{\mathrm{crys}} := (\mathcal{E}|_{(\mathfrak{X}_k)_{\Delta}})^{\mathrm{crys}}$. The crystal $\mathcal{E}^{\mathrm{crys}}$ enjoys a concrete description when evaluated on base $\mathcal{O}_K$-algebras.

Proposition 3.1. *Let $R = R_0 \otimes_W \mathcal{O}_K$ be a base $\mathcal{O}_K$-algebra and $t \colon T_d \to R_0$ a formal framing. For $\mathcal{F}$ in $\mathbf{Vect}((R_k)_{\Delta}, \mathcal{O}_{\Delta})$ there is a canonical isomorphism*

$$\vartheta_t = \vartheta_{R,t} \colon (\phi^*\mathcal{F})(R_0^{(\phi_t)}, (p)) \xrightarrow{\sim} \mathcal{F}^{\mathrm{crys}}(R_0).$$

If $\mathcal{F}$ carries a Frobenius structure, then ϑ_t is Frobenius-equivariant.

Before we begin the proof, we recall the following basic descent result. Let A be a quasi-syntomic p-adically complete ring. Then, as $\mathrm{Spf}(A)$ has a cover in $\mathrm{Spf}(A)_{\mathrm{fl}}$ whose entire Čech cover is qrsp, we see by p-adic faithfully flat descent that the natural map

$$M \to \varprojlim_{B \in A_{\mathrm{qrsp}}} (M \otimes_A B), \tag{3.1.1}$$

is an isomorphism for any p-adically complete A-module M.

Proof of Proposition 3.1. For any object T of $(R_k)_{\mathrm{qrsp}}$, there exists a canonical identification of modules $\mathcal{F}(\mathrm{A}_{\mathrm{crys}}(T), (p)) \xrightarrow{\sim} \mathcal{F}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(T) \twoheadrightarrow T)$ by the definition of $(-)^{\mathrm{crys}}$. For each object S of R_{qrsp}, let $t_i^{\flat}$ in $S^{\flat}$ be compatible sequences of p^{th}-power roots of t_i in S/p. We then have maps $\beta \colon R_0 \to \mathrm{A}_{\mathrm{crys}}(S)$ as in §1.1.5. The map β induces morphisms $(R_0 \twoheadrightarrow R_k) \to (\mathrm{A}_{\mathrm{crys}}(S) \twoheadrightarrow S_k)$

and $(R_0,(p),F_{R_k}\circ q)\to(\mathrm{A}_{\mathrm{crys}}(S),(p),\widetilde{\mathrm{nat.}})$ in $(R_k)_{\mathrm{crys}}$ and R_{Δ}, respectively. Thus, from the crystal property, and the discussion in Remark 1.18, we obtain isomorphisms

$$\begin{aligned}\phi^*\mathcal{F}(R_0,(p))\otimes_{R_0}S &\xrightarrow{\sim}\mathcal{F}(\mathrm{A}_{\mathrm{crys}}(S),(p))\otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta}S,\\ \mathcal{F}^{\mathrm{crys}}(R_0)\otimes_{R_0}S &\xrightarrow{\sim}\mathcal{F}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(S)\twoheadrightarrow S_k)\otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta}S,\end{aligned}$$

compatible in S. Then by the isomorphism in (3.1.1), we get canonical isomorphisms

$$\begin{aligned}(\phi^*\mathcal{F})(R_0,(p)) &\xrightarrow{\sim}\varprojlim_{S\in R_{\mathrm{qrsp}}}\mathcal{F}(\mathrm{A}_{\mathrm{crys}}(S),(p))\otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta}S\\ &\xrightarrow{\sim}\varprojlim_{S\in R_{\mathrm{qrsp}}}\mathcal{F}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(S)\twoheadrightarrow S_k)\otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta}S\\ &\xrightarrow{\sim}\mathcal{F}^{\mathrm{crys}}(R_0),\end{aligned}$$

which proves the assertion. The second claim follows from the first, via the natural identification of $(\phi^*\mathcal{E})^{\mathrm{crys}}$ and $\phi^*(\mathcal{E}^{\mathrm{crys}})$ for an object $\mathcal{E}$ of $\mathbf{Vect}(\mathfrak{X}_{\Delta},\mathcal{O}_{\Delta})$. □

We now define the *crystalline realization functor*

$$\mathbb{D}_{\mathrm{crys}}\colon\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})\to\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}),\qquad(\mathcal{E},\varphi_{\mathcal{E}})\mapsto(\mathcal{E}^{\mathrm{crys}},\varphi_{\mathcal{E}^{\mathrm{crys}}}).$$

While $\mathbb{D}_{\mathrm{crys}}$ is far from full, it is faithful.

Proposition 3.2. *The functors*

$$(-)^{\mathrm{crys}}\colon\mathbf{Vect}(\mathfrak{X}_{\Delta})\to\mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}}),\qquad\mathbb{D}_{\mathrm{crys}}\colon\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})\to\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$$

are faithful.

Proof. It suffices to prove the first functor is faithful. As passing to a cover is a faithful operation, we see by Proposition 1.10 and the definition of $(-)^{\mathrm{crys}}$ that it is sufficient to show the following: if R is a perfectoid ring, then the base change functor $\mathbf{Vect}(\mathrm{A}_{\mathrm{inf}}(R))\to\mathbf{Vect}(\mathrm{A}_{\mathrm{crys}}(R))$ given by $M\mapsto M\otimes_{\mathrm{A}_{\mathrm{inf}}(R)}\mathrm{A}_{\mathrm{crys}}(R)$ is faithful. But, as M is flat over $\mathrm{A}_{\mathrm{inf}}(R)$, and $\mathrm{A}_{\mathrm{inf}}(R)\to\mathrm{A}_{\mathrm{crys}}(R)$ is injective (see Example 1.8), we deduce that $M\to M\otimes_{\mathrm{A}_{\mathrm{inf}}(R)}\mathrm{A}_{\mathrm{crys}}(R)$ is injective, from where the claim follows. □

Lastly, we give a calculation of $\mathbb{D}_{\mathrm{crys}}$ in terms of Breuil–Kisin modules.

Proposition 3.3. *For a base $\mathcal{O}_K$-algebra $R=R_0\otimes_W\mathcal{O}_K$, there is a canonical Frobenius-equivariant isomorphism*

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi)(R_0)\xrightarrow{\sim}(\phi^*\mathcal{E})(\mathfrak{S}_R,(E))/(u).$$

Proof. As E is an Eisenstein polynomial, the map $\mathfrak{S}_R\to R_0$ sending u to 0 defines a morphism $(\mathfrak{S}_R,(E))\to(R_0,(p))$ in R_{Δ}. The desired isomorphism then follows from applying the crystal property in conjunction with Proposition 3.1. □

Remark 3.4. The Frobenius structure on $(\phi^*\mathcal{E})(\mathfrak{S}_R,(E))/(u)$ in Proposition 3.3 is taken in the sense of Remark 1.18 (either before or after quotienting by (u)).

Example 3.5. For $R=\mathcal{O}_K$ we abuse notation and define the exact $\mathbb{Z}_p$-linear ⊗-functor

$$\mathbb{D}_{\mathrm{crys}}\colon\mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\Gamma_K)\to\mathbf{Vect}^{\varphi}(k_{\mathrm{crys}})=\mathbf{Vect}^{\varphi}(W,(p)),$$

to be $\mathbb{D}_{\mathrm{crys}}\circ T^{-1}_{\mathrm{Spf}(\mathcal{O}_K)}$. Then, there is a natural identification between $\mathbb{D}_{\mathrm{crys}}(\Lambda)$ and the Frobenius module $\phi^*\mathfrak{M}(\Lambda)/(u)$ over W, where $\mathfrak{M}$ is the functor from [Kis06]. Indeed, this follows from Proposition 3.3 and [BS23, Remark 7.11].

3.1.2. *The de Rham realization functor.* We now discuss the notion of a *de Rham realization functor* $\mathbb{D}_{\mathrm{dR}}\colon\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})\to\mathbf{Vect}(\mathfrak{X})$.

Hodge–Tate crystals. Recall that an object $\mathcal{V}$ of $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \overline{\mathcal{O}}_{\Delta})$ is called a *Hodge–Tate crystal* on $\mathfrak{X}$. The same method of proof used in [BS23, Proposition 2.7] shows that Hodge–Tate crystals are crystals of vector bundles on $(\mathfrak{X}_{\Delta}, \overline{\mathcal{O}}_{\Delta})$: $\mathcal{V}(A, I)$ is a projective A/I-module for all (A, I), and for any morphism of prisms $(A, I) \to (B, J)$ the induced map $\mathcal{V}(A, I) \otimes_{A/I} B/J \to \mathcal{V}(B, J)$ is an isomorphism of B/J-modules. We are particularly interested in Hodge–Tate crystals of the form $\overline{\mathcal{E}}$, where $\mathcal{E}$ is an object of $\mathbf{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$. Here we set

$$\overline{\mathcal{E}}(A, I) := (\mathcal{E}/\mathcal{I}_{\Delta})(A, I) = \mathcal{E}(A, I)/I,$$

where the last equality follows from Lemma 1.17.

Let us say that a presheaf $\mathcal{F}$ on $\mathfrak{X}_{\Delta}$ is *residual* if whenever $f_i \colon (A, I) \to (B, J)$ are maps in $\mathfrak{X}_{\Delta}$ for $i = 1, 2$ with $f_1 = f_2 \mod I$, then $f_1^* = f_2^*$ as maps $\mathcal{F}(A, I) \to \mathcal{F}(B, J)$.

Lemma 3.6. *Let $\mathcal{V}$ be a Hodge–Tate crystal on $\mathfrak{X}$. Then, $\mathcal{V}$ is residual.*

Proof. Observe that the exact same method of proof as in [ALB23, Proposition 4.4] shows that we have a pair of quasi-inverse pair

$$(v_*, v^*) \colon \mathbf{Vect}(\mathfrak{X}_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \to \mathbf{Vect}(\mathfrak{X}_{\mathrm{qsyn}}, v_*(\overline{\mathcal{O}}_{\Delta})),$$

with notation in loc. cit. In particular, we have a natural isomorphism of $\overline{\mathcal{O}}_{\Delta}$-modules

$$\mathcal{V} \xrightarrow{\sim} \overline{\mathcal{O}}_{\Delta} \otimes_{v^{\natural}(v_*(\overline{\mathcal{O}}_{\Delta}))} v^{\natural}(v_*(\mathcal{V})).$$

For a sheaf $\mathcal{F}$ in $\mathbf{Shv}(\mathfrak{X}_{\mathrm{qsyn}})$, $v^{\natural}(\mathcal{F})$ is the sheafification of the functor sending (A, I) to $\epsilon^{\natural}(\mathcal{F})(A/I)$ (with notation as in [ALB23, §4.1]), which is evidently residual. As $\overline{\mathcal{O}}_{\Delta}$ is residual, and the presheaf tensor product of residual presheaves is residual, we are reduced to showing that the residualness is preserved under sheafification. But, this follows from a Čech cohomology calculation. $\square$

The de Rham specialization. Consider the functor $\nu_* \colon \mathbf{Shv}(\mathfrak{X}_{\Delta}) \to \mathbf{Shv}(\mathfrak{X}_{\text{ét}})$ defined as

$$\nu_*(\mathcal{F}) := v_*(\mathcal{F})|_{\mathfrak{X}_{\text{ét}}} = u_*(\mathcal{F})_{\mathfrak{X}_{\text{ét}}},$$

where v_* and u_* are as in §1.1.3.

Proposition 3.7. *The following statements hold.*

1. *The natural morphism $\mathcal{O}_{\mathfrak{X}} \to \nu_*(\overline{\mathcal{O}}_{\Delta})$ is an isomorphism of sheaves of rings.*
2. *For a Hodge–Tate crystal $\mathcal{V}$ on $\mathfrak{X}$, the $\mathcal{O}_{\mathfrak{X}}$-module $\nu_*(\mathcal{V})$ is a vector bundle.*
3. *For a formally framed base $\mathcal{O}_K$-algebra R, and a Hodge–Tate crystal $\mathcal{V}$ on R, the map*

$$\mathrm{ev}_{\mathfrak{S}_R} \colon \nu_*(\mathcal{V})(R) = \Gamma(R_{\Delta}, \mathcal{V}) \to \mathcal{V}(\mathfrak{S}_R, (E))$$

given by evaluation is an isomorphism of R-modules.

Proof. Let R be a (formally framed) base $\mathcal{O}_K$-algebra. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{S}_R \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend left, "i_1"] \arrow[rr, bend right, "i_2"'] \arrow[d] & \mathfrak{S}_R \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_R \arrow[r] & \mathfrak{S}_R^{(2)} \arrow[d] \\
\mathfrak{S}_R/(E) \arrow[r, "\sim"] & (\mathfrak{S}_R \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_R)/J \arrow[r] & \mathfrak{S}_R^{(2)}/(E),
\end{tikzcd}$$

where $\mathfrak{S}_R^{(2)}$ is the self-product of $\mathfrak{S}_R$ in $\mathbf{Shv}(R_{\Delta})$, and J denotes the kernel of the surjection $\mathfrak{S}_R \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_R \to \mathfrak{S}_R \to \mathfrak{S}_R/(E) = R$ (see [DLMS22, Example 3.4]). As $\mathfrak{S}_R$ is a cover of $*$ in $\mathbf{Shv}(R_{\Delta})$ (see Proposition 1.15), for any Hodge–Tate crystal $\mathcal{V}$ on R, there is an equalizer diagram

$$\nu_*(\mathcal{V})(R) = \Gamma(R_{\Delta}, \mathcal{V}) \longrightarrow \mathcal{V}(\mathfrak{S}_R, (E)) \underset{i_2^*}{\overset{i_1^*}{\rightrightarrows}} \mathcal{V}(\mathfrak{S}_R^{(2)}, (E)).$$

As R was arbitrary, and a basis of $\mathfrak{X}_{\text{ét}}$ consist of the formal spectra of such R, the first and third claims follow from Lemma 3.6. The second claim follows from the third. $\square$

For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ we call the vector bundle $\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}) := \nu_*(\overline{\phi^*\mathcal{E}})$ the *de Rham specialization* of $\mathcal{E}$, which forms a functor $\mathbb{D}_{\mathrm{dR}} \colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}(\mathfrak{X})$. We next observe that $\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}})$ comes equipped with a natural filtration.

Definition 3.8. We make the following definitions.

(1) For a prism (A, I), and an object (M, φ_M) of $\mathbf{Vect}^{\varphi}(A, I)$, set

$$\mathrm{Fil}^r_{\mathrm{Nyg}}(\phi_A^* M) := \{x \in \phi_A^* M : \varphi_M(x) \in I^r M\},$$

which defines a filtration $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*\mathcal{E})$ by A-submodules, called the *Nygaard filtration*.

(2) For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, we define the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*\mathcal{E}) \subseteq \phi^*\mathcal{E}$ in $\mathcal{O}_{\Delta}$-submodules, called the *Nygaard filtration*, so that $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*\mathcal{E})(A, I) = \mathrm{Fil}^r_{\mathrm{Nyg}}(\phi^*\mathcal{E}(A, I))$, functorially in an object (A, I) of $\mathfrak{X}_{\Delta}$.

From the Nygaard filtration on $\phi_A^* M$ (resp. $\phi^*\mathcal{E}$), we obtain a filtration $\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}(\phi_A^*(M)/I)$ (resp. $\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}(\overline{\phi^*\mathcal{E}})$) as the image of the Nygaard filtration in this quotient. If (M, φ_M) or $(\mathcal{E}, \varphi_{\mathcal{E}})$ is clear from context, we often omit them from the notation, just writing $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}$ or $\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}$.

We then define the filtration $\mathrm{Fil}^r_{\mathbb{D}_{\mathrm{dR}}}(\mathcal{E}, \varphi_{\mathcal{E}}) \subseteq \mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}})$ to be the image of the map

$$\nu_*\left(\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}(\overline{\phi^*\mathcal{E}})\right) \to \nu_*(\overline{\phi^*\mathcal{E}}) = \mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}),$$

a filtration by $\mathcal{O}_{\mathfrak{X}}$-submodules of $\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}})$.

3.1.3. *The crystalline-de Rham comparison.* We now compare the vector bundles $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ and $\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}})$ on $\mathfrak{X}$. To do this, we now assume that $K = K_0$.

Residualness on the crystalline site. We begin by establishing an analogue of Lemma 3.6 on the crystalline site. Fix an adic morphism $\mathfrak{Z} \to \mathrm{Spf}(W)$. Consider the category $\mathbf{Vect}((\mathfrak{Z}/W)_{\mathrm{crys}}, \overline{\mathcal{O}}_{\mathrm{crys}})$ where $\overline{\mathcal{O}}_{\mathrm{crys}} := \mathcal{O}_{\mathrm{crys}}/\mathcal{J}_{\mathrm{crys}}$. As in [SP, Tag 07IT], objects $\mathcal{V}$ of this category satisfy the crystal property: for a morphism $(f, g) \colon (i' \colon \mathfrak{U}' \hookrightarrow \mathfrak{T}', \gamma') \to (i \colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$, the natural map

$$(f, g)^* \otimes 1 \colon \mathcal{V}(i \colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) \otimes_{\mathcal{O}_{\mathfrak{U}}(\mathfrak{U})} \mathcal{O}_{\mathfrak{U}'}(\mathfrak{U}') \to \mathcal{V}(i' \colon \mathfrak{U}' \hookrightarrow \mathfrak{T}', \gamma')$$

is an isomorphism of $\mathcal{O}_{\mathfrak{U}'}(\mathfrak{U}')$-modules. We will be interested in objects of $\mathbf{Vect}((\mathfrak{Z}/W)_{\mathrm{crys}}, \overline{\mathcal{O}}_{\mathrm{crys}})$ that come from objects of $\mathbf{Vect}(\mathfrak{Z}_{\mathrm{crys}})$ via the following functor:

$$\mathbf{Vect}((\mathfrak{Z}/W)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}}) \to \mathbf{Vect}((\mathfrak{Z}/W)_{\mathrm{crys}}, \overline{\mathcal{O}}_{\mathrm{crys}}), \qquad \mathcal{V} \mapsto \overline{\mathcal{V}} = \mathcal{V} \otimes_{\mathcal{O}_{\mathrm{crys}}} \overline{\mathcal{O}}_{\mathrm{crys}}.$$

The values of $\overline{\mathcal{V}}$ can be computed in the naive way on $(i \colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ when $\mathfrak{T}$ is affine, as in this case $H^i((i \colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma), \mathcal{J}_{\mathrm{crys}}) = 0$ for $i > 0$ (see [SP, Tag 07JJ]).

Consider the cocontinuous functor $w_{\mathrm{crys}} \colon (\mathfrak{Z}/W)_{\mathrm{crys}} \to \mathfrak{Z}_{\mathrm{ZAR}}$ sending $(i \colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)$ to $\mathfrak{U}$. Then,

$$(w_{\mathrm{crys}})_*(\overline{\mathcal{O}}_{\mathrm{crys}})(\mathfrak{U}) = \varprojlim_{((i' \colon \mathfrak{U}' \hookrightarrow \mathfrak{T}', \gamma'), \psi) \in {}_{\mathfrak{U}}\mathcal{J}^{\mathrm{opp}}} \mathcal{O}_{\mathfrak{U}'}(\mathfrak{U}') = \varprojlim_{\mathfrak{U}' \in \mathfrak{Z}^{\mathrm{adic}}_{\mathrm{fl}}} \mathcal{O}_{\mathfrak{U}'}(\mathfrak{U}') = \mathcal{O}_{\mathfrak{U}}(\mathfrak{U}).$$

Thus, we have the equality $(w_{\mathrm{crys}})_* \overline{\mathcal{O}}_{\mathrm{crys}} = \mathcal{O}_{\mathfrak{Z}}$.

Lemma 3.9. *There is a pair of quasi-inverse equivalences*

$$((w_{\mathrm{crys}})_*, w^*_{\mathrm{crys}}) \colon \mathbf{Vect}((\mathfrak{Z}/W)_{\mathrm{crys}}, \overline{\mathcal{O}}_{\mathrm{crys}}) \xrightarrow{\sim} \mathbf{Vect}(\mathfrak{Z}_{\mathrm{ZAR}}, \mathcal{O}_{\mathfrak{Z}}) \cong \mathbf{Vect}(\mathfrak{Z}_{\mathrm{fl}}, \mathcal{O}_{\mathfrak{Z}}).$$

Proof. From Proposition A.9 it suffices to show that if $\mathcal{V}$ is a vector bundle on $((\mathfrak{Z}/W)_{\mathrm{crys}}, \overline{\mathcal{O}}_{\mathrm{crys}})$ then $(w_{\mathrm{crys}})_*(\mathcal{V})$ is a vector bundle. We may take an open cover $\{(\mathrm{id} \colon \mathfrak{U}_j \to \mathfrak{U}_j, \gamma_j)\}$ of $*$ in $\mathbf{Shv}((\mathfrak{Z}/W)_{\mathrm{crys}})$, with $\{\mathfrak{U}_j\}$ a Zariski cover of $\mathfrak{Z}$, and such that $\mathcal{V}|_{(\mathrm{id} \colon \mathfrak{U}_j \to \mathfrak{U}_j, \gamma_j)}$ is trivial for all j. We then claim that $(w_{\mathrm{crys}})_*(\mathcal{V})|_{\mathfrak{U}_j}$ is trivial. By the crystal property for $\mathcal{V}$, this follows as in the proof that $(w_{\mathrm{crys}})_*(\overline{\mathcal{O}})_{\mathrm{crys}} = \mathcal{O}_{\mathfrak{Z}}$. $\square$

Let us call an object $\mathcal{F}$ of $\mathbf{Shv}((\mathfrak{Z}/W)_{\text{crys}})$ *residual* if for any pair of morphisms

$$(f_j, g_j)\colon (i'\colon \mathfrak{U}' \hookrightarrow \mathfrak{T}', \gamma') \to (i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma), \qquad (j = 1, 2),$$

with $f_1 = f_2$, then $(f_1, g_1)^* = (f_2, g_2)^*$ as morphisms $\mathcal{F}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) \to \mathcal{F}(i'\colon \mathfrak{U}' \hookrightarrow T', \gamma')$. Given Lemma 3.9, the following is proved much the same way as Lemma 3.6.

Lemma 3.10. *Any object $\mathcal{V}$ of $\mathbf{Vect}((\mathfrak{Z}/W)_{\text{crys}}, \overline{\mathcal{O}}_{\text{crys}})$ is residual.*

The crystalline-de Rham comparison. We begin by establishing a local version of the comparison over a base W-algebra.

Lemma 3.11 (Local crystalline-de Rham comparison)**.** *Let R be a base W-algebra, and $t\colon T_d \to R$ a formal framing. For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}(R_{\mathbb{\Delta}})$, the following statements hold.*

(1) There is a canonical isomorphism of R-modules

$$\iota_{R,t}\colon \mathcal{E}^{\text{crys}}(R) \xrightarrow{\sim} \Gamma(R_{\mathbb{\Delta}}, \overline{\phi^*\mathcal{E}}).$$

(2) for each p-adically étale W-algebra map $\alpha\colon R \to R'$, the scalar extension of $\iota_{R,t}$ to R' is identified with $\iota_{R',\alpha\circ t}$.

(3) The map $\iota_{R,t}$ does not depend on the choice of the formal framing t.

Proof. Using Diagram (1.1.2), Lemma 1.13, and the crystal property, we obtain isomorphisms:

$$\phi_t^*\mathcal{E}(R^{(\phi_t)}, (p)) \otimes_R S_R \xrightarrow{\sim} \mathcal{E}(S_R^{(\phi_t)}, (p)) \xleftarrow{\sim} \phi_t^*\mathcal{E}(\mathfrak{S}_R, (E)) \otimes_{\mathfrak{S}_R} S_R. \tag{3.1.2}$$

Reducing this isomorphism along $S_R \twoheadrightarrow S_R/\operatorname{Fil}^1_{\text{PD}} \xrightarrow{\sim} R$, we obtain an isomorphism of R-modules

$$\phi_t^*\mathcal{E}((R^{(\phi_t)}, (p)) \xrightarrow{\sim} \phi_t^*\mathcal{E}((\mathfrak{S}_R, (E))/(E). \tag{3.1.3}$$

The isomorphism in the first claim follows by combining the above isomorphism with Proposition 3.1 and Proposition 3.7. The second assertion is then clear from this construction.

To prove third assertion, we consider the following diagram (where we suppress the base change to $\check{R}$ in the labeling of several of the arrows):

$$\begin{tikzcd}
\mathcal{E}^{\text{crys}}(R) \otimes_R \check{R} \arrow[rr, "\sim", "\iota_{R,t}"'] \arrow[d, "\vartheta_{R,t}", "\wr"'] & & \Gamma(R_{\mathbb{\Delta}}, \overline{\phi^*\mathcal{E}}) \otimes_R \check{R} \arrow[d, "\mathrm{ev}_{\mathfrak{S}_R}"', "\wr"] \\
\phi_t^*\mathcal{E}(R^{(\phi_t)}, (p)) \otimes_R \check{R} \arrow[r, "\sim"] \arrow[dr, "\sim", "\beta^*_{t^\flat}"'] & \mathcal{E}(S_R^{(\phi_t)}, (p)) \otimes_{S_R} \check{R} \arrow[d, "\wr"] & \phi_t^*\mathcal{E}(\mathfrak{S}_R^{(\phi_t)}, (E)) \otimes_{\mathfrak{S}_R} \check{R} \arrow[l, "\sim"'] \arrow[d, "\alpha^*_{\mathrm{inf},t^\flat}"', "\wr"] \\
& \mathcal{E}(\mathrm{A}_{\text{crys}}(\check{R}), (p)) \otimes_{\mathrm{A}_{\text{crys}}(\check{R})} \check{R} & \mathcal{E}(\mathrm{A}_{\text{inf}}(\check{R}), (\tilde{\xi})) \otimes_{\mathrm{A}_{\text{inf}}(\check{R})} \check{R}. \arrow[l, "\sim"']
\end{tikzcd}$$

By the definition of $\iota_{R,t}$, the top rectangle commutes. The lower triangle and lower rectangle commute as they are induced by the morphisms of prisms in Diagram (1.1.2). Note that the horizontal map on the bottom is independent of t, and that the composite $\alpha^*_{\mathrm{inf},t^\flat} \circ \mathrm{ev}_{\mathfrak{S}_R}$ is identified with the evaluation map $\mathrm{ev}_{\mathrm{A}_{\text{inf}}(\check{R})}$, which is independent of t and its p-power roots $t^\flat$. To show the composition $\beta^*_{t^\flat} \circ \vartheta_{R,t}$ is independent of t and $t^\flat$, we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\text{crys}}(R) \otimes_R \check{R} \arrow[r, "\sim", "\beta^*_{t^\flat}"'] \arrow[d, "\vartheta_{R,t}"', "\wr"] & \mathcal{E}^{\text{crys}}(\mathrm{A}_{\text{crys}}(\check{R}) \twoheadrightarrow \check{R}/p) \otimes_{\mathrm{A}_{\text{crys}}(\check{R}),\theta} \check{R} \arrow[d, "\wr"] \\
\phi_t^*\mathcal{E}(R^{(\phi_t)}, (p)) \otimes_R \check{R} \arrow[r, "\sim", "\beta^*_{t^\flat}"'] & \mathcal{E}(\mathrm{A}_{\text{crys}}(\check{R}), (p)) \otimes_{\mathrm{A}_{\text{crys}}(\check{R}),\theta} \check{R}.
\end{tikzcd}$$

Since $\beta_{t^\flat}$ is defined so that $\theta \circ \beta_{t^\flat}$ coincides with the natural inclusion $R \to \check{R}$, Lemma 3.10 implies that the upper horizontal map $\beta^*_{t_\flat}$ in the diagram is independent of t and $t^\flat$.

Thus, $\iota_{R,t}$ is independent of the choice of t, after base change to $\check{R}$ which implies that $\iota_{R,t}$ is independent of t as $R \to \check{R}$ is injective. $\square$

Remark 3.12. We make the following observations.

(1) Combining Proposition 3.3 and Proposition 3.7, we see that ultimately Lemma 3.11 amounts to a canonical isomorphism of R-modules

$$\phi^*\mathcal{E}(\mathfrak{S}_R,(E))/(u) \xrightarrow{\sim} \phi^*\mathcal{E}(\mathfrak{S}_R,(E))/(E).$$

(2) When $R = W$, and after inverting p, the isomorphism in (3.1.2) is compatible with that from [Kis06, Lemma 1.2.6]. More precisely, it is equal to the base change of the isomorphism ξ from [Kis06, Lemma 1.2.6] along the map $\phi\colon \mathcal{O} \to S_R[1/p]$, where $\mathcal{O}$ is as in loc. cit. This follows from the uniqueness property discussed in the proof of loc. cit.

(3) Kisin constructs in [Kis06, 1.2.7] an isomorphism as in (1) of this remark, after inverting p. By item (2) of this remark, this agrees with our isomorphism after inverting p.

Proposition 3.13 (Crystalline-de Rham comparison)**.** *Let $\mathfrak{X}$ be a base formal W-scheme, and $\mathcal{E}$ a prismatic F-crystal on $\mathfrak{X}$. Then there exists a canonical isomorphism*

$$\iota_{\mathfrak{X}}\colon \underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})_{\mathfrak{X}} \xrightarrow{\sim} \mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}}) \tag{3.1.4}$$

of vector bundles on $\mathfrak{X}$.

Proof. For each affine open formal subscheme $\mathfrak{U}$ of $\mathfrak{X}$ with $R = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ a base W-algebra, the choice of a formal framing t of R determines an isomorphism $\iota_{R,t}$ from the first assertion of Lemma 3.11. By the second and third assertions of Lemma 3.11, these isomorphisms are independent of the choice of t and thus glue to define the desired isomorphism globally. □

Remark 3.14. Our assumption that $K = K_0$ was necessary in the proof of Lemma 3.11 so that the arrow labeled $(*)$ in (1.1.2) was a morphism in $R_{\mathbb{\Delta}}$. But, using Lemma 1.13, one may adjust this for arbitrary $\mathcal{O}_K$ giving, as in Remark 3.12, an isomorphism of R-modules

$$(\phi^e)^*\mathcal{E}(\mathfrak{S}_R,(E))/(E) \xrightarrow{\sim} (\phi^e)^*\mathcal{E}(\mathfrak{S}_R,(E))/(u).$$

In fact, such an isomorphism should hold, by the same method of proof, with $(\mathcal{E},\varphi_{\mathcal{E}})$ replaced by an object of $\mathbf{D}^{\varphi}_{\mathrm{perf}}(\mathfrak{X}_{\mathbb{\Delta}})$, where the quotients should now be considered in the derived sense. One interesting implication of this would be the existence of canonically matched lattices under

$$Rf_*(\Omega^{\bullet}_{\mathfrak{X}/\mathfrak{Y}}) \otimes^L_{\mathcal{O}_K} K \xrightarrow{\sim} Rf_*(\mathcal{O}_{(\mathfrak{X}_k/W)_{\mathrm{crys}}})|_{\mathfrak{Y}_{\mathrm{Zar}}} \otimes^L_W K,$$

the isomorphism of Berthelot–Ogus (see [BO83, Theorem 2.4]), where $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a smooth proper morphism of formal $\mathcal{O}_K$-schemes, where $\mathfrak{Y}$ is smooth.

To give an idea of what this would look like, we make the following observation.

Proposition 3.15. *Let K be a finite extension of $K_0 := W(k)[1/p]$, where k is a perfect field, of absolute ramification index e and $\mathfrak{Y}$ be a formal scheme that is smooth and proper over* $\mathrm{Spf}(\mathcal{O}_K)$*. Then, for any integer a with $p^a \geq e$, there is a canonical isomorphism*

$$(\phi^a)^* R\Gamma_{\mathrm{crys}}(\mathfrak{Y}_k/W(k)) \otimes^L_{W(k)} \mathcal{O}_K \xrightarrow{\sim} (\phi^{a+1})^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/(\mathfrak{S},(E))) \otimes^L_{\mathfrak{S}} \mathfrak{S}/(E).$$

By inverting p, it induces a canonical isomorphism

$$R\Gamma_{\mathrm{crys}}(\mathfrak{Y}_k/W(k)) \otimes_{W(k)} K \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(\mathfrak{Y}/\mathcal{O}_K)[1/p].$$

Proof. Let f denote the structural morphism $\mathfrak{Y} \to \mathrm{Spf}(\mathcal{O}_K)$. In this proof, we write $\mathfrak{S}$ (resp. S) for $\mathfrak{S}_{\mathcal{O}_K}$ (resp. $S_{\mathcal{O}_K}$). Then, by [GR22, Corollary 5.16], the complex $Rf_*\mathcal{O}_{\mathbb{\Delta}}$ belongs to $D^{\varphi}_{\mathrm{perf}}((\mathcal{O}_K)_{\mathbb{\Delta}})$. In particular, using Lemma 1.13 (compare with the proof of Lemma 3.11), we obtain a canonical isomorphism of complexes

$$(\phi^{a+1})^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}_k/(W(k),(p)) \otimes^L_{W(k)} S \xrightarrow{\sim} (\phi^{a+1})^*(R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/(\mathfrak{S},(E)) \otimes^L_{\mathfrak{S}} S.$$

By (derived) scalar extension along $S \to S/\mathrm{Fil}^1_{\mathrm{PD}} \xrightarrow{\sim} \mathcal{O}_K$, it gives

$$(\phi^{a+1})^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}_k/(W(k),(p)) \otimes^L_{W(k)} \mathcal{O}_K \xrightarrow{\sim} (\phi^{a+1})^*(R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/(\mathfrak{S},(E)) \otimes^L_{\mathfrak{S}} \mathcal{O}_K.$$

By [BS22, Theorem 1.8 (1) and (5)], we get the first desired isomorphism.

Note that we have the following canonical identifications given by the Frobenius structures:

$$\begin{aligned}(\phi^a)^* R\Gamma_{\mathrm{crys}}(\mathfrak{Y}_k/W(k))[1/p] &\xrightarrow{\sim} R\Gamma_{\mathrm{crys}}(\mathfrak{Y}_k/W(k))[1/p],\\ (\phi^{a+1})^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/(\mathfrak{S},(E)))[1/\phi(E)\phi^2(E)\cdots\phi^a(E)] &\xrightarrow{\sim} \phi^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/(\mathfrak{S},(E)))[1/\phi(E)\phi^2(E)\cdots\phi^a(E)].\end{aligned}$$

Note also that $\phi^i(E)$ are invertible in $\mathfrak{S}/(E)[1/p]$. Then [BS22, Theorem 1.8 (3)] induces the second desired isomorphism. □

The authors are interested in pursuing this idea in the future.

3.2. **The functor $\mathbb{D}_{\mathrm{crys}}$.** We now apply the crystalline-de Rham comparison theorem to define our integral analogue $\mathbb{D}_{\mathrm{crys}}$ of the functor D_{crys}. Throughout we assume that $K = K_0$.

3.2.1. *The category of filtered F-crystals.* We begin by explicating several categories of F-crystals with filtration that will be important in the sequel.

Naive filtered F-crystals. By a *naive filtered F-crystal* on $\mathfrak{X}$ we mean a triple $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ where $(\mathcal{F}, \varphi_{\mathcal{F}})$ is an object of $\mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}})$ and $\mathrm{Fil}^\bullet_{\mathcal{F}}$ a filtration by $\mathcal{O}_{\mathfrak{X}}$-submodules of $\mathcal{F}_{\mathfrak{X}}$.[18] By a morphism of naive filtered F-crystals $\alpha\colon (\mathcal{F}', \varphi_{\mathcal{F}'}, \mathrm{Fil}^\bullet_{\mathcal{F}'}) \to (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$, we mean a morphism $\alpha\colon \mathcal{F}' \to \mathcal{F}$ of F-crystals with $\alpha(\mathrm{Fil}^i_{\mathcal{F}'}) \subseteq \mathrm{Fil}^i_{\mathcal{F}}$ for all i. Denote the category of naive filtered F-crystals by $\mathbf{VectNF}^\varphi(\mathfrak{X}_{\mathrm{crys}})$, which has a $\mathbb{Z}_p$-linear $\otimes$-structure where

$$\mathrm{Fil}^k_{\mathcal{F}_1\otimes\mathcal{F}_2} = \sum_{i+j=k} \mathrm{Fil}^i_{\mathcal{F}_1} \otimes \mathrm{Fil}^j_{\mathcal{F}_2}.$$

It has an exact structure where a sequence is exact if its associated sequence in $\mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}})$ is exact, and for all i the sequence of $\mathcal{O}_{\mathfrak{X}}$-modules on $\mathfrak{X}$ given by the i^{th}-graded piece is exact. We say that a naive filtered F-crystal has *level* in $[0, a]$ if $\mathrm{Fil}^0_{\mathcal{F}} = \mathcal{F}_{\mathfrak{X}}$ and $\mathrm{Fil}^{a+1}_{\mathcal{F}} = 0$.

Filtered F-crystals. We now examine several refinements of the notion of a naive filtered F-crystal that will play an important role in our discussion below.

Definition 3.16. Let $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ be a naive filtered F-crystal on the base formal W-scheme $\mathfrak{X}$. We say that $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$:

(1) is a *weakly filtered F-crystal* if $\mathrm{Fil}^\bullet_{\mathcal{F}}$ satisfies Griffiths transversality with respect to $\nabla_{\mathcal{F}}$ after inverting p, and the filtration $\mathrm{Fil}^\bullet_{\mathcal{F}}[1/p] \subset \mathcal{F}_{\mathfrak{X}}[1/p]$ is locally split,
(2) is *graded p-torsion-free (gtf)* if $\mathrm{Gr}^r(\mathrm{Fil}^\bullet_{\mathcal{F}})$ is a p-torsion-free $\mathcal{O}_{\mathfrak{X}}$-module for all r,
(3) is a *filtered F-crystal* if $\mathrm{Fil}^\bullet_{\mathcal{F}}$ satisfies Griffiths transversality with respect to $\nabla_{\mathcal{F}}$, and the filtration $\mathrm{Fil}^\bullet_{\mathcal{F}} \subset \mathcal{F}_{\mathfrak{X}}$ is locally split,
(4) is *strongly divisible* if for any affine open $\mathrm{Spf}(R) \subset \mathfrak{X}$ with R a base W-algebra, and any formal framing $t\colon T_d \to R$, the equality $\varphi_{\mathcal{F}}\left(\sum_{r\in\mathbb{Z}} p^{-r}\phi_t^*\,\mathrm{Fil}^r_{\mathcal{F}}(R)\right) = \mathcal{F}_{\mathfrak{X}}(R)$ holds.

Remark 3.17. Our terminology is quite at odds with [Lov17a, §2.4.6]. Namely, in op. cit., a filtered F-crystal (resp. a strongly divisible filtered F-crystal) is called a 'weak filtered F-crystal' (resp. a 'filtered F-crystal').

We give notation to the categories of these objects as follows. Namely, we have the following full subcategories of $\mathbf{VectNF}^\varphi(\mathfrak{X}_{\mathrm{crys}})$:

- $\mathbf{VectWF}^\varphi(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory consisting of weakly filtered F-crystals,
- $\mathbf{VectNF}^{\mathrm{gtf}}(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory consisting of gtf naive filtered F-crystals,
- $\mathbf{VectF}^\varphi(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory consisting of filtered F-crystals,
- $\mathbf{VectNF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory of strongly divisible naive filtered F-crystals.

[18] The moniker 'naive' is warranted as there is no imposed relation between the filtration $\mathbf{Fil}^\bullet_{\mathcal{F}}$ and the crystal $\mathcal{F}$ giving rise to the vector bundle $\mathcal{F}_{\mathfrak{X}}$ (e.g. satisfying Griffiths transversality).

We obtain further full subcategories of $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ by intersecting the above-defined full subcategories, which we denote by concatenating the relevant symbols in the obvious way. For any of these full subcategories, we use the subscript $[0,a]$ to denote the further full subcategory obtained by intersecting with $\mathbf{VectNF}^{\varphi}_{[0,a]}(\mathfrak{X}_{\mathrm{crys}})$. Lastly, we observe that $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ and $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$ are stable under tensor products and duals and so inherit an exact $\mathbb{Z}_p$-linear $\otimes$-structure from $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$.

Lastly, we note that if $\mathfrak{X} \to \mathrm{Spf}(W)$ is smooth, there is an exact $\mathbb{Z}_p$-linear $\otimes$-functor $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{IsocF}^{\varphi}(\mathfrak{X})$ sending $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ to the filtered F-isocrystal $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})[1/p]$ which is defined to be $(\mathcal{F}[1/p], \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}}[1/p]) = (\mathcal{F}[1/p], \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{F})$.

The Faltings morphism. To study the relationship between our various conditions on a naive filtered F-crystal, it is useful to recall the following construction of Faltings.

Fix a naive filtered F-crystal $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet})$ on $\mathrm{Spf}(R)$, where R is a small base W-algebra with formal framing t. Consider the following module as in [Fal89, II.c), p. 30]:

$$\widetilde{\mathcal{F}}_{\mathfrak{X}} := \operatorname{colim}\left(\cdots \to \mathrm{Fil}^{r+1}_{\mathcal{F}} \xleftarrow{\times p} \mathrm{Fil}^{r+1}_{\mathcal{F}} \to \mathrm{Fil}^{r}_{\mathcal{F}} \xleftarrow{\times p} \mathrm{Fil}^{r}_{\mathcal{F}} \to \mathrm{Fil}^{r-1}_{\mathcal{F}} \leftarrow \cdots\right). \tag{3.2.1}$$

The maps $\mathrm{Fil}^r_{\mathcal{F}} \to \mathcal{F}_{\mathfrak{X}}[1/p]$ sending x to $p^{-r}x$ induce a natural map $\widetilde{\mathcal{F}}_{\mathfrak{X}} \to \mathcal{F}_{\mathfrak{X}}[1/p]$ whose image is the sum $\sum_{r\in\mathbb{Z}} p^{-r}\,\mathrm{Fil}^r_{\mathcal{F}}$. We then have the following *Faltings morphism*

$$\phi_t^*\widetilde{\mathcal{F}}_{\mathfrak{X}} \to \phi_t^*\mathcal{F}_{\mathfrak{X}}[1/p] \xrightarrow{\varphi_{\mathcal{F}}} \mathcal{F}[1/p]. \tag{3.2.2}$$

Observe that if $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is gtf then the map $\widetilde{\mathcal{F}}_{\mathfrak{X}} \to \mathcal{F}_{\mathfrak{X}}[1/p]$ is injective. Thus, as ϕ_t is flat, we see that if $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is gtf and strongly divisible, then the Faltings morphism is an isomorphism.

The reason that $\widetilde{\mathcal{F}}_{\mathfrak{X}}$ is a useful object, is that modulo p it is highly related to the grading on $\mathcal{F}_{\mathfrak{X}}$. More precisely, observe that

$$\widetilde{\mathcal{F}}_{\mathfrak{X}}/p \cong \operatorname{colim}\left(\cdots \to \mathrm{Fil}^{r+1}_{\mathcal{F}}/p \xleftarrow{0} \mathrm{Fil}^{r+1}_{\mathcal{F}}/p \to \mathrm{Fil}^{r}_{\mathcal{F}}/p \xleftarrow{0} \mathrm{Fil}^{r}_{\mathcal{F}}/p \to \mathrm{Fil}^{r-1}_{\mathcal{F}}/p \leftarrow \cdots\right),$$

and so is isomorphic to $\bigoplus_r \mathrm{Gr}^r(\mathrm{Fil}^{\bullet}_{\mathcal{F}})/p$. Leveraging this, we show the following.

Proposition 3.18 (cf. [Fal89, Theorem 2.1], [LMP20, Theorem 2.2.1])**.** *Let $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ be a gtf strongly divisible naive filtered F-crystal. Then, the filtration $\mathrm{Fil}^{\bullet}_{\mathcal{F}} \subseteq \mathcal{F}_{\mathfrak{X}}$ is locally split. If $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is further a weakly filtered F-crystal, then it is a strongly divisible filtered F-crystal.*

In symbols the latter part of this proposition demonstrates the following equality of categories

$$\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}}) = \mathbf{VectWF}^{\varphi,\mathrm{gtf},\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}}),$$

for a base formal W-scheme $\mathfrak{X}$.

Proof of Proposition 3.18. It clearly suffices to assume that $\mathfrak{X} = \mathrm{Spf}(R)$, where R is a base $\mathcal{O}_K$-algebra. In what follows we fix a formal framing t.

Let us first verify that the filtration $\mathrm{Fil}^{\bullet}_{\mathcal{F}}$ is locally split. In other words, we must show that $\mathrm{Gr}^r(\mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is a locally free R-module for all r. As $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is gtf and strongly divisible, the Faltings morphism is an isomorphism. This implies that $\phi_t^*\widetilde{\mathcal{F}}(R)$ is a projective R-module. But, as observed above, this implies that $\phi_t^*\,\mathrm{Gr}^r(\mathrm{Fil}^r_{\mathcal{F}}(R))/p$ is a projective R/p-module for all r, and since ϕ_t is faithfully flat this implies that $\mathrm{Gr}^r(\mathrm{Fil}^{\bullet}_{\mathcal{F}}(R))/p$ is a locally free R/p-module for all r. This implies that $\mathrm{Gr}^r(\mathrm{Fil}^{\bullet}_{\mathcal{F}}(R))$ is a projective R-module for all R, by the following lemma.

Lemma 3.19. *Let A be a Noetherian p-adically complete ring and Q a finitely generated p-torsion-free A-module such that Q/p is a projective A/p-module. Then, Q is a projective A-module.*

Proof. Take a short exact sequence

$$P\colon \quad 0 \to K \xrightarrow{\iota} A^n \to Q \to 0.$$

As Q is p-torsion-free, this remains exact after reducing modulo p thus giving an exact sequence

$$0 \to K/p \xrightarrow{\bar{\iota}} (A/p)^n \to Q/p \to 0.$$

As Q/p is projective, there exists a retraction $\overline{\rho}\colon (A/p)^n \to K/p$ to $\overline{\iota}$. Consider the composition $A^n \to (A/p)^n \to K/p$, and lift it (arbitrarily) to a map $\rho\colon A^n \to K$. Note that $\rho \circ \iota$ is the identity modulo p. As K is p-adically complete (see [Mat80, (23.L), Corollary 2]) this implies that $\rho \circ \iota$ is an automorphism, with inverse $\sum_{k\geqslant 0}(-1)^k((\rho\circ\iota) - \mathrm{id})^k$. So then, $(\rho\circ\iota)^{-1}\circ\rho$ is a retraction to ι. Thus, P splits, and so Q is a direct summand of a free module, so projective. ☐

Suppose further that $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ is a weakly filtered F-crystal. To show that it is a strongly divisible F-crystal, it remains to show that $\mathrm{Fil}^\bullet_{\mathcal{F}}$ satisfies Griffiths transversality with respect to $\nabla_{\mathcal{F}}$. But, let us observe that by assumption, for each r we have that

$$\nabla_{\mathcal{F}}(\mathrm{Fil}^i_{\mathcal{F}}) \subseteq (\mathcal{F}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \cap (\mathrm{Fil}^{r-1}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K})[1/p].$$

But, the right-hand of this containment is just $\mathrm{Fil}^{r-1}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}$. Indeed, this follows from the observation that

$$(\mathcal{F}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K})/(\mathrm{Fil}^{r-1}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \cong (\mathcal{F}_{\mathfrak{X}}/\mathrm{Fil}^{r-1}_{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}$$

is p-torsion-free, being the tensor product of two vector bundles on $\mathfrak{X}$. ☐

The above observations also allow us to explain the relationship between the categories of naive filtered F-crystals we considered above, and the categories $\mathbf{MF}^{\nabla}_{[0,a]}(R)$ as in [Fal89, II.c) and p.34], when a in $[0, p-2]$.[19]

Proposition 3.20. *Let R be a base W-algebra. Then, for any a in $[0, p-2]$, the functor*

$$\mathbf{VectF}^{\varphi,\mathrm{div}}(R_{\mathrm{crys}}) \to \mathbf{MF}^{\nabla}_{[0,a]}(R), \qquad (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}}) \mapsto (\mathcal{F}(R), \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}}),$$

is an equivalence of categories.

The category $\mathbf{VectF}^{\nabla}(\mathfrak{X})$. We next discuss some results of Tsuji which allows one to understand the filtrations of some naive filtered F-crystals via the crystalline site.

First, we review some elementary terminology.

(1) A *filtered ring* is a pair $(R, \mathrm{Fil}^\bullet_R)$, where R is a ring and $\mathrm{Fil}^\bullet_R \subseteq R$ is filtration by ideals with $\mathrm{Fil}^0_R = R$ and $\mathrm{Fil}^a_R \cdot \mathrm{Fil}^b_R \subseteq \mathrm{Fil}^{a+b}_R$.
(2) A *filtered module* $(M, \mathrm{Fil}^\bullet_M)$ over $(R, \mathrm{Fil}^\bullet_R)$ consists of an R-module M, and a filtration $\mathrm{Fil}^\bullet_M$ by R-submodules such that $\mathrm{Fil}^a_R \cdot \mathrm{Fil}^b_M \subseteq \mathrm{Fil}^{a+b}_M$ for all a and b in $\mathbb{Z}$. With the obvious notion of morphisms, denote by $\mathbf{MF}(R, \mathrm{Fil}^\bullet_R)$ the category of filtered modules over $(R, \mathrm{Fil}^\bullet_R)$.
(3) A filtered ring map $f\colon (R, \mathrm{Fil}^\bullet_R) \to (S, \mathrm{Fil}^\bullet_S)$ is a ring map $f\colon R \to S$ with $f(\mathrm{Fil}^i_R) \subseteq \mathrm{Fil}^i_S$ for all i. We then define the *filtered tensor product/base change* functor

$$(-) \otimes_{(R,\mathrm{Fil}^\bullet_R)} (S, \mathrm{Fil}^\bullet_S)\colon \mathbf{MF}(R, \mathrm{Fil}^\bullet_R) \to \mathbf{MF}(S, \mathrm{Fil}^\bullet_S)$$

by equipping $M \otimes_R S$ with the filtration where

$$\mathrm{Fil}^r_{M\otimes_R S} := \sum_{a+b=r} \mathrm{im}(\mathrm{Fil}^a_M \otimes_R \mathrm{Fil}^b_S \to M \otimes_R S).$$

Moreover, the following freeness condition will play an important role in many of our proofs.

Definition 3.21 (cf. [Tsu20, Definition 10])**.** A filtered module $(M, \mathrm{Fil}^\bullet_M)$ over a filtered ring $(R, \mathrm{Fil}^\bullet_R)$ is *free* if there exists a *filtered basis*: a collection $(e_\nu, r_\nu)_{\nu=1}^n$ where $(e_\nu)_{\nu=1}^n$ is a basis of M as an R-module, and r_ν are integers, such that

$$\mathrm{Fil}^r_M = \sum_{\nu=1}^n \mathrm{Fil}^{r-r_\nu}_R \cdot e_\nu. \tag{3.2.3}$$

[19] In loc. cit. one considers categories $\mathbf{MF}^{\nabla}_{[0,a]}(R, \Phi)$ for some Frobenius lift Φ on R. But, as a is in $[0, p-2]$, this category is independent of Φ by [Fal89, Theorem 2.3].

We say that a filtered module $(M, \mathrm{Fil}^\bullet_M)$ over $(R, \mathrm{Fil}^\bullet_R)$ is *locally free* if there exists an open cover $\{\mathrm{Spec}(R_i)\}$ of $\mathrm{Spec}(R)$ such that filtered base change

$$(M, \mathrm{Fil}^\bullet_M) \otimes_{(R, \mathrm{Fil}^\bullet R)} (R_i, \mathrm{Fil}^\bullet_R \cdot R_i),$$

is free for all i. When $\mathrm{Fil}^\bullet_R = \mathrm{Fil}^\bullet_{\mathrm{triv}}$, it is simple to show that $(M, \mathrm{Fil}^\bullet_M)$ is (locally) free over $(R, \mathrm{Fil}^\bullet_{\mathrm{triv}})$ if and only if each graded piece of $\mathrm{Fil}^\bullet_M$ is a (locally) free R-module, i.e., that $\mathrm{Fil}^\bullet_M \subseteq M$ is a locally split filtration.

Basic properties concerning these notions can be found in [Tsu20, §3]. Moreover, loc. cit. explains that each of the above notions admit generalizations to modules over a ringed topos, which recovers the above objects when applied to the ringed topos $(\mathrm{Spec}(R)_{\mathrm{Zar}}, \mathcal{O}_{\mathrm{Spec}(R)})$.

Consider now a pair $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$, where $\mathcal{E}$ is an object of $\mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}})$, and $\mathrm{Fil}^\bullet_{\mathcal{E}} \subseteq \mathcal{E}$ is a filtration by locally quasi-coherent (see [SP, Tag 07IS]) $\mathcal{O}_{\mathrm{crys}}$-submodules. We say that $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$ is a *filtered crystal (of vector bundles)* if

(1) $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})(x)$ is a filtered module over $(A, \mathrm{Fil}^\bullet_{\mathrm{PD}})$ for all $x = (i\colon A \twoheadrightarrow B, \gamma)$ in $(\mathfrak{X}/W)_{\mathrm{crys}}$,
(2) for a morphism $x = (i\colon A \to B, \gamma) \to (i'\colon A' \to B', \gamma') = y$ in $(\mathfrak{X}/W)_{\mathrm{crys}}$, the natural map $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})(x) \otimes_{(A, \mathrm{Fil}^\bullet_{\mathrm{PD}}(A))} (A', \mathrm{Fil}^\bullet_{\mathrm{PD}}(A')) \to (\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})(y)$ is an isomorphism.

A morphism of filtered crystals $\alpha\colon (\mathcal{E}', \mathrm{Fil}^\bullet_{\mathcal{E}'}) \to (\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$ is a morphism $\alpha\colon \mathcal{E}' \to \mathcal{E}$ in $\mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}})$ such that $\alpha(\mathrm{Fil}^i_{\mathcal{E}}) \subseteq \mathrm{Fil}^i_{\mathcal{E}'}$ for all i. In other words, $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$ is (locally quasi-coherent) filtered module over $(\mathcal{O}_{\mathrm{crys}}, \mathrm{Fil}^\bullet_{\mathrm{PD}})$ with $\mathcal{E}$ a vector bundle. We will mostly be interested in the case when $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$ is locally free over $(\mathcal{O}_{\mathrm{crys}}, \mathrm{Fil}^\bullet_{\mathrm{PD}})$, and we denote the category of such by $\mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}})$.

On the other hand, let us define $\mathbf{VectF}^\nabla(\mathfrak{X})$ to consist of triples $(\mathcal{V}, \nabla_{\mathcal{V}}, \mathrm{Fil}^\bullet_{\mathcal{V}})$ where $(\mathcal{V}, \nabla_{\mathcal{V}})$ is an object $\mathbf{Vect}^\nabla(\mathfrak{X})$ and $\mathrm{Fil}^\bullet_{\mathcal{V}}$ is a locally split filtration on $\mathcal{V}$ which satisfies the Griffiths transversality condition: $\nabla_{\mathcal{V}}(\mathrm{Fil}^i_{\mathcal{V}}) \subseteq \mathrm{Fil}^{i-1}_{\mathcal{V}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/W}$, for all $i \geqslant 1$.

We then have the following result of Tsuji.[20]

Proposition 3.22 (cf. [Tsu20, Theorem 29])**.** *The functor*

$$\mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{VectF}^\nabla(\mathfrak{X}), \qquad \mathcal{E} \mapsto (\mathcal{E}_{\mathfrak{X}}, \nabla_{\mathcal{E}}, (\mathrm{Fil}^\bullet_{\mathcal{E}})_{\mathfrak{X}}),$$

is an equivalence.

Remark 3.23. Recall here that by definition, $\mathcal{E}_{\mathfrak{X}}$ has value on an open $\mathfrak{U} \subseteq \mathfrak{X}$ given by $\mathcal{E}(\mathrm{id}\colon \mathfrak{U} \to \mathfrak{U}, \gamma)$. In particular, while the value of $\mathcal{E}$ on $(\mathrm{id}\colon \mathfrak{U} \to \mathfrak{U}, \gamma)$ is the same as that on $(\mathfrak{U}_0 \hookrightarrow \mathfrak{U}, \gamma)$, this is not true for $\mathrm{Fil}^\bullet_{\mathcal{E}}$, as the former is a filtered module over $(\mathcal{O}_{\mathfrak{X}}, \mathrm{Fil}^\bullet_{\mathrm{triv}})$, while the latter is a filtered module over $(\mathcal{O}_{\mathfrak{X}}, \mathrm{Fil}^\bullet_{\mathrm{PD}})$.

Note that, by definition, we have

$$\mathbf{VectF}^\varphi(\mathfrak{X}_{\mathrm{crys}}) = \mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \times_{\mathbf{Vect}^\nabla(\mathfrak{X})} \mathbf{VectF}^\nabla(\mathfrak{X}).$$

Thus, by Proposition 3.22, we obtain an equivalence of categories

$$\mathbf{VectF}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \xrightarrow{\sim} \mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \times_{\mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}})} \mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}}).$$

For this reason, we will often consider the evaluation of $\mathrm{Fil}^\bullet_{\mathcal{F}}$ at objects of $(\mathfrak{X}/W)_{\mathrm{crys}}$, by which we mean the evaluation of associated object of $\mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}})$.

3.2.2. *The functor* $\mathbb{D}_{\mathrm{crys}}$. We now apply the crystalline-de Rham comparison theorem to define our integral analogue $\mathbb{D}_{\mathrm{crys}}$ of the functor D_{crys}. Throughout we assume that $K = K_0$.

[20]See [Tsu20, Remark 19] for the relationship to previous work of Ogus.

The definition. We now come to the definition of the naive filtered F-crystal $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$.

Definition 3.24. Let $(\mathcal{E},\varphi_{\mathcal{E}})$ be a prismatic F-crystal on $\mathfrak{X}$. We define $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ to have underlying F-crystal $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ and to have the filtration

$$\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}}) \supseteq \mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{crys}}}(\mathcal{E},\varphi_{\mathcal{E}}) := \iota_{\mathfrak{X}}^{-1}(\mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{dR}}}(\mathcal{E},\varphi_{\mathcal{E}})).$$

When $(\mathcal{E},\varphi_{\mathcal{E}})$ is clear from context, we often omit it from the notation, writing $\mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{crys}}}$ instead. We observe the following simple fact.

Proposition 3.25. *The functor*

$$\mathbb{D}_{\mathrm{crys}} \colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}),$$

is a $\mathbb{Z}_p$-linear $\otimes$-functor, which preserves duals, and maps $\mathbf{Vect}^{\varphi}_{[0,a]}(\mathfrak{X}_{\Delta})$ into $\mathbf{VectNF}^{\varphi}_{[0,a]}(\mathfrak{X}_{\mathrm{crys}})$.

In the rest of this subsection, we shall sort out what extra properties the naive filtered F-crystal $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ and the functor $\mathbb{D}_{\mathrm{crys}}$ possesses when various assumptions are made on $\mathfrak{X}$ and $(\mathcal{E},\varphi_{\mathcal{E}})$. In particular, we aim to single out a good subcategory of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ whose essential image under $\mathbb{D}_{\mathrm{crys}}$ lies inside of $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$.

Comparison to D_{crys} and Griffiths transversality. We first show that in the situation when $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth, that $\mathbb{D}_{\mathrm{crys}}$ agrees rationally with D_{crys}, with no assumptions on the prismatic F-crystal, thus justifying that it is an integral analogue of D_{crys}.

Proposition 3.26. *If $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth, then $\mathbb{D}_{\mathrm{crys}}$ takes values in $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$, and the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \arrow[r, "\mathbb{D}_{\mathrm{crys}}"] \arrow[d, "T_{\mathrm{\acute{e}t}}"'] & \mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}) \arrow[d, "(-)[1/p]"] \\
\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X) \arrow[r, "D_{\mathrm{crys}}"] & \mathbf{IsocF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}).
\end{tikzcd}$$

Obviously this latter commutativity implies the former claim, as D_{crys} takes values in $\mathbf{IsocF}^{\varphi}(\mathfrak{X})$. Thus, we shall focus entirely on this latter commutativity.

To prove Proposition 3.26, we first need some setup. To describe this, we first set some notational shorthand. First, shorten the notation for the composition $D_{\mathrm{crys}} \circ T_{\mathrm{\acute{e}t}}$ to just D_{crys}. Let R be a framed small W-algebra. Then, we further set

- $\mathrm{A}_{\mathrm{inf}} := \mathrm{A}_{\mathrm{inf}}(\widetilde{R})$,
- $\mathrm{A}_{\mathrm{crys}} := \mathrm{A}_{\mathrm{crys}}(\widetilde{R})$,
- $\widetilde{\mathrm{A}}_{\mathrm{crys}} := \mathrm{A}_{\mathrm{inf}}[\tilde{\xi}^n/n!]^\wedge_p$,
- $\mathrm{B}^+_{\mathrm{dR}} := \mathrm{B}^+_{\mathrm{dR}}(\widetilde{R})$,
- $\widetilde{\mathrm{B}}^+_{\mathrm{dR}} := \mathrm{A}_{\mathrm{inf}}[1/p]^\wedge_{(\tilde{\xi})}$.

These rings are arranged in the following diagram

$$\begin{tikzcd}
 & \mathrm{A}_{\mathrm{crys}} \arrow[r, "\phi"] \arrow[dl] \arrow[dr, "\widetilde{\psi}"'] \arrow[rrr, bend left, "\psi"] & \widetilde{\mathrm{A}}_{\mathrm{crys}} \arrow[r, hook] \arrow[d, hook] & \mathrm{A}_{\mathrm{crys}} \arrow[r, hook] & \mathrm{B}^+_{\mathrm{dR}} \\
\mathrm{B}^+_{\mathrm{dR}} \arrow[rr, "\phi_{\mathrm{dR}}"'] & & \widetilde{\mathrm{B}}^+_{\mathrm{dR}}. & &
\end{tikzcd}$$

Here we use the following notation:

- here $\widetilde{\mathrm{A}}_{\mathrm{crys}} \hookrightarrow \mathrm{A}_{\mathrm{crys}}$ is the natural inclusion,
- $\phi_{\mathrm{dR}} \colon \mathrm{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}$ and $\phi \colon \mathrm{A}_{\mathrm{crys}} \xrightarrow{\sim} \widetilde{\mathrm{A}}_{\mathrm{crys}}$ are the maps induced by $\phi \colon \mathrm{A}_{\mathrm{inf}} \to \mathrm{A}_{\mathrm{inf}}$,
- and the maps ψ and $\widetilde{\psi}$ are defined uniquely to make the diagram commute.

Set $\widetilde{\mathrm{B}}_{\mathrm{dR}} := \mathrm{A}_{\mathrm{inf}}[1/p]^{\wedge}_{(\tilde{\xi})}[1/\tilde{\xi}]$. Then, a *$\varphi$-module* over $\mathrm{B}^+_{\mathrm{dR}}$ is a triple $(M, \widetilde{M}, \varphi_M)$ with M (resp. $\widetilde{M}$) a finitely generated projective $\mathrm{B}^+_{\mathrm{dR}}$-module (resp. $\widetilde{\mathrm{B}}^+_{\mathrm{dR}}$-module) and φ_M a $\widetilde{\mathrm{B}}_{\mathrm{dR}}$-linear isomorphism $(\phi^*_{\mathrm{dR}}M)[1/\tilde{\xi}] \to \widetilde{M}[1/\tilde{\xi}]$. With the obvious notion of morphism, let us denote the category of φ-modules over $\mathrm{B}^+_{\mathrm{dR}}$ by $\mathbf{Vect}^{\varphi}(\mathrm{B}^+_{\mathrm{dR}})$.

We now observe that there is a natural functor

$$\mathcal{M}\colon \mathbf{IsocF}^{\varphi}(R) \to \mathbf{Vect}^{\varphi}(\mathrm{B}^+_{\mathrm{dR}}), \quad (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_F) \mapsto (\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}},\psi} \mathrm{B}^+_{\mathrm{dR}}, \widetilde{\mathrm{Fil}}^0_F, \varphi_{\mathcal{F}} \otimes 1).$$

Here we define the filtration $\widetilde{\mathrm{Fil}}^{\bullet}_F$ on $\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}},\tilde{\psi}} \widetilde{\mathrm{B}}_{\mathrm{dR}}$ as follows:

$$\widetilde{\mathrm{Fil}}^r_F = \sum_{i+j=r} \mathrm{Fil}^i_F(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}},\tilde{\psi}} \tilde{\xi}^j \widetilde{\mathrm{B}}^+_{\mathrm{dR}}. \tag{3.2.4}$$

Note that the Frobenius map $\varphi_{\mathcal{F}} \otimes 1$ is sensible as $\widetilde{\mathrm{Fil}}^0_F$ is a lattice in $\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \check{R}) \otimes_{\mathrm{A}_{\mathrm{crys}},\tilde{\psi}} \widetilde{\mathrm{B}}_{\mathrm{dR}}$. On the other hand, consider the functor

$$\mathrm{Nyg}_{\mathrm{dR}}\colon \mathbf{Vect}^{\varphi}(\mathrm{B}^+_{\mathrm{dR}}) \to \mathbf{MF}(\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^{\bullet}_{\tilde{\xi}}), \quad (M, \widetilde{M}, \varphi_M) \mapsto (\phi^*_{\mathrm{dR}}M, \mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}}M)),$$

where $\mathrm{Fil}^r_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}}M) := \phi^*_{\mathrm{dR}}M \cap \varphi_M^{-1}(\tilde{\xi}^r \widetilde{M})$. Then we have the following result.

Lemma 3.27. *The following diagram commutes.*

$$\begin{tikzcd}
\mathbf{Vect}^{\varphi}(R_{\mathbb{\Delta}}) \arrow[r, "D_{\mathrm{crys}}"] \arrow[d, "\mathrm{ev}_{\mathrm{A}_{\mathrm{inf}}}"'] & \mathbf{IsocF}^{\varphi}(R) \arrow[r] \arrow[d, "\mathcal{M}"] & \mathbf{MF}(\mathrm{A}_{\mathrm{crys}}[1/p], \mathrm{Fil}^{\bullet}_{\xi}) \arrow[d] \\
\mathbf{Vect}^{\varphi}(\mathrm{A}_{\mathrm{inf}}) \arrow[r, "-\otimes_{\mathrm{A}_{\mathrm{inf}}} \mathrm{B}^+_{\mathrm{dR}}"] & \mathbf{Vect}^{\varphi}(\mathrm{B}^+_{\mathrm{dR}}) \arrow[r, "\mathrm{Nyg}_{\mathrm{dR}}"] & \mathbf{MF}(\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^{\bullet}_{\tilde{\xi}}).
\end{tikzcd}$$

The top-right arrow is the evaluation of $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_F)$ at $\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}$ and the right vertical arrow is scalar extension along the map of filtered rings $\tilde{\psi}\colon (\mathrm{A}_{\mathrm{crys}}[1/p], \mathrm{Fil}^{\bullet}_{\xi}) \to (\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^{\bullet}_{\tilde{\xi}})$.

Proof of Lemma 3.27. The left square commutes by the proof of [GR22, Theorem 4.8] (cf. the proof of Lemma 2.25). Let $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_F)$ be an object of $\mathbf{IsocF}^{\varphi}(R_{\mathrm{crys}})$. As the category $\mathbf{MF}(\widetilde{B}^+_{\mathrm{dR}}, \mathrm{Fil}^{\bullet}_{\tilde{\xi}})$ satisfies descent (cf. [SW20, Corollary 17.1.9]), to show the commutativity of the right square, we are free to localize on $\mathrm{Spf}(R)$. But, note that as the graded pieces of the filtration on F are locally free, we may localize on $\mathrm{Spf}(R)$ to assume that they are free.[21] Note that this condition implies that the evaluation of $(\mathcal{F}, \mathrm{Fil}^{\bullet}_F)$ at any object $A \twoheadrightarrow R'$ of $(R/W)_{\mathrm{crys}}$ admitting a map to $\mathrm{id}_R\colon R \to R$ is free as a filtered module over $(A[1/p], \mathrm{Fil}^{\bullet}_{\mathrm{PD}}[1/p])$ in the sense of Definition 3.21.

In particular, take a filtered basis $(e_\nu, r_\nu)_{\nu=1}^n$ of $\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R})[1/p]$. Then, $\mathcal{M}(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_F) = (M, \widetilde{M}, \varphi_M)$ where:

$$M = \mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}},\psi} \mathrm{B}^+_{\mathrm{dR}}, \quad \widetilde{M} = \sum_{\nu=1}^n \tilde{\xi}^{-r_\nu} \widetilde{\mathrm{B}}^+_{\mathrm{dR}} \cdot e_\nu \subset \bigoplus_{\nu=1}^n \widetilde{\mathrm{B}}_{\mathrm{dR}} \cdot e_\nu,$$

and φ_M is the scalar extension of $\varphi_{\mathcal{F}}$. Thus, the Nygaard filtration on $\phi^*_{\mathrm{dR}}M$ is

$$\mathrm{Fil}^r_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}}M) = \phi^*_{\mathrm{dR}}M \cap \varphi_M^{-1}\left(\sum_{\nu=1}^n \tilde{\xi}^{r-r_\nu} \widetilde{\mathrm{B}}^+_{\mathrm{dR}} \cdot e_\nu\right).$$

[21] Indeed, suppose that $\mathcal{E}$ is a vector bundle on $\mathrm{Spa}(R[1/p])$. Then, $\mathcal{E}(R[1/p])$ is a projective $R[1/p]$-module (see [Ked19, Theorem 1.4.2], and [Kie67]). Thus, there exists an open cover $\mathrm{Spec}(R[1/f_i p])$ of $\mathrm{Spec}(R[1/p])$ such that $\mathcal{E}(R[1/p]) \otimes_{R[1/p]} R[1/pf_i]$ is trivial, where f_i is a collection of elements of R. So, replacing $\mathrm{Spf}(R)$ by $\mathrm{Spf}(R[1/f_i]^{\wedge}_p)$, one may Zariski localize on $\mathrm{Spf}(R)$ to assume that $\mathcal{E}$ is free.

On the other hand, if the object $(M', \mathrm{Fil}^\bullet_{M'})$ denotes the image under the other composition in the right-hand square, then it may be described as follows:

$$M' = \mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}},\widetilde{\psi}} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \qquad \mathrm{Fil}^r_{M'} = \sum_{\nu=1}^{n} \mathrm{Fil}^{r-r_\nu}_{\tilde{\xi}} \cdot e_\nu.$$

Thus, φ_M induces an isomorphism $(\phi^*_{\mathrm{dR}} M, \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}} M)) \xrightarrow{\sim} (M', \mathrm{Fil}^\bullet_{M'})$. ◻

Proof of Proposition 3.26. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta})$. We will show that there is a canonical isomorphism $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})[1/p] \xrightarrow{\sim} \underline{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ in $\mathbf{Isoc}^\varphi(\mathfrak{X})$ respecting filtrations. To prove the existence of this canonical isomorphism, note that by the proof of [GR22, Theorem 4.8], the underlying F-isocrystals are identified, and so it suffices to show that the filtrations are matched. So, we may assume that $\mathfrak{X} = \mathrm{Spf}(R)$ where R is a framed small W-algebra. For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^\varphi(R_{\Delta})$, let $\mathbb{D}_{\mathrm{crys},R}[1/p](\mathcal{E}, \varphi_{\mathcal{E}})$ (resp. $\underline{\mathbb{D}}_{\mathrm{crys},R}[1/p](\mathcal{E}, \varphi_{\mathcal{E}})$) denote the filtered $R[1/p]$-module $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(R)[1/p]$ (resp. its underlying $R[1/p]$-module). Define $D_{\mathrm{crys},R}(\mathcal{E}, \varphi_{\mathcal{E}})$ and $\underline{D}_{\mathrm{crys},R}(\mathcal{E}, \varphi_{\mathcal{E}})$ similarly.

Define the functor

$$\begin{aligned} \mathrm{Nyg}_{\mathfrak{S}_R} \colon \mathbf{Vect}^\varphi(\mathfrak{S}_R, (E)) &\to \mathbf{MF}(\mathfrak{S}_R, \mathrm{Fil}^\bullet_E), \\ (\mathfrak{M}, \varphi_{\mathfrak{M}}) &\mapsto (\phi^*\mathfrak{M}, \mathrm{Fil}^\bullet_{\mathrm{Nyg}}). \end{aligned}$$

Then we have the following diagram.

$$\begin{tikzcd}
\mathbf{Vect}^\varphi(R_{\Delta}) \arrow[rr, "{\mathbb{D}_{\mathrm{crys},R}[1/p]}"] \arrow[d, "\mathrm{ev}_{\mathfrak{S}_R}"] & & \mathbf{MF}(R[1/p], \mathrm{Fil}^\bullet_{\mathrm{triv}}) \arrow[d, equal] \\
\mathbf{Vect}^\varphi(\mathfrak{S}_R) \arrow[r, "\mathrm{Nyg}_{\mathfrak{S}_R}"] \arrow[d, "(-)\otimes_{\mathfrak{S}_R}\mathrm{B}^+_{\mathrm{dR}}"] & \mathbf{MF}(\mathfrak{S}_R, \mathrm{Fil}^\bullet_E) \arrow[r, "{(-)\otimes_{\mathfrak{S}_R} R[1/p]}"] \arrow[d, "(-)\otimes_{\mathfrak{S}_R}\widetilde{\mathrm{B}}^+_{\mathrm{dR}}"] & \mathbf{MF}(R[1/p], \mathrm{Fil}^\bullet_{\mathrm{triv}}) \arrow[d, "(-)\otimes_R \widetilde{R}"] \\
\mathbf{Vect}^\varphi(\mathrm{B}^+_{\mathrm{dR}}) \arrow[r, "\mathrm{Nyg}_{\mathrm{dR}}"] & \mathbf{MF}(\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^\bullet_{\tilde{\xi}}) \arrow[r, "\bmod \tilde{\xi}"] & \mathbf{MF}(\widetilde{R}[1/p], \mathrm{Fil}^\bullet_{\mathrm{triv}}).
\end{tikzcd}$$

Here $\mathfrak{S}_R \to \mathrm{B}^+_{\mathrm{dR}}$ (resp. $\mathfrak{S}_R \to \widetilde{B}^+_{\mathrm{dR}}$) is the composition $\mathfrak{S}_R \xrightarrow{\alpha_{\mathrm{crys}}} \mathrm{A}_{\mathrm{crys}} \xrightarrow{\psi} \mathrm{B}^+_{\mathrm{dR}}$ (resp. $\mathfrak{S}_R \xrightarrow{\alpha_{\mathrm{crys}}} \mathrm{A}_{\mathrm{crys}} \xrightarrow{\widetilde{\psi}} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}$).

The upper rectangle and the right lower square commute by definition. Noting that the map $\mathfrak{S}_R \to \mathrm{B}^+_{\mathrm{dR}}$ is flat as $\mathfrak{S}_R$ is noetherian and it is E-adically flat (indeed, the image of E, which is ξ is a nonzerodivisor and the map is flat mod E by Lemma 1.14), we get that the left lower square commutes as well. Thus, for an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^\varphi(R_{\Delta})$, we have a canonical identification

$$\mathbb{D}_{\mathrm{crys},R}[1/p](\mathcal{E}, \varphi_{\mathcal{E}}) \otimes_R \widetilde{R} \cong \mathrm{Nyg}_{\mathrm{dR}}\left(\mathcal{E}(\mathfrak{S}_R, (E)) \otimes_{\mathfrak{S}_R} \mathrm{B}^+_{\mathrm{dR}}\right) \otimes_{\widetilde{\mathrm{B}}^+_{\mathrm{dR}}} \widetilde{R}[1/p]$$

On the other hand, by Lemma 3.27, we also have a canonical identification

$$D_{\mathrm{crys},R}(\mathcal{E}, \varphi_{\mathcal{E}}) \otimes_R \widetilde{R} \cong \mathrm{Nyg}_{\mathrm{dR}}\left(\mathcal{E}(\mathfrak{S}_R, (E)) \otimes_{\mathfrak{S}_R} \mathrm{B}^+_{\mathrm{dR}}\right) \otimes_{\widetilde{\mathrm{B}}^+_{\mathrm{dR}}} \widetilde{R}[1/p].$$

These identifications in particular induce an isomorphism $\underline{\mathbb{D}}_{\mathrm{crys},R}[1/p](\mathcal{E}) \otimes_R \widetilde{R} \xrightarrow{\sim} \underline{D}_{\mathrm{crys},R}(\mathcal{E}) \otimes_R \widetilde{R}$ which agrees with that obtained from [GR22, Theorem 4.8]. Thus, $\underline{\mathbb{D}}_{\mathrm{crys},R}[1/p](\mathcal{E}) \xrightarrow{\sim} \underline{D}_{\mathrm{crys},R}(\mathcal{E})$ preserves filtrations after base change along the faithfully flat map $R[1/p] \to \widetilde{R}[1/p]$ (see Lemma 1.14), and thus preserves filtrations. ◻

Example 3.28. Similar to Example 3.5, when $\mathfrak{X} = \mathrm{Spf}(W)$, we define

$$\mathbb{D}_{\mathrm{crys}} \colon \mathbf{Rep}^{\mathrm{crys}}(\Gamma_{K_0}) \to \mathbf{VectWF}^{\varphi,\mathrm{div}}(W), \qquad \Lambda \mapsto \mathbb{D}_{\mathrm{crys}}(T^{-1}_{\mathrm{Spf}(W)}(\Lambda)).$$

By Proposition 3.26, we have an identification of filtered F-isocrystals $\mathbb{D}_{\mathrm{crys}}(\Lambda)[1/p] \xrightarrow{\sim} D_{\mathrm{crys}}(\Lambda)$. By Remark 3.12, this agrees with the composition

$$\underline{\mathbb{D}}_{\mathrm{crys}}(\Lambda)[1/p] \xrightarrow{\sim} \phi^*(\mathfrak{M}(\Lambda))/(u)[1/p] \xrightarrow{\sim} (\mathfrak{M}(\Lambda)/(u))[1/p] \xrightarrow{\sim} D_{\mathrm{crys}}(\Lambda),$$

where the first isomorphism is that in Example 3.5, the second is from the Frobenius structure, and the last is from the definition of $\mathfrak{M}$ (see [Kis10, Theorem (1.2.1) (1)]).

Locally filtered free prismatic F-crystals. We now single out the conditions on a prismatic F-crystal such that its associated naive filtered F-crystal is a filtered F-crystal.

The main content of this is the following omnibus criterion, relating various notions of when a prismatic F-crystal over a base formal scheme may be considered to have a 'locally free filtration'.

Proposition 3.29. *Let $\mathfrak{X}$ be a base formal W-scheme, and $(\mathcal{E}, \varphi_{\mathcal{E}})$ a prismatic F-crystal on $\mathfrak{X}$. Let $\{\mathrm{Spf}(R_i)\}$ an open cover with each R_i a base (formally framed) W-algebra, and for each i let $\mathfrak{M}_i := \mathcal{E}(\mathfrak{S}_{R_i}, (E))$. Then, the following conditions are equivalent:*

1. *the filtration $\mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*\mathcal{E}) \subseteq \phi^*\mathcal{E}$ is locally free over $(\mathcal{O}_{\Delta}, \mathrm{Fil}^\bullet_{\mathcal{J}_{\Delta}})$,*
2. *the filtration $\mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*\mathfrak{M}_i)$ is locally free over $(\mathfrak{S}_{R_i}, \mathrm{Fil}^\bullet_E)$,*
3. *the filtration $\mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{crys}}} \subseteq \mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ is locally free over $(\mathcal{O}_{\mathfrak{X}}, \mathrm{Fil}^\bullet_{\mathrm{triv}})$,*
4. *and $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ is an object of $\mathbf{VectNF}^{\varphi,\mathrm{div},\mathrm{gtf}}(\mathfrak{X}_{\mathrm{crys}})$.*

If $\mathfrak{X} \to \mathrm{Spf}(W)$ is smooth, then these conditions are further equivalent to

5. *$\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ is an object of $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$,*

We begin by establishing the following lemma, showing that we may lift freeness of the quotient Nygaard filtration to the actual Nygaard filtration.

Lemma 3.30. *Let $(A, (d))$ be a bounded prism, and (M, φ_M) an object of $\mathbf{Vect}^{\varphi}(A, (d))$ with M free over A. Set $\mathrm{Fil}^\bullet := \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*M)$ and $\overline{\mathrm{Fil}}^\bullet := \overline{\mathrm{Fil}}^\bullet_{\mathrm{Nyg}}(\phi^*(M)/(d))$. Let $(f_\nu, r_\nu)_{\nu=1}^n$ be a filtered basis of $(\phi^*(M)/(d), \overline{\mathrm{Fil}}^\bullet)$ over $(A/(d), \mathrm{Fil}^\bullet_{\mathrm{triv}})$. Choose e_ν in Fil^{r_ν} such that $\overline{e_\nu} = f_\nu$ (which exist as f_ν is in $\overline{\mathrm{Fil}}^{r_\nu}$). Then, the following is true:*

1. *$(e_\nu, r_\nu)_{\nu=1}^n$ is a filtered basis of $(\phi^*M, \mathrm{Fil}^\bullet)$ over $(A, \mathrm{Fil}^\bullet_d)$,*
2. *$\varphi_M(e_\nu)$ is in $d^{r_\nu}M$, and $(\frac{\varphi_M(e_\nu)}{d^{r_\nu}})_\nu$ is a basis of M.*

Proof. To see that (e_ν) is a basis of ϕ^*M we observe that the map $A^n \to \phi^*M$ sending $(a_1, \ldots, a_n)$ to $\sum_{\nu=1}^n a_\nu e_\nu$ is surjective modulo d by assumption, and so surjective by Nakayama's lemma as A is d-adically complete. It is then an isomorphism as the source and target are finite projective A-modules of the same rank.

Now, by a standard twisting argument (using the Breuil–Kisin twist as in [BS23, Example 4.5]), we may assume that (M, φ_M) is effective . To prove the first assertion, we check by induction on $r \geqslant 0$ that the equality in (3.2.3) holds. When $r = 0$, the assertion is obvious as both sides are the entirety of ϕ^*M. Assume the assertion holds for r, so that we have

$$\mathrm{Fil}^r = \sum_{\nu\,:\,r\geqslant r_\nu} d^{r-r_\nu}A \cdot e_\nu + \sum_{\nu\,:\,r<r_\nu} A \cdot e_\nu.$$

Note that the second term $\sum_{\nu\,:\,r<r_\nu} A \cdot e_\nu$ is included in Fil^{r+1} by our assumptions. Let x be an element of Fil^r of the form $\sum_{\nu\,:\,r\geqslant r_\nu} d^{r-r_\nu}a_\nu e_\nu + \sum_{\nu\,:\,r<r_\nu} a_\nu e_\nu$. As the second term is contained in Fil^{r+1}, the element x is in Fil^{r+1} if and only if the first term $x_1 = \sum_{\nu\,:\,r\geqslant r_\nu} d^{r-r_\nu}a_\nu e_\nu$ is in Fil^{r+1}. If a_ν is divisible by d for every ν with r at least r_ν, then $\varphi_M(x_1) = d \cdot \varphi_M(\sum_\nu d^{r-r_\nu}(\frac{a_\nu}{d})e_\nu)$ belongs to $d \cdot d^rM$, and hence x_1 is in Fil^{r+1} by definition. On the other hand, if x_1 belongs to Fil^{r+1}, then its image in $\phi^*M/(d)$ is in $\overline{\mathrm{Fil}}^{r+1}$, and hence a_ν is divisible by d for ν with $r = r_\nu$. Then, $\sum_{\nu\,:\,r>r_\nu} d^{r-r_\nu}a_\nu e_\nu$ is in Fil^{r+1}, which implies that $\sum_{\nu\,:\,r>r_\nu} d^{r-r_\nu-1}a_\nu e_\nu$ is in Fil^r. Thus, by the induction hypothesis, we get a_ν is divisible by d. This proves the assertion for $r+1$.

We now prove the second assertion. We take an integer r large enough so that $r \geq r_\nu$ for any ν and that $\varphi_M^{-1}(d^rM) \subset \phi^*M$. In particular, φ_M induces $\mathrm{Fil}^r = \sum_\nu d^{r-r_\nu}A \cdot e_\nu \xrightarrow{\sim} d^rM$, which implies that $d^{r-r_\nu}\varphi(e_\nu)$ forms a basis of d^rM. Dividing it by d^r, we obtain assertion (2). □

We next observe that for such 'filtered free' prismatic F-crystals $(\mathcal{E},\varphi_{\mathcal{E}})$, the strong divisibility condition on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is automatic. To make this precise, fix a base W-algebra R. We define the category $\mathbf{Vect}^{\varphi}_{\mathrm{free}}(R_{\Delta})$ to be the full subcategory $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ consisting of those $(\mathcal{E},\varphi_{\mathcal{E}})$ such that $(\mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}}),\mathrm{Fil}^{\bullet}_{\mathbb{D}_{\mathrm{dR}}}(\mathcal{E},\varphi_{\mathcal{E}}))$ (equiv. $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$) is free over $(R,\mathrm{Fil}^{\bullet}_{\mathrm{triv}})$ (i.e., the graded pieces are free).

Lemma 3.31. *Let R be a base W-algebra and $(\mathcal{E},\varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi}_{\mathrm{free}}(R_{\Delta})$. Then, $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is strongly divisible.*

Proof. Write $M=\mathcal{E}(\mathfrak{S}_R,(E))$. By construction of $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$, we see that $\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}(\phi^*M/(E))$ is free. Choose a filtered basis $(\overline{e_\nu},r_\nu)$ of $\phi^*M/(E)$ over $(R,\mathrm{Fil}^{\bullet}_{\mathrm{triv}})$. Then, we have the filtered basis of $(\iota^{-1}(\overline{e_\nu}),r_\nu)$ of $\phi_R^*\mathcal{E}(R,(p))$, where ι is the isomorphism as in (3.1.3). Unraveling the definitions, we must show that

$$\sum_{r\in\mathbb{Z}}\sum_{\nu=1}^{n}p^{-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))=\phi_R^*\mathcal{E}(R,(p)).$$

But, observe that by Lemma 3.30 we have that $\varphi_{\mathcal{E}}(e_\nu)\in E^{r_\nu}M$, and that $\frac{\varphi_{\mathcal{E}}(e_\nu)}{E^{r_\nu}}$ is a basis of M. Pulling back along $\phi_{\mathfrak{S}_R}$ we see that $\phi_{\mathfrak{S}_R}^*(e_\nu)$ is a basis of $(\phi_{\mathfrak{S}_R}^2)^*M$, that $\varphi_{\mathcal{E}}(\phi_{\mathfrak{S}_R}^*(e_\nu))$ belongs to $\phi_{\mathfrak{S}_R}(E)^{r_\nu}\phi_{\mathfrak{S}_R}^*M$, and that $\frac{\varphi_{\mathcal{E}}(\phi_{\mathfrak{S}_R}^*(e_\nu))}{\phi_{\mathfrak{S}_R}(E)^{r_\nu}}$ is a basis of $\phi_{\mathfrak{S}_R}^*M$.

Now, recall that from the crystal property we have a diagram of isomorphisms

$$\begin{tikzcd}
(\phi_R^2)^*\mathcal{E}(R,(p))\otimes_R S_R \arrow[r, "\sim"] \arrow[d, dotted, "\varphi_{\mathcal{E}}"] & \phi_{S_R}^*\mathcal{E}(S_R,(p)) \arrow[d, dotted, "\varphi_{\mathcal{E}}"] & (\phi_{\mathfrak{S}_R}^2)^*M\otimes_{\mathfrak{S}_R}S_R \arrow[l, "\sim"'] \arrow[d, dotted, "\varphi_{\mathcal{E}}"] \\
\phi_R^*\mathcal{E}(R,(p))\otimes_R S_R \arrow[r, "\sim"'] & \mathcal{E}(S_R,(p)) & \phi_{\mathfrak{S}_R}^*M\otimes_{\mathfrak{S}_R}S_R. \arrow[l, "\sim"]
\end{tikzcd}$$

The dotted arrow indicates that these arrows only exist after inverting p. Moreover, the induced isomorphisms between the outer objects of the first (resp. second) row is precisely the isomorphism $\phi_R^*(\iota)$ (resp. ι). As $(\phi_{\mathfrak{S}_R}(E))=(p)$ in S_R, we deduce from this diagram and the contents of the previous paragraph that $\phi_R^*(\iota^{-1}(\overline{e_\nu}))$ is a basis of $(\phi_R^2)^*\mathcal{E}(R,(p))$, that $\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))$ belongs to $p^{r_\nu}\phi_R^*\mathcal{E}(R,(p))$, and that $\frac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}$ is a basis of $\phi^*\mathcal{E}(R,(p))$. Thus, we see that

$$\begin{aligned}
\sum_{r\in\mathbb{Z}}\sum_{\nu=1}^{n}p^{-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu}))) &= \sum_{r\in\mathbb{Z}}\sum_{\nu=1}^{n}p^{r_\nu-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\frac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}\\
&= \sum_{\nu=1}^{n}\sum_{r\in\mathbb{Z}}p^{r_\nu-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\frac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}\\
&= \sum_{\nu=1}^{n}R\cdot\frac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}\\
&= \phi_R^*\mathcal{E}(R,(p)),
\end{aligned}$$

as desired. □

We are now ready to prove Proposition 3.29.

Proof of Proposition 3.29. The equivalence of (1) and (2) follows by combining Proposition 1.15 and [Ito23b, Proposition 3.1.9]. The equivalence of (2) and (3) follows from Lemma 3.30. To show that (3) implies (4), it suffices to consider a cover $\{\mathrm{Spf}(R_i)\}$ of $\mathfrak{X}$ where each R_i is a base W-algebra, and the restriction of $(\mathcal{E},\varphi_{\mathcal{E}})$ to $(R_i)_{\Delta}$ belongs to $\mathbf{Vect}^{\varphi}((R_i)_{\Delta})$. The desired implication then follows from Lemma 3.31. That (4) implies (3) is given by Proposition 3.18. Finally, when $\mathfrak{X}\to\mathrm{Spf}(\mathcal{O}_K)$ is smooth, the equivalence of (4) and (5) follows by combining Proposition 3.18 and Proposition 3.26. □

We now codify this notion of 'locally free filtration' as follows.

Definition 3.32. Let $\mathfrak{X}$ be a base (resp. smooth) formal W-scheme. We call a prismatic F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ on $\mathfrak{X}$ *locally filtered free* if it satisfies the equivalent conditions (1)–(4) (resp. (1)–(5)) of Proposition 3.29. We denote by $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ the full subcategory of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ consisting of locally filtered free prismatic F-crystals.

Remark 3.33. The notion of lff F-crystals was originally included as a technical assumption needed to make various results work, notably Proposition 3.36. After the first draft of this article the authors realized that such a notion is quite natural from the perspective of prismatic F-gauges (see [Bha23] and [GL23]). Indeed, being lff is precisely the condition necessary to upgrade a prismatic F-crystal (in vector bundles) to a prismatic F-gauge in vector bundles (see Remark 2).

Example 3.34. Let $\mathfrak{X}$ be a base formal W-scheme. Then, $\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,1]}(\mathfrak{X}_{\Delta}) = \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{X}_{\Delta})$. In fact, a stronger claim holds: for any object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{X}_{\Delta})$, the conditions (1)–(5) from Proposition 3.29 holds for $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$.

Indeed, Condition (2) holds by [Ito23b, Proposition 3.1.6]. Proposition 3.29 then implies that all of conditions (1)–(4) hold, and so it suffices to show that $\mathrm{Fil}^{\bullet}_{\mathcal{F}}$ satisfies Griffiths transversality, but this is tautological as the filtration is concentrated in $[0, 1]$.

Example 3.35. Let $\mathfrak{X}$ be a base formal W-scheme. If ω belongs to $\mathcal{G}$-$\mathbf{Vect}^{\varphi,\mu}(\mathfrak{X}_{\Delta})$ for a cocharacter $\mu\colon \mathbb{G}_m \to \mathcal{G}$ (see Definition 5.8), then $\omega(\Lambda)$ belongs to $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ for all objects Λ of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$.

Exactness in the lff case. We end with the observation that exactness of $\mathbb{D}_{\mathrm{crys}}$ holds when restricted to the category $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$.

Proposition 3.36. *The functor*

$$\mathbb{D}_{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta}) \to \mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}),$$

is exact.

To prove this, it is useful to first make a simple observation about filtered modules. To state this, first recall that a map $f\colon (M_1, \mathrm{Fil}^{\bullet}_1) \to (M_2, \mathrm{Fil}^{\bullet}_2)$ of filtered $(R, \mathrm{Fil}^{\bullet})$-modules is called *strict* if the equality $f(\mathrm{Fil}^j_1) = f(M) \cap \mathrm{Fil}^j_2$ for all j (cf. [SP, Tag 0120] and [SP, Tag 05SI]). We then have the following simple lemma.

Lemma 3.37. *Let $(R, \mathrm{Fil}^{\bullet})$ be a filtered ring, and let*

$$0 \to (M_1, \mathrm{Fil}^{\bullet}_1) \to (M_2, \mathrm{Fil}^{\bullet}_2) \to (M_3, \mathrm{Fil}^{\bullet}_3) \to 0,$$

be a sequence of finitely supported (i.e., the set of j such that $\mathrm{Gr}^j(\mathrm{Fil}^{\bullet}_i)$ is non-zero is finite) filtered modules over $(R, \mathrm{Fil}^{\bullet})$, which is a short exact sequence on the underlying R-modules. Then, the following are equivalent:

(1) the sequence

$$0 \to \mathrm{Gr}^j(\mathrm{Fil}^{\bullet}_1) \to \mathrm{Gr}^j(\mathrm{Fil}^{\bullet}_2) \to \mathrm{Gr}^j(\mathrm{Fil}^{\bullet}_3) \to 0$$

is exact for all j,

(2) the maps $(M_1, \mathrm{Fil}^{\bullet}_1) \to (M_2, \mathrm{Fil}^{\bullet}_2)$ and $(M_2, \mathrm{Fil}^{\bullet}_2) \to (M_3, \mathrm{Fil}^{\bullet}_3)$ are strict,

(3) the sequence

$$0 \to \mathrm{Fil}^j_1 \to \mathrm{Fil}^j_2 \to \mathrm{Fil}^j_3 \to 0$$

is exact for all j.

Proof. To show the equivalence of (2) and (3), observe that the strictness of $(M_2, \mathrm{Fil}^{\bullet}_2) \to (M_3, \mathrm{Fil}^{\bullet}_3)$ is precisely equivalent to the claim that $\mathrm{Fil}^j_2 \to \mathrm{Fil}^j_3$ is surjective for all j. On the other hand, observe that the strictness of $(M_1, \mathrm{Fil}^{\bullet}_1) \to (M_2, \mathrm{Fil}^{\bullet}_2)$ by definition means that $M_1 \cap \mathrm{Fil}^{\bullet}_2 = \mathrm{Fil}^{\bullet}_1$, but as $M_1 = \ker(M_2 \to M_3)$, this is equivalent to $\ker(\mathrm{Fil}^j_2 \to \mathrm{Fil}^j_3) = \mathrm{Fil}^j_1$ for all j. As $\mathrm{Fil}^j_1 \to \mathrm{Fil}^j_2$ is obviously injective for all j, the equivalence of (2) and (3) follows.

Observe that (3) implies (1) by the snake lemma, and so we now show that (1) implies (3). To see that $\mathrm{Fil}_2^j \to \mathrm{Fil}_3^j$ is surjective, we observe that from the surjectivity of $\mathrm{Gr}^j(\mathrm{Fil}_2^\bullet) \to \mathrm{Gr}^j(\mathrm{Fil}_3^\bullet)$, for any y_j in Fil_3^j there exists some y_{j+1} in Fil_3^{j+1} and some x_j in Fil_2^j such that x_j maps to $y_j + y_{j+1}$. If $\mathrm{Fil}_2^{j+1} \to \mathrm{Fil}_3^{j+1}$ is surjective, there exists some x_{j+1} mapping to y_{j+1} and thus $x_j - x_{j+1}$ belongs to Fil_2^j and maps to y_j. Thus, we see that it suffices to prove surjectivity for j sufficiently large. But, by our finite-support hypotheses, we know that $\mathrm{Fil}_3^j = \mathrm{Fil}_2^j = 0$ for $j \gg 0$, so the claim follows. A similar argument using the finite support hypothesis shows that $\ker(\mathrm{Fil}_2^j \to \mathrm{Fil}_2^j) = \mathrm{Fil}_1^j$ for all j, from where the claim follows. □

Proof of Proposition 3.36. For this we may reduce to the case when $\mathfrak{X} = \mathrm{Spf}(R)$, where R is a (formally framed) base W-algebra. Consider an exact sequence

$$0 \to (\mathcal{E}_1, \varphi_{\mathcal{E}_1}) \to (\mathcal{E}_2, \varphi_{\mathcal{E}_2}) \to (\mathcal{E}_3, \varphi_{\mathcal{E}_3}) \to 0$$

in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$. Set $M_i := \mathcal{E}_i(\mathfrak{S}_R, (E))$ and write $\mathrm{Fil}_i^\bullet$ for $\iota_i^{-1}(\overline{\mathrm{Fil}}^\bullet_{\mathrm{Nyg}}(\phi^* M_i/(E))$, where ι_i is as in Lemma 3.11, a filtration on $\phi^* M_i/(u)$ (cf. Proposition 3.3). Then, as evaluation at $(\mathfrak{S}_R, (E))$ is exact (Lemma 1.17), and an exact sequence of vector bundles is universally exact, we see that we obtain a sequence of filtered modules over $(R, \mathrm{Fil}^\bullet_{\mathrm{triv}})$

$$0 \to (\phi^* M_1/(u), \mathrm{Fil}_1^\bullet) \to (\phi^* M_2/(u), \mathrm{Fil}_2^\bullet) \to (\phi^* M_3/(u), \mathrm{Fil}_2^\bullet) \to 0,$$

which is short exact on the underlying R-modules. Then, by the definition of $\mathbb{D}_{\mathrm{crys}}$, it suffices to show this is an exact sequence of filtered modules over $(R, \mathrm{Fil}^\bullet_{\mathrm{triv}})$. As each $(\phi^* M_i/(u), \mathrm{Fil}_i^\bullet)$ is finitely supported, it suffices by Lemma 3.37 to show that $(\phi^* M_1/(u), \mathrm{Fil}_1^\bullet) \to (\phi^* M_2/(u), \mathrm{Fil}_2^\bullet)$ and $(\phi^* M_2/(u), \mathrm{Fil}_2^\bullet) \to (\phi^* M_3/(u), \mathrm{Fil}_3^\bullet)$ are strict. But, as each $(\mathcal{E}_i, \varphi_{\mathcal{E}_i})$ is in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(R_{\Delta})$, the Faltings morphism (3.2.2) is an isomorphism, and thus, the claim follows from [Fal89, Theorem 2.1 (3)] (cf. [LMP20, Theorem 2.2.1 (3)]). □

3.3. **Relationship to Fontaine–Laffaille theory.** We assume $p \neq 2$ in this subsection. We now relate the functor $\mathbb{D}_{\mathrm{crys}}$ to relative Fontaine–Laffaille theory as first developed in [Fal89], when our base scheme is smooth over W.

Throughout this section we fix a perfect field k of characteristic p, and set $W = W(k)$ and $K = W[1/p]$. We also fix a smooth formal W-scheme $\mathfrak{X}$, and let X be its generic fiber over K.

3.3.1. *Statement of the comparison.* We are interested in studying the functor $\mathbb{D}_{\mathrm{crys}}$ in the Fontaine–Laffaille range, and its relationship to the theory developed in [Fal89].

More precisely, on the one hand we have the functor

$$\mathbb{D}_{\mathrm{crys}} \colon \mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \to \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}).$$

On the other hand, we have the functor

$$T^*_{\mathrm{crys}} \colon \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{Loc}_{\mathbb{Z}_p}(X),$$

constructed by Faltings in [Fal89, II.e), pp. 35–37] (see also [Tsu20, §4]), which is described as follows. For an object $\mathcal{F} = (\mathcal{F}, \varphi, \mathrm{Fil}^\bullet_{\mathcal{F}})$ of $\mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}})$, the reductions $\mathcal{F}_m$ of $\mathcal{F}$ modulo p^m define a projective system of objects of the category $\mathfrak{MF}^{\nabla}(\mathfrak{X})$ defined in [Fal89, 2.c)–d), pp. 30–33]. With the notation of loc. cit., $T^*_{\mathrm{crys}}(\mathcal{F})$ is then given as the inverse limit $\varprojlim_m \mathbf{D}(\mathcal{F}_m)$.

We prove the following compatibility between $\mathbb{D}_{\mathrm{crys}}$, T^*_{crys}, and $T^*_{\text{ét}}(\mathcal{E}, \varphi_{\mathcal{E}}) := T(\mathcal{E}, \varphi_{\mathcal{E}})^\vee$.

Proposition 3.38. *For a smooth formal W-scheme $\mathfrak{X}$, the following diagram commutes*

$$\begin{tikzcd} & \mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[dl, "T^*_{\text{ét}}"'] \arrow[dr, "\mathbb{D}_{\mathrm{crys}}"] & \\ \mathbf{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_\eta) & & \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}). \arrow[ll, "T^*_{\mathrm{crys}}"] \end{tikzcd}$$

In words, Proposition 3.38 says that if $(\mathcal{E},\varphi_\mathcal{E})$ is an lff effective prismatic F-crystal with height in $[0,p-2]$, then the étale local system $T^*_{\text{ét}}(\mathcal{E},\varphi_\mathcal{E})$ is Fontaine–Laffaille with associated strongly divisible filtered F-crystal given by $\mathbb{D}_{\text{crys}}(\mathcal{E},\varphi_\mathcal{E})$.

Remark 3.39. The restriction $\mathbf{Vect}^{\varphi}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\varphi,\text{an}}_{[0,p-2]}(\mathfrak{X}_{\Delta})$ is an equivalence by [DLMS22, Remark 3.37]. So, while we have only defined $\mathbb{D}_{\text{crys}}$ for prismatic F-crystals and not analytic prismatic F-crystals, this distinction disappears in the Fontaine–Laffaille range.

3.3.2. *Several functors of Tsuji.* To prove Proposition 3.38, we make a detailed analysis in the small affine case. A key to this analysis are certain results in [Tsu20]. Let us fix a small W-algebra R, and a framing $t\colon T_d \to R$. To describe Tsuji's results, we first must describe certain subcategories of $\mathbf{Vect}^{\varphi,\text{lff}}_{[0,p-2]}(R_{\Delta})$ and $\mathbf{VectF}^{\varphi,\text{div}}_{[0,p-2]}(R_{\text{crys}})$.

First, recall (see Proposition 3.20) that there is a natural equivalence of categories

$$\mathbf{VectF}^{\varphi,\text{div}}_{[0,p-2]}(R_{\text{crys}}) \xrightarrow{\sim} \mathbf{MF}^{\nabla}_{[0,p-2]}(R), \qquad (\mathcal{F},\varphi_\mathcal{F},\text{Fil}^\bullet_\mathcal{F}) \mapsto (\mathcal{F}_\mathfrak{X}(R),\varphi_{\mathcal{F}_\mathfrak{X}(R)},\nabla_\mathcal{F},\text{Fil}^\bullet_\mathcal{F}).$$

Following [Tsu20, §4], we consider the full subcategory $\mathbf{MF}^{\nabla}_{[0,p-2],\text{free}}(R)$ consisting of those $(M,\varphi_M,\nabla_M,\text{Fil}^\bullet_M)$ such that $\text{Gr}^r(\text{Fil}^\bullet_M)$ is free over R for every $r\in\mathbb{Z}$. Then, unraveling the definition of $\mathbb{D}_{\text{crys}}$ we obtain a functor

$$\mathbf{Vect}^{\varphi}_{[0,p-2],\text{free}}(R_{\Delta}) \to \mathbf{MF}^{\nabla}_{[0,p-2],\text{free}}(R),$$

which we denote again by $\mathbb{D}_{\text{crys}}$.

Now, for notational simplicity, we again use abbreviations:

$$\mathrm{A}_{\text{inf}} := \mathrm{A}_{\text{inf}}(\check{R}), \quad \mathrm{A}_{\text{crys}} := \mathrm{A}_{\text{crys}}(\check{R}),$$

with notation as in §1.1.5. We then further consider the categories

$$\mathbf{M}^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R), \quad \mathbf{MF}^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R), \quad \mathbf{MF}^{p,\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{crys}},\varphi,\Gamma_R),$$

defined in [Tsu20, Definition 51 and §8].[22] By [Tsu20, Equation (49) and Proposition 59], we have equivalences of categories

$$\mathbf{M}^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R) \xleftarrow{\sim} \mathbf{MF}^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{p,\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{crys}},\varphi,\Gamma_R). \quad (3.3.1)$$

The first functor is that forgetting the filtration, and the second is defined by

$$(M,\text{Fil}^\bullet_M,\varphi_M) \mapsto \Big((M,\text{Fil}^\bullet_M)\otimes_{(\mathrm{A}_{\text{inf}},\text{Fil}^\bullet_{\tilde{\xi}})}(\mathrm{A}_{\text{crys}},\text{Fil}^\bullet_{\text{PD}}),\varphi_M\otimes 1\Big)$$

with the semi-linearly extended action of Γ_R.

In [Tsu20, Equations (23) and (36)], Tsuji constructs functors

$$T\mathrm{A}_{\text{crys}}\colon \mathbf{MF}^{\nabla}_{[0,p-2],\text{free}}(R) \to \mathbf{MF}^{p,\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{crys}},\varphi,\Gamma_R),$$

$$T\mathrm{A}_{\text{inf}}\colon \mathbf{MF}^{\nabla}_{[0,p-2],\text{free}}(R) \to \mathbf{M}^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R),$$

described as follows. Let $(M,\varphi_M,\nabla_M,\text{Fil}^\bullet_M)$ be an object of $\mathbf{MF}^{\nabla}_{[0,p-2]\text{free}}(R)$, and $(\mathcal{F},\varphi_\mathcal{F},\text{Fil}^\bullet_\mathcal{F})$ denote the corresponding object of $\mathbf{VectF}^{\varphi}(\mathfrak{X}_{\text{crys}})$. Then, $T\mathrm{A}_{\text{crys}}(M,\varphi_M,\nabla_M,\text{Fil}^\bullet_M)$ is defined as the evaluation $(\mathcal{F},\varphi_\mathcal{F},\text{Fil}^\bullet_\mathcal{F})(\mathrm{A}_{\text{crys}}\twoheadrightarrow\check{R})$ (recall our conventions concerning F-crystals as in §2.3.1), and $T\mathrm{A}_{\text{inf}}$ is the result of translating $T\mathrm{A}_{\text{crys}}$ through (3.3.1).

We have a natural functor

$$\mathbb{D}\mathrm{A}_{\text{inf}}\colon \mathbf{Vect}^{\varphi}_{[0,p-2],\text{free}}(R_{\Delta}) \to \mathbf{M}^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R), \quad (\mathcal{E},\varphi_\mathcal{E}) \mapsto (\mathcal{E}(\mathrm{A}_{\text{inf}},(\tilde{\xi})),\varphi_\mathcal{E}),$$

where we equip $\mathcal{E}(\mathrm{A}_{\text{inf}},\tilde{\xi})$ with the Γ_R-action induced by the Γ_R-action on $(\mathrm{A}_{\text{inf}},(\tilde{\xi}))$. That $(\mathcal{E}(\mathrm{A}_{\text{inf}},(\tilde{\xi})),\varphi_\mathcal{E})$ is an object of $M^{\tilde{\xi},\text{cont}}_{[0,p-2],\text{free}}(\mathrm{A}_{\text{inf}},\varphi,\Gamma_R)$ follows from Lemma 3.30.

[22] In [Tsu20, §2 and §8], the element $\tilde{\xi}$ of A_{inf} is denoted by q.

Let $\mathbb{D}^{\mathrm{F}}\mathrm{A}_{\mathrm{inf}}$ and $\mathbb{D}\mathrm{A}_{\mathrm{crys}}$ denote the compositions

$$\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\mathbb{\Delta}}) \xrightarrow{\mathbb{D}\mathrm{A}_{\mathrm{inf}}} \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R),$$

$$\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\mathbb{\Delta}}) \xrightarrow{\mathbb{D}\mathrm{A}_{\mathrm{inf}}} \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R),$$

respectively. Note that for an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\mathbb{\Delta}})$, the underlying φ-module of $\mathbb{D}\mathrm{A}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is given by $\mathcal{E}(\mathrm{A}_{\mathrm{crys}},(p))$. The action of Γ_R is induced by that on the object $(\mathrm{A}_{\mathrm{crys}},(p),\widetilde{\mathrm{nat.}})$ of $R_{\mathbb{\Delta}}$. The filtration on these objects is described as follows.

Lemma 3.40. *Let $(e_\nu,r_\nu)_{\nu=1}^n$ be a filtered basis of $(\phi^*\mathcal{E}(\mathfrak{S}_R,(E)),\mathrm{Fil}^\bullet_{\mathrm{Nyg}})$ over the filtered ring $(\mathfrak{S}_R,\mathrm{Fil}^\bullet_E)$ in the sense of Definition 3.21. Then we have:*

(1) $(\alpha^*_{\mathrm{inf}}(e_\nu),r_\nu)_{\nu=1}^n$ is a flitered basis of $\mathbb{D}^{\mathrm{F}}\mathrm{A}_{\mathrm{inf}}(\mathcal{E},\varphi_{\mathcal{E}})$ over the filtered ring $(\mathrm{A}_{\mathrm{inf}},\mathrm{Fil}^\bullet_{\tilde{\xi}})$,
(2) $(\alpha^*_{\mathrm{crys}}(e_\nu),r_\nu)_{\nu=1}^n$ is a flitered basis of $\mathbb{D}\mathrm{A}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ over the filtered ring $(\mathrm{A}_{\mathrm{crys}},\mathrm{Fil}^\bullet_{\mathrm{PD}})$.

Proof. The first assertion follows from the description of the equivalence

$$\mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R)$$

given in the proof of [Tsu20, Lemma 46] combined with Lemma 3.30, and the second follows from the first. ☐

3.3.3. *The proof of the comparison.* To begin, we consider the functor

$$T^*_{\mathrm{inf}}\colon \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \to \mathbf{Rep}_{\mathbb{Z}_p}(\Gamma_R), \quad (M,\varphi_M)\mapsto \mathrm{Hom}((M,\varphi_M),(\mathrm{A}_{\mathrm{inf}},\phi)),$$

where Γ_R acts on $\mathrm{Hom}((M,\varphi_M),(\mathrm{A}_{\mathrm{inf}},\phi))$ via its action on (M,φ_M). By [Tsu20, Theorem 63 (2)], the composition $T^*_{\mathrm{inf}}\circ T\mathrm{A}_{\mathrm{inf}}$ is identified with T^*_{crys}. Thus, the proof of Proposition 3.38 is reduced to showing that the following diagram of categories commutes:

$$\begin{tikzcd}
 & \mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\mathbb{\Delta}}) \arrow[dl, "T^*_{\text{ét}}"'] \arrow[d, "\mathbb{D}\mathrm{A}_{\mathrm{inf}}"] \arrow[dr, "\mathbb{D}_{\mathrm{crys}}"] & \\
\mathbf{Rep}_{\mathbb{Z}_p}(\Gamma_R) & \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \arrow[l, "T^*_{\mathrm{inf}}"] \arrow[d, "\wr"] & \mathbf{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(R) \arrow[l, "T\mathrm{A}_{\mathrm{inf}}"] \arrow[dl, "T\mathrm{A}_{\mathrm{crys}}"] \\
 & \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R). &
\end{tikzcd} \tag{3.3.2}$$

Moreover, the lower-right triangle of (3.3.2) commutes by the definition of $T\mathrm{A}_{\mathrm{inf}}$.

Lemma 3.41. *The upper-right triangle of (3.3.2) commutes.*

Proof. It suffices to show that the right large triangle in (3.3.2) commutes. Moreover, note that by definition the composition of the two vertical arrows is precisely $\mathbb{D}\mathrm{A}_{\mathrm{crys}}$.

So, let $(\mathcal{E},\varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\mathbb{\Delta}})$. We first check that the underlying (φ,Γ_R)-modules of $(T\mathrm{A}_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})$ and $\mathbb{D}\mathrm{A}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ are canonically identified. The latter is by definition $\mathcal{E}(\mathrm{A}_{\mathrm{crys}},(p))$ with Γ_R-action induced by that on $(\mathrm{A}_{\mathrm{crys}},(p),\widetilde{\mathrm{nat.}})$ via its normal action on $\mathrm{A}_{\mathrm{crys}}$. To describe $(T\mathrm{A}_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})$, recall that the underlying F-crystal of $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is $(\mathcal{E}^{\mathrm{crys}},\varphi_{\mathcal{E}^{\mathrm{crys}}})$. So, by definition, the underlying (φ,Γ_R)-module of $(T\mathrm{A}_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})$ is $\mathcal{E}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}\twoheadrightarrow\check{R}/p)$ with Γ_R-action induced by that on $\mathrm{A}_{\mathrm{crys}}\twoheadrightarrow\check{R}/p$ via its normal action on $\mathrm{A}_{\mathrm{crys}}$. But, these two coincide by the definition of $(-)^{\mathrm{crys}}$.

We now compare the filtrations. We put $\mathcal{M}:=\mathcal{E}(S_R,(p))$ which is naturally isomorphic to $\mathcal{E}^{\mathrm{crys}}(S_R\twoheadrightarrow R)$ (cf. Proposition 3.1). Let $\mathrm{Fil}^\bullet_1$ denote the filtration on $\mathcal{M}$ obtained from the Nygaard filtration on $\phi^*\mathcal{E}(\mathfrak{S},(E))$ by filtered scalar extension along $(\mathfrak{S}_R,\mathrm{Fil}^\bullet_E)\to(S_R,\mathrm{Fil}^\bullet_{\mathrm{PD}})$, where $\mathfrak{S}_R\to S_R$ is the natural inclusion. On the other hand, consider the filtration $\mathrm{Fil}^\bullet_2$ on $\mathcal{M}$ obtained from the filtration on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E})(R)$ via the filtered map $(R,\mathrm{Fil}^\bullet_{\mathrm{triv}})\to(S_R,\mathrm{Fil}^\bullet_{\mathrm{PD}})$, where $R\to S_R$ is the natural inclusion. Note that the filtration on $\mathbb{D}A_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})=\mathcal{E}(A_{\mathrm{crys}},(p))$ (resp.

$(T\mathrm{A}_{\mathrm{crys}} \circ \mathbb{D}_{\mathrm{crys}})(\mathcal{E}, \varphi_{\mathcal{E}}) = \mathcal{E}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \check{R}))$ is obtained from $\mathrm{Fil}_1^\bullet$ (resp. $\mathrm{Fil}_2^\bullet$) by scalar extension along the faithfully flat map $S_R \to \mathrm{A}_{\mathrm{crys}}$. Thus, it suffices to show the equality $\mathrm{Fil}_1^\bullet = \mathrm{Fil}_2^\bullet$.

Here we note that we have the equality $\mathrm{Fil}^r_{\mathrm{PD}} = \mathrm{Fil}^r_{\mathrm{PD}}[1/p] \cap S_R$, which follows from the fact that the graded quotients of $\mathrm{Fil}^\bullet_{\mathrm{PD}}$ is p-torsion free (see the sentence before [Tsu20, Lemma 8]). Then, using [Tsu20, Lemma 42.(1)], the problem is reduced to showing $\mathrm{Fil}_1^\bullet[1/p] = \mathrm{Fil}_2^\bullet[1/p]$.

We consider the filtration $\mathrm{F}^\bullet\mathcal{M}[1/p]$ (resp. $\mathrm{Fil}^\bullet\,\mathcal{D}$) from the paragraph before [DLMS22, Lemma 4.31] (for the p-adic representation $V = T_{\text{ét}}(\mathcal{E})[1/p]$). We claim that the $\mathrm{Fil}_1^\bullet[1/p]$ (resp. $\mathrm{Fil}_2^\bullet[1/p]$) coincides with the filtration $\mathrm{F}^\bullet\mathcal{M}[1/p]$ (resp. $\mathrm{Fil}^\bullet\,\mathcal{D}$).

To see $\mathrm{Fil}_1^\bullet[1/p] = \mathrm{F}^\bullet\mathcal{M}[1/p]$, we take a filtered basis $(e_\nu, r_\nu)_\nu$ of $(\phi^*\mathcal{E})(\mathfrak{S}, (E))$ over $(\mathfrak{S}, \mathrm{Fil}^\bullet_E)$ in the sense of Definition 3.21. We denote again by e_ν the image of e_ν under the map

$$\phi^*\mathcal{E}(\mathfrak{S}_R, (E)) \to \mathcal{E}(S_R, (p)) = \mathcal{M}$$

induced by the natural inclusion $\mathfrak{S}_R \hookrightarrow S_R$. Let $x = \sum_{\nu=1}^n a_\nu e_\nu$, with a_ν in $S_R[1/p]$, be an arbitrary element of $\mathcal{M}[1/p]$. Then x belongs to $\mathrm{F}^r\mathcal{M}[1/p]$ if and only if the element

$$\sum_\nu a_\nu \varphi_{\mathcal{M}}(e_\nu) = \sum_\nu a_\nu E^{r_\nu}\left(\frac{\varphi_{\mathcal{M}}(e_\nu)}{E^{r_\nu}}\right)$$

belongs to $E^r\mathcal{M}[1/p]$. By the second part of Lemma 3.30, this is equivalent to the claim that a_ν belongs to $E^{r-r_\nu}S_R[1/p] = \mathrm{Fil}^{r-r_\nu}_{\mathrm{PD}}\,S_R[1/p]$ for all ν. Thus, we have $\mathrm{Fil}_1^\bullet[1/p] = \mathrm{F}^\bullet\mathcal{M}[1/p]$.

We show $\mathrm{Fil}_2^\bullet[1/p] = \mathrm{Fil}^\bullet\,\mathcal{D}$ by induction on r. We take a filtered basis $(e'_\nu, r'_\nu)_\nu$ of $D_{\mathrm{crys}}(V)(R)$ over $(R, \mathrm{Fil}^\bullet_{\mathrm{triv}})$. Through the identification $D_{\mathrm{crys}}(V) \otimes_{R[1/p]} S_R[1/p] \xrightarrow{\sim} \mathcal{M}[1/p]$ induced by Proposition 3.26 and the crystal property, the collection $(e'_\nu \otimes 1, r'_\nu)$ gives a filtered basis of $(\mathcal{M}[1/p], \mathrm{Fil}_2^\bullet[1/p])$ over $(S_R[1/p], \mathrm{Fil}^\bullet_{\mathrm{PD}})$. Here we are implicitly using that the filtrations on $S_R[1/p]$ induced by $\mathrm{Fil}^\bullet_{\mathrm{PD}}$ and $\mathrm{Fil}^\bullet_E$ agree, and we use both notations below depending on which is convenient.

Now we assume the equality $\mathrm{Fil}_2^{r-1}[1/p] = \mathrm{Fil}^{r-1}\,\mathcal{D}$ and take an arbitrary element $x = \sum_\nu a_\nu e'_\nu$ (with a_ν in $S_R[1/p]$ for all ν) from $\mathrm{Fil}_2^{r-1}[1/p]$, so that we have $a_\nu \in \mathrm{Fil}^{r-1-r_\nu}_{\mathrm{PD}}$ for all ν. We use the notation as in [DLMS22, §4]. Noting that we have $N_u(x) = \sum_\nu N_u(a_\nu)e'_\nu$, we see that x is in $\mathrm{Fil}^r\,\mathcal{D}$ if and only if the following two conditions hold:

- $N_u(a_\nu)$ belongs to $\mathrm{Fil}^{r-1-r_\nu}_E$ for ν with $r-1 > r_\nu$,
- $a_\nu \in \mathrm{Fil}^1_E$ for ν with $r-1 = r_\nu$.

For ν with $r-1 > r_\nu$, writing the element a_ν of $\mathrm{Fil}^{r-1-r_\nu}_E$ as $a_\nu = \sum_{i \geq r-1-r_\nu} b_i E^{[i]}$, with b_i in $R[1/p]$ and where $E^{[i]}$ denotes $E^i/i!$, we obtain $N_u(a_\nu) = -u\sum_{i\geq r-1-r_\nu} b_i E^{[i-1]}$. Hence, the second condition is equivalent to the claim that $b_{r-1-r_\nu} = 0$, which happens if and only if a_ν belongs to $\mathrm{Fil}^{r-r_\nu}_E$. Thus, $\mathrm{Fil}_2^\bullet[1/p] = \mathrm{Fil}^\bullet\,\mathcal{D}$. Finally, the equality $\mathrm{Fil}_1^\bullet[1/p] = \mathrm{Fil}_2^\bullet[1/p]$ follows from [DLMS22, Lemma 4.31] (cf. [Bre97, Proposition 6.2.2.1]). □

Remark 3.42. The method of proof in Lemma 3.41 shows the following, which will be useful later. Let R be a base W-algebra, and $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi,\mathrm{lff}}(R_{\Delta})$. Then, there is a tautological identification of filtered Frobenius modules

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R}) \xrightarrow{\sim} (\phi^*\mathcal{E}(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)), \mathrm{Fil}^\bullet_{\mathrm{Nyg}}) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^\bullet_\xi)} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^\bullet_{\mathrm{PD}})$$

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(S_R, (p)) \xrightarrow{\sim} (\phi^*\mathcal{E}(\mathfrak{S}_R, (E)), \mathrm{Fil}^\bullet_{\mathrm{Nyg}}) \otimes_{(\mathfrak{S}_R, \mathrm{Fil}^\bullet_E)} (S_R, \mathrm{Fil}^\bullet_{\mathrm{PD}}).$$

Lemma 3.43. *The upper left triangle of the diagram* (3.3.2) *commutes.*

Proof. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta})$. By definition, we have

$$(T^*_{\mathrm{inf}} \circ \mathbb{D}\mathrm{A}_{\mathrm{inf}})(\mathcal{E}, \varphi_{\mathcal{E}}) \cong \mathrm{Hom}((\mathcal{E}(\mathrm{A}_{\mathrm{inf}}, (\tilde{\xi})), \varphi_{\mathcal{E}}), (\mathrm{A}_{\mathrm{inf}}, \phi)).$$

On the other hand, by Example 2.13, we have

$$T^*_{\text{ét}}(\mathcal{E}, \varphi_{\mathcal{E}}) \cong \mathrm{Hom}((\mathcal{E}(\mathrm{A}_{\mathrm{inf}}, (\tilde{\xi})), \varphi_{\mathcal{E}}), (\mathrm{A}_{\mathrm{inf}}[1/\tilde{\xi}]^\wedge_p, \phi)).$$

The obvious map $(T^*_{\mathrm{inf}} \circ T\mathrm{A}_{\mathrm{inf}})(\mathcal{E}, \varphi_{\mathcal{E}}) \to T^*_{\text{ét}}(\mathcal{E}, \varphi_{\mathcal{E}})$ is an isomorphism. Indeed, it is an injective map between free $\mathbb{Z}_p$-modules of the same rank by [Tsu20, Proposition 66] and Lemma 3.41. So, the cokernel is killed by a power of p, but also embedds into $\mathrm{Hom}(\mathcal{E}(\mathrm{A}_{\mathrm{inf}},(\tilde{\xi})),\mathrm{A}_{\mathrm{inf}}[1/\tilde{\xi}]^\wedge_p/\mathrm{A}_{\mathrm{inf}})$, which is p-torsion free as $\mathrm{A}_{\mathrm{inf}}[1/\tilde{\xi}]^\wedge_p/\mathrm{A}_{\mathrm{inf}}$ is, and so the cokernel is zero as desired. $\square$

With these observations, the proof of Proposition 3.38 is now an exercise in parts assembly.

Proof of Proposition 3.38. Let $(\mathcal{E},\varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta})$. By taking an open covering $(\mathfrak{U}_i)_i$ by small affine opens of $\mathfrak{X}$ such that $\mathcal{E}_{\mathfrak{U}_{i,\Delta}}$ is in $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(\mathfrak{U}_{i,\Delta})$, the proof is reduced to constructing, for a small affine formal scheme smooth $\mathfrak{X}=\mathrm{Spf}(R)$ over W and an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta})$, an isomorphism

$$(T^*_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})\xrightarrow{\sim}T^*_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})$$

functorial in $(\mathcal{E},\varphi_{\mathcal{E}})$ and compatible with open immersions $\mathrm{Spf}(R')\to\mathrm{Spf}(R)$. Such an isomorphism is obtained using Lemmas 3.41 and 3.43, together with [Tsu20, Theorem 63.(2)], which is seen to satisfy the desired compatibility. $\square$

As $T_{\text{ét}}$ and T_{crys} are fully faithful by [GR22, Theorem A] and [Fal89, Theorem 2.6], respectively, we immediately arrive at the following corollary to Proposition 3.38.

Corollary 3.44. *The functor* $\mathbb{D}_{\mathrm{crys}}\colon\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta})\to\mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}})$ *is fully-faithful.*

Remark 3.45. While we assumed that $\mathfrak{X}$ was smooth to prove that $\mathbb{D}_{\mathrm{crys}}$ (restricted to lff prismatic F-crystals) takes values in $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ and to prove Proposition 3.38, in both of these cases it likely suffices to consider only base formal W-schemes. To address the first claim, one can likely prove the compatibility with rational p-adic Hodge theory in roughly the same way for general base W-algebras, or employ [DLMS22, Lemma 4.31] to at least show that Griffiths transversality holds. For the second claim, one needs to define $T^*_{\mathrm{crys}}:=T_{\mathrm{inf}}\circ T\mathrm{A}_{\mathrm{inf}}$ in this generality, and then once this Giffiths transversality is established the proof of Proposition 3.38 should apply mutatis mutandis. The authors will likely pursue this in a future draft of this article.

Remark 3.46. Late into the writing of this article, we became aware of the work of Matti Würthen (see [Wü23]), and forthcoming work of Christian Hokaj. These papers deal with equivalences $\mathbf{Vect}^{\varphi}_{[0,p-2]}(R_{\Delta})$ and Fontaine–Laffaille-like objects. They have the advantages of showing their functors are essentially surjective and not making assumptions on the prismatic F-crystals involved. That said, their functors take values in more exotic categories than considered here, and so are harder to work with. In addition, our construction is significantly simpler, and works beyond the Fontaine–Laffaille range.

Suppose now that $\mathscr{X}\to\mathrm{Spec}(W)$ is smooth and proper. For $(\mathcal{E},\varphi_{\mathcal{E}})$ in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\widehat{\mathscr{X}}_{\Delta})$ consider the object $T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})^{\mathrm{alg}}$ of $\mathbf{Loc}_{\mathbb{Z}_p}(\mathscr{X}_K)$ (see §2.1.3). Then, for $i\geqslant 0$, we have a diagram

$$\begin{array}{ccc} D_{\mathrm{crys}}\left(H^i_{\text{ét}}(\mathscr{X}_{\overline{K}},T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})^{\mathrm{alg}}[1/p])\right) & \xrightarrow{c_{(\mathcal{E},\varphi_{\mathcal{E}})}} & H^i\left((\mathscr{X}_k/W)_{\mathrm{crys}},\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})[1/p]\right) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathbb{D}_{\mathrm{crys}}\left(H^i_{\text{ét}}(\mathscr{X}_{\overline{K}},T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})^{\mathrm{alg}})\right) & & H^i\left((\mathscr{X}_k/W)_{\mathrm{crys}},\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})\right) \end{array} \tag{3.3.3}$$

where $c_{(\mathcal{E},\varphi_{\mathcal{E}})}$ is the isomorphism obtained by combining [TT19, Theorem 5.5], Proposition 3.26, and [Hub96, Theorem 3.7.2], and we are using notation as in Example 3.28.

From Proposition 3.38 and [Fal89, Theorem 5.3] we deduce the following.

Corollary 3.47. *Let* $\mathscr{X}\to\mathrm{Spec}(W)$ *be smooth and proper. Then, for an object* $(\mathcal{E},\varphi_{\mathcal{E}})$ *of* $\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,a]}(\widehat{\mathscr{X}}_{\Delta})$ *and* b *in* $\mathbb{N}$ *with* $a+b\leqslant p-2$, *the map* $c_{(\mathcal{E},\varphi_{\mathcal{E}})}$ *restricts to an isomorphism*

$$c_{(\mathcal{E},\varphi_{\mathcal{E}})}\colon\mathbb{D}_{\mathrm{crys}}\left(H^b(\mathscr{X}_{\overline{K}},T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})^{\mathrm{alg}})\right)\xrightarrow{\sim}H^b\left((\mathscr{X}_k/W)_{\mathrm{crys}},\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})\right).$$

In words, this is equivalent to saying that the p-adic representation $H^b_{\text{ét}}(\mathscr{X}_{\overline{K}}, T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})^{\text{alg}})$ is Fontaine–Laffaille with associated strongly divisible module $H^b((\mathscr{X}_k/W)_{\text{crys}}, \mathbb{D}_{\text{crys}}(\mathcal{E},\varphi_{\mathcal{E}}))$.

3.4. **Relationship to Dieudonné theory.** We assume $p \neq 2$ in this subsection. Here we record the relationship between $\mathbb{D}_{\text{crys}}$ and the prismatic Dieudonné functor as developed in [ALB23]. Throughout, for a p-adically complete ring R we denote by $\mathbf{BT}_p(R)$ the category p-divisible groups over R (see [dJ95a, Lemma 2.4.4] for why this notation is not ambiguous).

3.4.1. *The prismatic Dieudonné functor.* Let R be a quasi-syntomic ring. Fix an object H of $\mathbf{BT}_p(R)$. We may then consider the sheaf[23]

$$H_{\overline{\mathcal{O}}_{\Delta}} \colon R_{\Delta} \to \mathbf{Grp}, \qquad (A, I) \mapsto H(A/I) = H(\overline{\mathcal{O}}_{\Delta}(A, I)).$$

Following [ALB23], consider the group presheaf

$$\mathcal{M}_{\Delta}(H) := \mathcal{E}xt^1_{\mathbf{Ab}(\mathfrak{X}_{\Delta})}(H_{\overline{\mathcal{O}}_{\Delta}}, \mathcal{O}_{\Delta}).$$

The abelian sheaf $\mathcal{M}_{\Delta}(H)$ has the structure of an $\mathcal{O}_{\Delta}$-module and a Frobenius morphism

$$\varphi_{\mathcal{M}_{\Delta}(H)} \colon \phi^*\mathcal{M}_{\Delta}(H) \to \mathcal{M}_{\Delta}(H),$$

inherited from those structures on $\mathcal{O}_{\Delta}$. As in [ALB23], we call $\mathcal{M}_{\Delta}(H)$ the *prismatic Dieudonné crystal* associated to H.

We then have the following result of Anschutz–Le Bras.

Theorem 3.48 ([ALB23])**.** *The functor $\mathcal{M}_{\Delta}$ defines a contravariant fully faithful embedding*

$$\mathcal{M}_{\Delta} \colon \mathbf{BT}_p(R) \to \mathbf{Vect}^{\varphi}_{[0,1]}(R_{\Delta}),$$

which is an anti-equivalence if R admits a quasi-syntomic cover $R \to \widetilde{R}$ with $\widetilde{R}$ perfectoid.

Proof. By [ALB23, Theorem 4.74] (and Proposition 1.30), the functor $\mathcal{M}_{\Delta}$ forms an equivalence between $\mathbf{BT}_p(R)$ and the full subcategory of $\mathbf{Vect}^{\varphi}_{[0,1]}(R_{\Delta})$ consisting of so-called admissible objects (see [ALB23, Definition 4.5]). Thus, to prove the second part of the claim, it suffices to show that the existence of such a cover $R \to \widetilde{R}$ implies any object of $\mathbf{Vect}^{\varphi}(R_{\Delta})$ is admissible. As admissibility is clearly a local condition on R_{qsyn} (see [ALB23, Proposition 4.9]), it suffices to prove the claim over $\widetilde{R}$. But, this is the content of [ALB23, Proposition 4.12]. ∎

Finally, we record the compatibility of $\mathcal{M}_{\Delta}$ with the functors M^{SW} and M^{Lau} defined by [SW20, Theorem 17.5.2] and [Lau18a, Theorem 9.8], respectively. More precisely, suppose that R is perfectoid. Then, there is a natural identification between $M^{\text{SW}}(H)$ and $M^{\text{Lau}}(H)$, by the unicity part of [SW20, Theorem 17.5.2], and we have the following result of Anschutz–Le Bras.

Proposition 3.49 (cf. [ALB23, Proposition 4.48])**.** *Let R be a perfectoid ring and H an object of $\mathbf{BT}_p(R)$. Then, we have canonical identifications:*

$$M^{\text{SW}}(H)^* = M^{\text{Lau}}(H) = \mathcal{M}_{\Delta}(H)(\mathrm{A}_{\text{inf}}(R), (\tilde{\xi})).$$

3.4.2. *Crystalline Dieudonné theory.* For $\mathfrak{X} = \operatorname{Spf}(R)$, where R is a formally framed base W-algebra, a *filtered Dieudonné crystal* on R is an object $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ of $\mathbf{VectNF}^{\varphi}_{[0,1]}(\mathfrak{X}_{\text{crys}})$ with:

- $(\mathcal{F}, \varphi_{\mathcal{F}})$ an effective F-crystal,
- $\mathrm{Fil}^1_{\mathcal{F}}$ a direct factor of $\mathcal{F}_{\mathfrak{X}}$,
- $\phi^*(\mathrm{Fil}^1_{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_k}$ equal to the kernel of $\varphi_{\mathcal{F}} \colon \phi^*\mathcal{F}_{\mathfrak{X}_k} \to \mathcal{F}_{\mathfrak{X}_k}$,
- and there existing $V \colon \mathcal{F}_0 \to \phi^*\mathcal{F}_0$ with $\varphi_{\mathcal{F}} \circ V = [p]_{\mathcal{F}_0}$ and $V \circ \varphi_{\mathcal{F}} = [p]_{\phi^*\mathcal{F}_0}$,

[23] To see that this is a sheaf, we combine the following observations: $\overline{\mathcal{O}}_{\Delta}$ is a sheaf and H is finitely continuous (because $H = \varinjlim H[p^n]$, with each $H[p^n]$ representable, and finite limits commute with filtered colimits in **Set**).

(cf. [Kim15, Definition 3.1]), the category of which we denote $\mathbf{DDCrys}(R)$. One may check that $\mathbf{DDCrys}(R) = \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,1]}(R_{\mathrm{crys}})$, and so we shall identify these two in practice.

Let us now fix an object H of $\mathbf{BT}_p(R)$. We then consider the sheaf

$$H_{\overline{\mathcal{O}}_{\mathrm{crys}}}\colon (R/W)_{\mathrm{crys}} \to \mathbf{Grp}, \quad (i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) \mapsto H(\mathfrak{U}) = H(\overline{\mathcal{O}}_{\mathrm{crys}}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)),$$

(which is a sheaf as in Footnote 23). We then consider the sheaf

$$\underline{\mathbb{D}}(H) := \mathcal{E}xt^1_{\mathbf{Ab}((R/W)_{\mathrm{crys}})}(H_{\overline{\mathcal{O}}_{\mathrm{crys}}}, \mathcal{O}_{\mathrm{crys}}),$$

which comes with the structure of an $\mathcal{O}^{\mathrm{crys}}$-module from the second entry. By [BBM82, Théorème 3.3.3], the $\mathcal{O}_{\mathrm{crys}}$-module $\underline{\mathbb{D}}(H)$ is an object of $\mathbf{Vect}(R_{\mathrm{crys}})$. It is simple to check that

$$(\iota_{0,\infty})_*\underline{\mathbb{D}}(H_k) = \underline{\mathbb{D}}(H), \qquad \iota^*_{0,\infty}\underline{\mathbb{D}}(H) = \underline{\mathbb{D}}(H_k),$$

(with notation as in §2.3.1). As in [BBM82, 1.3.5] we have a Frobenius morphism

$$\varphi\colon \phi^*\underline{\mathbb{D}}(H_k) \to \underline{\mathbb{D}}(H_k).$$

So, $\underline{\mathbb{D}}(H)$ is an object of $\mathbf{Vect}^{\varphi}(R_{\mathrm{crys}})$ called the *Dieudonné crystal* associated to H.

One defines a *Hodge filtration* (see [BBM82, Corollaire 3.3.5])

$$\mathrm{Fil}^1_{H,\mathrm{Hodge}} := \mathcal{E}xt^1_{\mathbf{Ab}((R/W)_{\mathrm{crys}})}(H_{\overline{\mathcal{O}}_{\mathrm{crys}}}, \mathcal{J}_{\mathrm{crys}})_{\mathfrak{X}} \subseteq \underline{\mathbb{D}}(H)_{\mathfrak{X}}.$$

This defines a functor

$$\mathbb{D}\colon \mathbf{BT}_p(R) \to \mathbf{DDCrys}(R), \qquad H \mapsto \mathbb{D}(H) = (\underline{\mathbb{D}}(H), \mathrm{Fil}^1_{H,\mathrm{Hodge}}),$$

where $\mathbb{D}(H)$ is called the *filtered Dieudonné crystal* associated to H (cf. [Kim15, Lemma 3.2]). By work of de Jong and Grothendieck–Messing, $\mathbb{D}$ is an anti-equivalence of categories (see [Kim15, Theorem 3.17]). As in §3.2.1, we will consider the evaluation of $\mathbb{D}(H)$ on objects of $(R/W)_{\mathrm{crys}}$.

Proposition 3.50. *There is a natural equivalence $\mathbb{D}_{\mathrm{crys}} \circ \mathcal{M}_{\mathbb{\Delta}} \xrightarrow{\sim} \mathbb{D}$.*

Proof. We begin by observing that there is a tautological identification between $\underline{\mathbb{D}}_{\mathrm{crys}} \circ \mathcal{M}_{\mathbb{\Delta}}$ and $\underline{\mathbb{D}}$ (see [ALB23, Theorem 4.44]). Thus, it suffices to check that the submodules $\mathrm{Fil}^1_{\mathbb{D}_{\mathrm{crys}}}$ and $\mathrm{Fil}^1_{H,\mathrm{Hodge}}$ agree. To check this, it suffices to pass to the faithfully flat cover $R \to \widetilde{R}$ (see Lemma 1.14). But, observe that we have a commutative diagram of filtered rings

$$\begin{tikzcd}
(R, \mathrm{Fil}^\bullet_{\mathrm{triv}}) \arrow[r, "\beta"] \arrow[dr] & (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^\bullet_{\mathrm{PD}}) \arrow[d] \\
 & (\widetilde{R}, \mathrm{Fil}^\bullet_{\mathrm{triv}}),
\end{tikzcd}$$

thus it further suffices by the crystal property to check the equality of filtrations after evaluation on $\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R}$.

We first observe that by Example 3.34 and Remark 3.42, for any object H of $\mathbf{BT}_p(R)$ there is a canonical identification of filtered $\mathrm{A}_{\mathrm{crys}}(\widetilde{R})$-modules

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{M}_{\mathbb{\Delta}}(H))(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R}) \xrightarrow{\sim} \phi^*\mathcal{M}_{\mathbb{\Delta}}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^\bullet_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^\bullet_{\mathrm{PD}}).$$

On the other hand, by Lemma 3.51, the filtration on $\mathbb{D}(H)(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R})$ is equal to the preimage of the filtration $\mathrm{Fil}^1 \subseteq \underline{\mathbb{D}}(H)(\check{R})$, defined by $\mathbb{D}(H)(\check{R})$, under the surjection

$$\Pi\colon \underline{\mathbb{D}}(H)(\mathrm{A}_{\mathrm{crys}}(\check{R}) \twoheadrightarrow \check{R}) \to \underline{\mathbb{D}}(H)(\check{R}).$$

Thus, it suffices to show the following equality of filtered $\mathrm{A}_{\mathrm{crys}}(\check{R})$-modules:

$$\phi^*\mathcal{M}_{\mathbb{\Delta}}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^\bullet_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^\bullet_{\mathrm{PD}}) = (\underline{\mathbb{D}}(H)(\mathrm{A}_{\mathrm{crys}}(\check{R}) \twoheadrightarrow \check{R}), \Pi^{-1}(\mathrm{Fil}^1)).$$

But combining Proposition 3.49 and the definition of M^{Lau} we have

$$
\begin{aligned}
(\underline{\mathbb{D}}(H)(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R}), \Pi^{-1}(\mathrm{Fil}^1)) &= \Phi^{\mathrm{cris}}_{\widetilde{R}}(H_{\widetilde{R}}) \\
&= \lambda^*(M^{\mathrm{Lau}}(H_{\widetilde{R}})) \\
&= \lambda^*(\phi^* \mathcal{M}_{\mathbb{A}}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi))) \\
&= \phi^* \mathcal{M}_{\mathbb{A}}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\mathrm{PD}}),
\end{aligned}
$$

where $\Phi^{\mathrm{cris}}_{\widetilde{R}}$ is as in [Lau18a, Theorem 6.3] and λ^* is as in [Lau18a, Proposition 9.3]. □

Lemma 3.51. *Let $\mathcal{F}$ be an object of $\mathbf{VectF}_{[0,1]}(R_{\mathrm{crys}})$ and $(A \twoheadrightarrow R') \to (B \twoheadrightarrow R')$ be a morphism in $(R/W)_{\mathrm{crys}}$, with $A \to B$ surjective. Then $\mathrm{Fil}^1_{\mathcal{F}}(A \twoheadrightarrow R') \subseteq \mathcal{F}(A \twoheadrightarrow R')$ is the preimage of the submodule $\mathrm{Fil}^1_{\mathcal{F}}(B \twoheadrightarrow R') \subseteq \mathcal{F}(B \twoheadrightarrow R')$ via the surjection $\mathcal{F}(A \twoheadrightarrow R') \to \mathcal{F}(B \twoheadrightarrow R')$.*

Proof. By the crystal property, we may work Zariski locally on A. But, Zariski locally on A, there is a basis $(e_\nu)_{\nu=1}^n$ of $\mathcal{F}(A \twoheadrightarrow R')$ and a subset I of $\{1, \ldots, n\}$ such that the filtration $\mathrm{Fil}^1(A \twoheadrightarrow R')$ is given by $\sum_{\nu \in I} A \cdot e_\nu + \sum_{\nu \notin I} \mathrm{Fil}^1_{\mathrm{PD}}(A) \cdot e_\nu$. The claim then follows as $\mathrm{Fil}^1_{\mathrm{PD}}(A)$ is the preimage of $\mathrm{Fil}^1_{\mathrm{PD}}(B)$ under the surjection $A \to B$. □

Remark 3.52. In the case when R is small, Proposition 3.50 follows from Example 3.34 and Proposition 3.38. Indeed, as T^*_{crys} is fully faithful, it suffices to show that $T^*_{\mathrm{crys}}(\mathbb{D}_{\mathrm{crys}}(\mathcal{M}_{\mathbb{A}}(H)))$ is isomorphic to $T^*_{\mathrm{crys}}(\mathbb{D}(H))$. But, by Proposition 3.38 and [DLMS22, Proposition 3.34] the former is $T_p(H)$, which is also the latter by [Kim15, Corollary 5.3 and §5.4]. Such a proof should work, with some minor extra considerations, for general base W-algebras R given the contents of Remark 3.45.

3.4.3. *The Breuil–Kisin theory.* Let R be a base W-algebra and $t \colon T_d \to R$ a formal framing.

Filtered Breuil modules. Consider quadruples $(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}, \varphi_{\mathcal{M}}, \nabla^0_{\mathcal{M}})$ with:

- $\mathcal{M}$ a finite projective S_R-module,
- $\mathrm{Fil}^1_{\mathcal{M}} \subseteq \mathcal{M}$ an S_R-submodule with $\mathrm{Fil}^1_{\mathrm{PD}} \cdot \mathcal{M} \subseteq \mathrm{Fil}^1_{\mathcal{M}}$ and $\mathcal{M}/\mathrm{Fil}^1_{\mathcal{M}}$ projective over R,
- $\varphi_{\mathcal{M}} \colon \phi_t^* \mathcal{M} \to \mathcal{M}$ is an S_R-linear map with $\varphi_M(\phi_t^* \mathrm{Fil}^1_{\mathcal{M}}) = p\mathcal{M}$,
- $\nabla^0_{\mathcal{M}}$ is a topologically quasi-nilpotent integrable connection (cf. [dJ95a, Remark 2.2.4]) on $\mathcal{M}_0 := \mathcal{M} \otimes_{S_R} R$, where $S_R \to R$ is induced by the map $\mathfrak{S}_R \to R$ sending u to 0, such that the induced Frobenius $\varphi_{\mathcal{M}_0}$ is horizontal.

These naturally form a category which we denote by $\mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0)$. By [Kim15, Proposition 3.8 and Lemma 3.15], there is a fully faithful embedding

$$
\mathbf{DDCrys}(R) \to \mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0), \quad (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}}) \mapsto ((\mathcal{F}, \mathrm{Fil}^1_{\mathcal{F}}, \varphi_{\mathcal{F}})(S_R \twoheadrightarrow R), \nabla_{\mathcal{F}}).
$$

We thus obtain a fully faithful embedding

$$
\mathcal{M}^{\mathrm{Br}} \colon \mathbf{BT}_p(R) \to \mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0), \qquad H \mapsto \mathcal{M}^{\mathrm{Br}}(H) = (\mathbb{D}(H)(S_R \twoheadrightarrow R), \nabla_{\underline{\mathbb{D}}(H)}),
$$

(cf. [Kim15, Theorem 3.17]).

Kisin modules. Further, consider triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla^0_{\mathfrak{M}})$ where $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is an object of the category $\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, (E))$ and $\nabla^0_{\mathfrak{M}}$ is topologically quasi-nilpotent integral connection on $\mathfrak{M}_0 := \phi^* \mathfrak{M} \otimes_{\mathfrak{S}_R} R$, where $\mathfrak{S}_R \to R$ is obtained by sending u to 0. Such triples form a natural category which we denote $\mathbf{Vect}^{\varphi}(\mathfrak{S}_R, \nabla^0_{\mathfrak{M}})$. There is a functor

$$
\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0) \to \mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0), \tag{3.4.1}
$$

where $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla^0_{\mathfrak{M}_0})$ is sent to the object whose underlying filtered S_R-module is

$$
(\phi_t^* \mathfrak{M}, \mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}) \otimes_{(\mathfrak{S}_R, \mathrm{Fil}^{\bullet}_E)} (S_R, \mathrm{Fil}^{\bullet}_{\mathrm{PD}})
$$

(forgetting everything but the 1-part of the filtration), with Frobenius given by $\varphi_{\phi_t^*\mathfrak{M}}\otimes 1$, and with connection $\nabla^0_{\mathfrak{M}}$, which is sensible via the identification of the R-modules $\phi_t^*\mathfrak{M}\otimes_{\mathfrak{S}_R} R$ and $\phi_t^*\mathfrak{M}\otimes_{\mathfrak{S}_R} S_R\otimes_{S_R} R$. This functor is fully faithful by [Kim15, Lemma 6.5].

In [Kim15, Corollary 6.7], Kim shows that for a p-divisible group H over R, the object $\mathcal{M}^{\mathrm{Br}}(H)$ lies in the essential image image of (3.4.1). Thus, there exists a unique object

$$\mathfrak{M}(H)=(\underline{\mathfrak{M}}(H),\varphi_{\underline{\mathfrak{M}}(H)},\nabla^0_{\mathbb{D}(H)})$$

of $\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R,\nabla^0)$ such that there exists a Frobenius-equivariant filtered isomorphism

$$(\phi_t^*\underline{\mathfrak{M}}(H),\mathrm{Fil}^1_{\mathrm{Nyg}})\otimes_{(\mathfrak{S}_R,\mathrm{Fil}^\bullet_E)}(S_R,\mathrm{Fil}^\bullet_{\mathrm{PD}})\cong\mathbb{D}(H)(S_R\twoheadrightarrow R),$$

i.e., has image $\mathcal{M}^{\mathrm{Br}}(H)$ under (3.4.1). When $R=\mathcal{O}_K$, this agrees with the functor from [Kis06].

Connection to the prismatic theory. Let us begin by observing that there is a functor

$$\mathrm{ev}_{\mathfrak{S}_R}\colon\mathbf{Vect}^{\varphi}_{[0,1]}(R_{\triangle})\to\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R,(E)),\qquad(\mathcal{E},\varphi_{\mathcal{E}})\mapsto(\mathcal{E},\varphi_{\mathcal{E}})(\mathfrak{S}_R,(E)).$$

We then upgrade this to a functor

$$\mathrm{ev}^{\mathrm{qK}}_{\mathfrak{S}_R}\colon:\mathbf{Vect}^{\varphi}_{[0,1]}(R_{\triangle})\to\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R,\nabla^0),\quad(\mathcal{E},\varphi_{\mathcal{E}})\mapsto(\underline{\mathfrak{M}}'(\mathcal{E},\varphi_{\mathcal{E}}),\nabla_{\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})}).$$

Here by $\nabla_{\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})}$ we abusively mean the pullback of $\nabla_{\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})}$ under the isomorphism

$$\phi^*\mathcal{E}(\mathfrak{S}_R,(E))/(u)\xrightarrow{\sim}\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})(R),$$

from Proposition 3.3.

Proposition 3.53. *There is a natural identification* $\mathrm{ev}^{\mathrm{qK}}_{\mathfrak{S}_R}\circ\mathcal{M}_{\triangle}\xrightarrow{\sim}\mathfrak{M}$.

Proof. Let H be an object of $\mathbf{BT}_p(R)$. Then, it suffices to show that $\mathrm{ev}^{\mathrm{qK}}_{\mathfrak{S}_R}(\mathcal{M}_{\triangle}(H))$ and $\mathfrak{M}(H)$ have image under (3.4.1) which are canonically identified. But, this follows by combining Remark 3.42, Proposition 3.50, and the definition of $\mathcal{M}^{\mathrm{Br}}(H)$. ☐

Remark 3.54. For $R=\mathcal{O}_K$ this was previously shown in [ALB23, Proposition 5.18].

Combining Theorem 3.48, [Kim15, Corollary 6.7 and Corollary 10.4], and Proposition 3.53 we deduce the following.

Corollary 3.55. *Let R be a base W-algebra. Then, the functor*

$$\mathrm{ev}^{\mathrm{qK}}_{\mathfrak{S}_R}\colon\mathbf{Vect}^{\varphi}_{[0,1]}(R_{\triangle})\to\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R,\nabla^0),$$

is an equivalence of categories. If $R=W[\![t_1,\ldots,t_d]\!]$ for some $d\geqslant 0$, then

$$\mathrm{ev}_{\mathfrak{S}_R}\colon\mathbf{Vect}^{\varphi}_{[0,1]}(R_{\triangle})\to\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R,(E)),$$

is an equivalence of categories.

Remark 3.56. We make the following remarks.

(1) The second claim of Corollary 3.55 was previously established by Anschutz–Le Bras (see [ALB23, Theorem 5.12]) and Ito (see [Ito23b, Proposition 7.1.1]).
(2) While we will not define it here, using [GL23, Theorem 2.54], one may immediately enhance Corollary 3.55 to include an equivalence with F-gauges (see op. cit.) with heights in $[0,1]$.

4. Applications to Shimura varieties of abelian type

We construct an object $\omega_{\mathsf{K}^p,\triangle}$ of $\mathcal{G}$-$\mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\triangle})$, called the *prismatic realization functor*, where $\mathscr{S}_{\mathsf{K}^p}$ is the integral canonical model of a Shimura variety of abelian type. We use this to deduce some results about certain natural local systems occurring on these Shimura varieties. Throughout this section we assume that p is odd.

4.1. **Notation and basic definitions.** Throughout this section, we fix the following data/notation:

- $\mathbf{G}$ is a reductive group over $\mathbb{Q}$,
- $\mathbf{Z}$ denotes the center $Z(\mathbf{G})$ of $\mathbf{G}$,
- $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ is the Deligne torus,
- $(\mathbf{G}, \mathbf{X})$ is a Shimura datum (see [Mil05, Definition 5.5]),
- $\mathbf{E} = \mathbf{E}(\mathbf{G}, \mathbf{X}) \subseteq \mathbb{C}$ denotes the reflex field of $(\mathbf{G}, \mathbf{X})$ (see [Mil05, Definition 12.2]),
- $\mathsf{K} \subseteq \mathbf{G}(\mathbb{A}_f)$ is a (variable) neat (cf. [Mil05, p. 288]) compact open subgroup.

As in [Del79] (cf. [Moo98]), associated to this data is the (canonical model of the) Shimura variety $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$, which is a smooth and quasi-projective $\mathbf{E}$-scheme. For K and K' of $\mathbf{G}(\mathbb{A}_f)$, and g in $\mathbf{G}(\mathbb{A}_f)$ such that $g^{-1}\mathsf{K}g \subseteq \mathsf{K}'$, denote by $t_{\mathsf{K},\mathsf{K}'}(g)$ the unique finite étale morphism of $\mathbf{E}$-schemes $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X}) \to \mathrm{Sh}_{\mathsf{K}'}(\mathbf{G}, \mathbf{X})$ given on $\mathbb{C}$-points by

$$t_{\mathsf{K},\mathsf{K}'}(g) \left(\mathbf{G}(\mathbb{Q})(x, g')\mathsf{K} \right) = \mathbf{G}(\mathbb{Q})(x, g'g)\mathsf{K}'.$$

We shorten $t_{\mathsf{K},\mathsf{K}'}(\mathrm{id})$ to $\pi_{\mathsf{K},\mathsf{K}'}$ and $t_{\mathsf{K},g^{-1}\mathsf{K}g}(g)$ to $[g]_K$. The morphisms $\pi_{\mathsf{K},\mathsf{K}'}$ form a projective system $\{\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})\}$ with finite étale transition maps, and the morphisms $[g]_K$ endow

$$\mathrm{Sh}(\mathbf{G}, \mathbf{X}) := \varprojlim_{\mathsf{K}} \mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})$$

(cf. [SP, Tag 01YX]) with a continuous action of $\mathbf{G}(\mathbb{A}_f)$ (in the sense of [Del79, 2.7.1]).

We shall often fix the following additional data/notation/assumptions:

- p is a rational prime and $\mathfrak{p}$ a prime of $\mathbf{E}$ lying over p,
- E is the completion $\mathbf{E}_{\mathfrak{p}}$, $\mathcal{O}_E$ its ring of integers, and k its residue field,
- $G := \mathbf{G}_{\mathbb{Q}_p}$, and $\mathcal{G}$ is a parahoric model of G over $\mathbb{Z}_p$,
- $\mathsf{K}_0 \subseteq G(\mathbb{Q}_p)$ the parahoric subgroup given by $\mathcal{G}(\mathbb{Z}_p)$,
- $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ a neat compact open subgroup.

The triple $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is a *parahoric Shimura datum*, and is an *unramified Shimura datum* if $\mathcal{G}$ is reductive. For an unramified Shimura datum, the extension $E/\mathbb{Q}_p$ is unramified (see [Mil94, Corollary 4.7]) and we identify $\mathcal{O}_E$ with $W = W(k)$. Moreover, G is quasi-split and split over $\breve{E}$. We shorten $\mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})_E$ (resp. $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_E$) to Sh_{K} (resp. Sh).

Let $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ be a parahoric Shimura datum. Associated to $\mathbf{X}$ is a unique conjugacy class of coharacters $\mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ (see [Mil05, p. 344]) whose field of definition is $\mathbf{E}$. Using [Kot84, Lemma 1.1.3] this corresponds to a unique conjugacy class $\boldsymbol{\mu}_h$ of cocharacters $\mathbb{G}_{m,\overline{E}} \to G_{\overline{E}}$ which one checks has field of definition E. If $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is unramified then one may use loc. cit. to show the existence of a unique conjugacy class μ_h of cocharacters $\mathbb{G}_{m,\breve{\mathbb{Z}}_p} \to \mathcal{G}_{\breve{\mathbb{Z}}_p}$ modeling $\boldsymbol{\mu}_h$.

We often denote other Shimura data with numerical subscripts (e.g. $(\mathbf{G}_1, \mathbf{X}_1)$) and use the same numerical subscripts to denote the objects defined above (or below) for this Shimura datum (e.g. $\mathrm{Sh}_{\mathsf{K}_{0,1}\mathsf{K}_1^p}$ or $\mathcal{G}_1$). By a morphism of Shimura data $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1) \to (\mathbf{G}, \mathbf{X})$ we mean a morphism of group $\mathbb{Q}$-schemes $\alpha\colon \mathbf{G}_1 \to \mathbf{G}$ such that $\alpha_{\mathbb{R}} \circ h$ belongs to $\mathbf{X}$ for h in $\mathbf{X}_1$. The morphism is an embedding if $\mathbf{G}_1 \to \mathbf{G}$ is a closed embedding. By [Del79, §5], for such an α, one has that $\mathbf{E} \subseteq \mathbf{E}_1$ and there is a unique morphism $\mathrm{Sh}(\mathbf{G}_1, \mathbf{X}_1) \to \mathrm{Sh}(\mathbf{G}, \mathbf{X})_{\mathbf{E}_1}$ of $\mathbf{E}_1$-schemes equivariant for the map $\alpha\colon \mathbf{G}_1(\mathbb{A}_f) \to \mathbf{G}(\mathbb{A}_f)$ and such that if $\alpha(\mathsf{K}_1) \subseteq \mathsf{K}$ then the induced map on the quotients $\alpha_{\mathsf{K}_1,\mathsf{K}}\colon \mathrm{Sh}_{\mathsf{K}_1}(\mathbf{G}, \mathbf{X}) \to \mathrm{Sh}_{\mathsf{K}}(\mathbf{G}, \mathbf{X})_{E_1}$ is given on $\mathbb{C}$-points by

$$\alpha_{\mathsf{K}_1,\mathsf{K}} \left(\mathbf{G}_1(\mathbb{Q})(x, g_1)\mathsf{K}_1 \right) = \mathbf{G}(\mathbb{Q})(\alpha \circ x, \alpha(g_1))\mathsf{K}.$$

If the induced map $\alpha\colon G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ is an isogeny, then each $\alpha_{K_1,K}$ is finite étale, as can be checked on connected components (cf. [She17, p. 6620]).

By a morphism $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1) \to (\mathbf{G}, \mathbf{X}, \mathcal{G})$ of parahoric Shimura data we mean a morphism $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1) \to (\mathbf{G}, \mathbf{X})$ of Shimura data together with a specified model $\mathcal{G}_1 \to \mathcal{G}$ of $G_1 \to G$, which we also denote α. We say that α is an embedding if $\mathcal{G}_1 \to \mathcal{G}$ is a closed embedding.

4.2. **Integral canonical models.** We consider the following objects:

- a symplectic space $\mathbf{\Lambda}_0$ over $\mathbb{Z}_{(p)}$,
- set $\mathbf{V}_0 := \mathbf{\Lambda}_0 \otimes_{\mathbb{Z}} \mathbb{Q}$,
- set $\Lambda_0 := \mathbf{\Lambda}_0 \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$,
- set $V_0 := \mathbf{V}_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p = \Lambda_0[1/p]$.

We then have the *Siegel Shimura datum* $(\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm})$ (see [Mil05, §6]) with reflex field $\mathbb{Q}$. For a neat compact open subgroup $\mathsf{L} \subseteq \mathrm{GSp}(\mathbf{V}_0)(\mathbb{A}_f)$ there is an identification of $\mathbf{Sh}_{\mathsf{L}}(\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm})$ with Mumford's moduli space $\mathsf{A}_{\mathsf{L}}(V_0)$ of principally polarized abelian schemes with level L-structure (see [Del71, §4]). Set $\mathsf{L}_0 = \mathrm{GSp}(\Lambda_0)$. Then, $\mathrm{Sh}_{\mathsf{L}_0\mathsf{L}^p}$ admits a smooth model $\mathscr{M}_{\mathsf{L}^p}(\Lambda_0)$ over $\mathbb{Z}_p$ with a similar moduli description (see loc. cit.).

Recall that $(\mathbf{G}, \mathbf{X})$ is of *Hodge type* if there exists an embedding (called a *Hodge embedding*) $(\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm})$ for some symplectic space $\mathbf{V}_0$ over $\mathbb{Q}$. Recall that $(\mathbf{G}, \mathbf{X})$ is called of *abelian type* if there exists a Shimura datum $(\mathbf{G}_1, \mathbf{X}_1)$ of Hodge type and an isogeny $\mathbf{G}_1^{\mathrm{der}} \to \mathbf{G}^{\mathrm{der}}$ inducing an isomorphism of adjoint Shimura data $(\mathbf{G}_1^{\mathrm{ad}}, \mathbf{X}_1^{\mathrm{ad}}) \to (\mathbf{G}^{\mathrm{ad}}, \mathbf{X}^{\mathrm{ad}})$. As in [Lov17b, 2.5.14] (cf. the proof of [Kis10, Corollary 3.4.14]), if $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is an unramified Shimura datum of abelian type then $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$ may be further chosen so that $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ admits a central isogeny model $\mathcal{G}_1^{\mathrm{der}} \to \mathcal{G}^{\mathrm{der}}$. For such well-chosen data, we say that $(\mathbf{G}_1, \mathbf{X}_1)$ (resp. $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$) is *adapted* to $(\mathbf{G}, \mathbf{X})$ (resp. $(\mathbf{G}, \mathbf{X}, \mathcal{G})$).

Shimura data of PEL type (see [Mil05, §8]) are of abelian type, but not conversely.[24]

Example 4.1. Let $\mathbf{V}$ be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$. The group $\mathrm{GSpin}(\mathbf{V})$ acts transitively on the space $\mathbf{X}$ of oriented negative definite 2-planes in $\mathbf{V}_{\mathbb{R}}$, and $\mathbf{X}$ can be identified with a $\mathrm{GSpin}(\mathbf{V})(\mathbb{R})$-conjugacy class of morphisms $\mathbb{S} \to \mathrm{GSpin}(\mathbf{V})_{\mathbb{R}}$ (see [MP16, §1]). The pair $(\mathrm{GSpin}(\mathbf{V}), \mathbf{X})$ is a Shimura datum of Hodge type which is not of PEL type (see [MP16, §3]).

Example 4.2. Let $F \supsetneq \mathbb{Q}$ be a totally real field, B a quaternion algebra over F, and $\mathbf{G}_B$ the algebraic $\mathbb{Q}$-group $B^{\times}$. There is a Shimura datum $(\mathbf{G}_B, \mathbf{X}_B)$ associated to B (see [Mil05, Example 5.24]). If B is not $\mathbb{R}$-split then $(\mathbf{G}_B, \mathbf{X}_B)$ is of abelian type, but not of Hodge type. The Shimura varieties associated to such $(\mathbf{G}_B, \mathbf{X}_B)$ include Shimura curves.

Example 4.3. For a $\mathbb{Q}$-torus $\mathbf{T}$ any homomorphism $h\colon \mathbb{S} \to \mathbf{T}_{\mathbb{R}}$ defines a Shimura datum $(\mathbf{T}, \{h\})$ of abelian type, which is rarely of Hodge type, called of *special type*.

Suppose now that $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is an unramified Shimura datum of abelian type. Set

$$\mathrm{Sh}_{\mathsf{K}_0} = \varprojlim_{\mathsf{K}^p} \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p} = \mathrm{Sh}/\mathsf{K}_0,$$

which is a scheme with a continuous action of $\mathbf{G}(\mathbb{A}_f^p)$. In [Kis10], there is constructed an $\mathcal{O}_E$-scheme $\mathscr{S}$ with a continuous action of $\mathbf{G}(\mathbb{A}_f^p)$ whose generic fiber recovers $\mathrm{Sh}_{\mathsf{K}_0}$ with its $\mathbf{G}(\mathbb{A}_f^p)$-action. For a neat compact open subgroup $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ write $\mathscr{S}_{\mathsf{K}^p} := \mathscr{S}/\mathsf{K}^p$, and for neat compact open subgroups K^p and K'^p of $\mathbf{G}(\mathbb{A}_f^p)$, and an element g^p of $\mathbf{G}(\mathbb{A}_f^p)$ such that $(g^p)^{-1}\mathsf{K}^p g^p \subseteq \mathsf{K}'^p$ denote by $t_{\mathsf{K}^p,\mathsf{K}'^p}(g^p)$ the induced map $\mathscr{S}_{\mathsf{K}^p} \to \mathscr{S}_{\mathsf{K}'^p}$, subject to the same notational shortenings as in the generic fiber case. Then, $\mathscr{S}$ is a so-called *integral canonical model*: the $\mathcal{O}_E$-schemes $\mathscr{S}_{\mathsf{K}^p}$ are smooth (and quasi-projective), the maps $t_{\mathsf{K}^p,\mathsf{K}'^p}(g^p)$ are finite étale, and for any regular and formally smooth $\mathcal{O}_E$-scheme $\mathscr{X}$ any morphism $\mathscr{X}_\eta \to \mathrm{Sh}_{\mathsf{K}_0}$ of E-schemes lifts uniquely to a morphism of $\mathcal{O}_E$-schemes $\mathscr{X} \to \mathscr{S}$ (the *extension property*).

Example 4.4. When $(\mathbf{G}, \mathbf{X}) = (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm})$, and $\mathsf{L}_0 = \mathrm{GSp}(\Lambda_0)$, then the integral canonical model is precisely the system $\{\mathscr{M}_{\mathsf{L}^p}(\Lambda_0)\}$ (cf. [Moo98, Corollary 3.8]).

If $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1) \to (\mathbf{G}, \mathbf{X}, \mathcal{G})$ is a morphism of unramified Shimura data of abelian type, then the morphism $\mathrm{Sh}_{\mathsf{K}_{0,1}} \to (\mathrm{Sh}_{\mathsf{K}_0})_{E_1}$ has a unique model $\mathscr{S}_1 \to \mathscr{S}_{\mathcal{O}_{E_1}}$ equivariant for the map $\mathbf{G}_1(\mathbb{A}_f^p) \to \mathbf{G}(\mathbb{A}_f^p)$. If $\alpha(\mathsf{K}_1^p) \subseteq \mathsf{K}^p$ we denote by $\alpha_{\mathsf{K}_1^p,\mathsf{K}^p}$ the induced morphism $\mathscr{S}_{\mathsf{K}_1^p} \to (\mathscr{S}_{\mathsf{K}^p})_{\mathcal{O}_{E_1}}$.

Lemma 4.5. *If $\alpha\colon \mathcal{G}_1^{\mathrm{der}} \to \mathcal{G}^{\mathrm{der}}$ is a central isogeny, then each $\alpha_{\mathsf{K}_1^p,\mathsf{K}^p}$ is finite étale.*

[24] An essentially full classification of Shimura varieties of abelian type is given in the appendix of [MS81].

Proof. It suffices to show the maps $\mathscr{S}_{\mathsf{K}_1^p}(\mathcal{G}_1^{\mathrm{der}}, \mathbf{X}_1^+) \to \mathscr{S}_{\mathsf{K}^p}(\mathcal{G}^{\mathrm{der}}, \mathbf{X}^+)$ (with notation as in [Kis10, (3.4.9)]) are finite étale. Let $(\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2)$ be an unramified Shimura datum of Hodge type adapted to $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$ and thus to $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ and fix a sufficiently small neat compact open subgroup K_2^p. As the map $\mathscr{S}_{\mathsf{K}_1^p}(\mathcal{G}_1^{\mathrm{der}}, \mathbf{X}_1^+) \to \mathscr{S}_{\mathsf{K}^p}(\mathcal{G}^{\mathrm{der}}, \mathbf{X}^+)$ fits into a commutative triangle with maps of the form $\mathscr{S}_{\mathsf{K}_2^p}(\mathcal{G}_2^{\mathrm{der}}, \mathbf{X}_2^+) \to \mathscr{S}_{\mathsf{K}_1^p}(\mathcal{G}_1^{\mathrm{der}}, \mathbf{X}_1^+)$ and $\mathscr{S}_{\mathsf{K}_2^p}(\mathcal{G}_2^{\mathrm{der}}, \mathbf{X}_2^+) \to \mathscr{S}_{\mathsf{K}^p}(\mathcal{G}^{\mathrm{der}}, \mathbf{X}^+)$ it suffices to show these maps are finite étale. But, this follows from [Lov17b, 2.5.14] as the group Δ^N is finite and acts freely by [Lov17b, Proposition 2.5.9 and Lemma 2.5.10]. □

For an unramified Shimura datum $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ of Hodge type, an *integral Hodge embedding* is an embedding $\iota\colon (\mathbf{G}, \mathbf{X}, \mathcal{G}) \hookrightarrow (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm}, \mathrm{GSp}(\Lambda_0))$. By [Kim18b, 3.3.1], such an integral Hodge embedding always exists. As each $\mathscr{M}_{\mathsf{L}^p}(\Lambda_0)$ is a fine moduli space of principally polarized abelian varieties it has a universal abelian scheme $\mathscr{A}_{\mathsf{L}^p}$ compatible in L^p. If $\iota(\mathsf{K}^p) \subseteq \mathsf{L}^p$, we (suppressing ι from the notation) denote by $\mathscr{A}_{\mathsf{K}^p} \to \mathscr{S}_{\mathsf{K}^p}$ the pullback of $\mathscr{A}_{\mathsf{L}^p}$ along $\iota_{\mathsf{K}^p,\mathsf{L}^p}$. Denote by $\widehat{\mathscr{A}_{\mathsf{K}^p}} \to \widehat{\mathscr{S}_{\mathsf{K}^p}}$ its p-adic completion (equiv. the pullback of $\mathscr{A}_{\mathsf{K}^p}$ along $\widehat{\mathscr{S}_{\mathsf{K}^p}} \to \mathscr{S}_{\mathsf{K}^p}$), and by $A_{\mathsf{K}^p} \to \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ the generic fiber of $\mathscr{A}_{\mathsf{K}^p} \to \mathscr{S}_{\mathsf{K}^p}$.

We finally observe that the connected components of $\mathscr{S}$ are homogeneous in a suitable sense.

Lemma 4.6 (cf. [Kis10, Lemma 2.2.5])**.** *The action of $\mathbf{G}(\mathbb{A}_f^p)$ on $\pi_0(\mathscr{S})$ is transitive.*

4.3. **Étale realization functors.** Following [KSZ21, Definition 1.5.4], for a multiplicative $\mathbb{Q}$-group $\mathbf{T}$ denote by $\mathbf{T}_{\mathrm{a}}$ the largest $\mathbb{Q}$-anisotropic subtorus of $\mathbf{T}$, and by $\mathbf{T}_{\mathrm{ac}}$ the smallest subtorus of $\mathbf{T}_{\mathrm{a}}$ whose base change to $\mathbb{R}$ contains the maximal split subtorus of $(\mathbf{T}_a)_{\mathbb{R}}$. Both of these constructions are functorial in the $\mathbb{Q}$-torus $\mathbf{T}$. For a reductive $\mathbb{Q}$-group $\mathbf{G}$ denote by $\mathbf{G}^c$ the $\mathbb{Q}$-group $\mathbf{G}/\mathbf{Z}_{\mathrm{ac}}$, and by G^c the group $\mathbf{G}^c_{\mathbb{Q}_p}$.

Fix a parahoric Shimura datum $(\mathbf{G}, \mathbf{X}, \mathcal{G})$. There is a canonical map of Bruhat–Tits buildings $B(G, F) \to B(G^c, F)$. Let x denote a point of $B(G, F)$ corresponding to $\mathcal{G}$, and x^c its image in $B(G^c, F)$ (see [KP18, §1.1–1.2] and the references therein). Set $\mathcal{G}^c$ to be the parahoric group scheme associated to x^c (denoted by $\mathcal{G}^{\circ}_{x^c}$ in [KP18, §1.2]). By [KP18, Proposition 1.1.4], $\mathcal{G}$ is a central extension of $\mathcal{G}^c$, and so one is reductive if and only if the other is.[25] Denote by $\boldsymbol{\mu}_h^c$ the conjugacy class of cocharacters of G^c induced by $\boldsymbol{\mu}_h$, and if $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is unramified let μ_h^c be conjugacy class of cocharacters of $\mathcal{G}^c_{\mathbb{Z}_p}$ induced by μ_h.

Lemma 4.7. *A morphism $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1) \to (\mathbf{G}, \mathbf{X})$ (resp. $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1) \to (\mathbf{G}, \mathbf{X}, \mathcal{G})$) of Shimura data (resp. parahoric Shimura data) induces a morphism $\mathbf{G}_1^c \to \mathbf{G}^c$ (resp. $\mathcal{G}_1^c \to \mathcal{G}^c$).*

Proof. The claim concerning Shimura data would follow from $\alpha(\mathbf{Z}_{1,\mathrm{ac}}) \subseteq \mathbf{Z}_{\mathrm{ac}}$. To show this, it suffices to show that if $\mathbf{S} = (\alpha^{-1}(\mathbf{Z}_{\mathrm{ac}}) \cap \mathbf{Z}_{1,\mathrm{ac}})^{\circ}$, then $\mathbf{S}_{\mathbb{R}}$ contains the split component of $(\mathbf{Z}_{1,\mathrm{a}})_{\mathbb{R}}$. Suppose not and that $\mathbb{R}^{\times} \subseteq (\mathbf{Z}_{1,\mathrm{a}})(\mathbb{R})$ is not contained in $\mathbf{S}(\mathbb{R})$. If $\alpha(\mathbb{R}^{\times})$ is not contained in $\mathbf{Z}_{\mathbb{R}}$ then we arrive at a contradiction as in the proof of [Lov17b, Lemma 3.1.3]. As $\mathbb{R}^{\times} \subseteq (\mathbf{Z}_{1,\mathrm{a}})(\mathbb{R})$ this then implies that $\alpha(\mathbb{R}^{\times}) \subseteq \mathbf{Z}_{\mathrm{ac}}(\mathbb{R})$ which is again a contradiction. The claim concerning parahoric Shimura data then follows by applying [KP18, Proposition 1.1.4]. □

If $(\mathbf{G}, \mathbf{X})$ is a Shimura datum of Hodge type, then $\mathbf{G}$ is equal to $\mathbf{G}^c$. Indeed, this can be checked explicitly for Siegel datum, and follows by functoriality for arbitrary $(\mathbf{G}, \mathbf{X})$. Shimura data of abelian type need not enjoy this equality in general, and $\mathcal{G} \to \mathcal{G}^c$ is not necessarily an isomorphism even in the unramified case.

For a Shimura datum $(\mathbf{G}, \mathbf{X})$, and a neat compact open $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$, the map

$$\varprojlim_{\mathsf{K}'_p \subseteq \mathsf{K}_p} \mathrm{Sh}_{\mathsf{K}'_p\mathsf{K}^p} \to \mathrm{Sh}_{\mathsf{K}}$$

is a $\mathsf{K}_p/\mathbf{Z}(\mathbb{Q})^-_{\mathsf{K}}$-torsor on $(\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p})_{\mathrm{pro\acute{e}t}}$, where $\mathbf{Z}(\mathbb{Q})^-_{\mathsf{K}}$ is the closure of $\mathbf{Z}(\mathbb{Q}) \cap \mathsf{K}$ in K (see [KSZ21, §1.5.8]). If K_p^c denotes the image of K_p in $G^c(\mathbb{Q}_p)$, loc. cit. shows that $K_p \to K_p^c$ factorizes

[25] For the reader less familiar with Bruhat–Tits theory, [KP18, Proposition 1.1.4] shows that when $\mathcal{G}$ is reductive, $\mathcal{G}^c = \mathcal{G}/\mathcal{Z}$, where $\mathcal{Z}$ is the Zariski closure of $(\mathbf{Z})_{\mathbb{Q}_p}$.

through $K_p/\mathbf{Z}(\mathbb{Q})^-_K$. Denote by T_K the K_p^c-torsor obtained by pushing forward $\varprojlim_{\mathsf{K}'_p\subseteq \mathsf{K}_p} \mathrm{Sh}_{\mathsf{K}'_p\mathsf{K}^p}$ along $\mathsf{K}_p/\mathbf{Z}(\mathbb{Q})^-_\mathsf{K} \to K_p^c$. We obtain an object $\nu_{\mathsf{K},\text{ét}}$ of G^c-$\mathbf{Loc}_{\mathbb{Q}_p}(\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p})$ given by sending $\rho\colon G^c \to \mathrm{GL}(V)$ to the pushforward of T_K along $\rho\colon K_p^c \to \mathrm{GL}(V)$. Fix g in $\mathbf{G}(\mathbb{A}_f)$, and suppose $g^{-1}\mathsf{K}g \subseteq \mathsf{K}'$. If $g = g_pg^p$, and $\mathrm{Int}(g_p^c)$ is the inner automorphism of G^c associated to the image g_p^c of g_p in $G^c(\mathbb{Q}_p)$, then

$$t_{\mathsf{K},\mathsf{K}'}(g)^*(\nu_{\mathsf{K}',\text{ét}}(\rho)) = \nu_{\mathsf{K},\text{ét}}(\rho \circ \mathrm{Int}((g_p^c)^{-1})). \tag{4.3.1}$$

We call the system $\nu_{\text{ét}} := \{\nu_{\mathsf{K}_p\mathsf{K}^p,\text{ét}}\}$ the *(rational p-adic) étale realization functor* on $\mathrm{Sh}_{\mathsf{K}_p}$.

Let $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1) \to (\mathbf{G}, \mathbf{X})$ be a morphism of Shimura data. If $\alpha(\mathsf{K}_1) \subseteq \mathsf{K}$ one obtains a morphism $\mathsf{T}_{\mathsf{K}_1} \to \mathsf{T}_\mathsf{K} \times_{(\mathrm{Sh}_\mathsf{K})_{E_1}} \mathrm{Sh}_{\mathsf{K}_1}$ equivariant for $\alpha^c\colon K_{p,1}^c \to K_p^c$ and, thus an isomorphism of K_p^c-torsors $\alpha^c_*(\mathsf{T}_{\mathsf{K}_1}) \to \mathsf{T}_\mathsf{K} \times_{(\mathrm{Sh}_\mathsf{K})_{E_1}} \mathrm{Sh}_{\mathsf{K}_1}$. This is compatible in K in the obvious way. Equivalently, for ρ in $\mathbf{Rep}_{\mathbb{Q}_p}(G^c)$ there is an identification

$$\alpha^*_{\mathsf{K}_1,\mathsf{K}}(\nu_{\mathsf{K},\text{ét}}(\rho)_{E_1}) = \nu_{\mathsf{K}_1,\text{ét}}(\rho \circ \alpha^c), \tag{4.3.2}$$

compatible in K_1, K, and ξ in the obvious sense.

For a parahoric Shimura datum $(\mathbf{G}, \mathbf{X}, \mathcal{G})$, and $\mathsf{K} = \mathsf{K}_0\mathsf{K}^p$ (recall $K_0 = \mathcal{G}(\mathbb{Z}_p)$), there are analogous integral objects. Again by [KSZ21, §1.5.8], $\mathbf{Z}(\mathbb{Q})_\mathsf{K} \subseteq \mathbf{Z}_{\text{ac}}(\mathbb{Q})$ and so $\mathsf{K}_0 \to \mathcal{G}^c(\mathbb{Z}_p)$ factorizes through $\mathsf{K}_0/\mathbf{Z}(\mathbb{Q})^-_\mathsf{K}$.

Denote by $\mathsf{S}_{\mathsf{K}^p}$ the push forward of $\varprojlim_{\mathsf{K}_p\subseteq \mathsf{K}_0} \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$ along $\mathsf{K}_0/\mathbf{Z}(\mathbb{Q})^-_\mathsf{K} \to \mathcal{G}^c(\mathbb{Z}_p)$. From the contents of §2.1.1, we obtain an associated object of $\mathcal{G}^c$-$\mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})$:

$$\omega_{\mathsf{K}^p,\text{ét}}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \to \mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}).$$

Fix $g = g_pg^p$ in $K_0\mathbf{G}(\mathbb{A}_f^p)$, and suppose $(g^p)^{-1}\mathsf{K}^pg^p \subseteq \mathsf{K}'^p$. If $\mathrm{Int}(g_p^c)$ is the inner automorphism of $\mathcal{G}^c$ associated to the image g_p^c of g_p in $\mathcal{G}^c(\mathbb{Z}_p)$, then

$$t_{\mathsf{K}^p\mathsf{K}_0,\mathsf{K}'^p\mathsf{K}_0}(g)^*(\omega_{\mathsf{K}'^p,\text{ét}}(\xi)) = \omega_{\mathsf{K}^p,\text{ét}}(\xi \circ \mathrm{Int}((g_p^c)^{-1})). \tag{4.3.3}$$

We call the system $\{\omega_{\mathsf{K}^p,\text{ét}}\}$ the *(integral) étale realization functor* on $\mathrm{Sh}_{\mathsf{K}_0}$. That this is an integral model of $\nu_{\mathsf{K}_0\mathsf{K}^p,\text{ét}}$ is made precise by the observation that

$$\omega_{\mathsf{K}^p,\text{ét}}[1/p] = \nu_{\mathsf{K}_0\mathsf{K}^p,\text{ét}} \tag{4.3.4}$$

compatibly in a neat compact open subgroup $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$.

Let $\alpha\colon (\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1) \to (\mathbf{G}, \mathbf{X}, \mathcal{G})$ be a morphism of parahoric Shimura data. If $\alpha(\mathsf{K}_1^p) \subseteq \mathsf{K}^p$ one obtains a morphism $\mathsf{S}_{\mathsf{K}_{0,1}\mathsf{K}_1^p} \to \mathsf{S}_{\mathsf{K}^p} \times_{(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{E_1}} \mathrm{Sh}_{\mathsf{K}_{0,1}\mathsf{K}_1^p}$ equivariant for $\alpha^c\colon \mathcal{G}_1^c(\mathbb{Z}_p) \to \mathcal{G}^c(\mathbb{Z}_p)$ and, thus an isomorphism of $\mathcal{G}^c(\mathbb{Z}_p)$-torsors $\alpha^c_*(\mathsf{S}_{\mathsf{K}^1_p}) \to \mathsf{S}_{\mathsf{K}^p} \times_{(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{E_1}} \mathrm{Sh}_{\mathsf{K}_{0,1}\mathsf{K}_1^p}$. This is compatible in K^p in the obvious way. Equivalently, for ξ in $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$ there is an identification

$$\alpha^*_{\mathsf{K}_1^p,\mathsf{K}^p}(\omega_{\mathsf{K}^p,\text{ét}}(\xi)_{E_1}) = \omega_{\mathsf{K}_1^p,\text{ét}}(\xi \circ \alpha^c), \tag{4.3.5}$$

compatible in K_1^p, K^p, and ξ in the obvious sense.

Proposition 4.8. *The $\mathcal{G}^c(\mathbb{Z}_p)$-local system $\omega_{\mathsf{K}^p,\text{ét}}$ belongs to $\mathcal{G}^c$-$\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}_h^c}_{\mathbb{Z}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})$.*

Proof. The fact that $\omega_{\mathsf{K}^p,\text{ét}}$ is de Rham follows from [LZ17, Corollary 4.9]. To verify the claim about cocharacters, note that by the analytic density of the special points (consider [Mil05, Lemma 13.5] and its proof), we are reduced to the case of special points, but the claim then follows from [LZ17, Lemma 4.8]. ☐

Remark 4.9. In the case of Shimura varieties of abelian type, one may directly verify Proposition 4.8 by reducing to the Siegel-type case (cf. the argument for Theorem 4.12).

Let $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ be an unramified Shimura datum of Hodge type, and fix an integral Hodge embedding $\iota\colon (\mathbf{G}, \mathbf{X}, \mathcal{G}) \hookrightarrow (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^\pm, \mathrm{GSp}(\Lambda_0))$. By Theorem A.14, there is a tensor package $(\Lambda_0, \mathbb{T}_0)$ with $\mathcal{G} = \mathrm{Fix}(\mathbb{T}_0)$. As in [Kim18b, §3.1.2], one may construct from $\mathbb{T}_0 \otimes 1 \subseteq V_0^\otimes$ tensors $\mathbb{T}_{0,p}^{\text{ét}}$ on $\mathcal{H}^1_{\mathbb{Q}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee$ as an object of $\mathbf{Loc}_{\mathbb{Q}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})$, which are compatible in K^p.

Proposition 4.10. *There is an isomorphism of $\mathbb{Z}_p$-local systems $\omega_{\mathsf{K}^p,\text{ét}}(\Lambda_0) \xrightarrow{\sim} \mathcal{H}^1_{\mathbb{Z}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee$ carrying $\omega_{\mathsf{K}^p,\text{ét}}(\mathbb{T}_0) \otimes 1$ in $\omega_{\mathsf{K}^p,\text{ét}}(\Lambda_0)[1/p]$ to $\mathbb{T}^{\text{ét}}_{0,p}$ in $\mathcal{H}^1_{\mathbb{Q}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee$.*

Proof. First suppose that $(\mathbf{G}, \mathbf{X}, \mathcal{G}) = (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^\pm, \mathrm{GSp}(\Lambda_0))$ and ι is the identity embedding. Then, by the moduli description of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ one observes that there is an identification

$$\mathsf{S}_{\mathsf{K}^p} = \underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \underline{\mathbb{Z}_p}, t_0), (T_p(A_{\mathsf{K}^p}), t)\right),$$

where t_0 is the tensor as in [Kim18a, Example 2.1.6], and t is the analogous tensor built from the Weil pairing coming from the principal polarization on A_{K^p}. We deduce a natural identification between $\omega_{\mathsf{K}^p,\text{ét}}(\Lambda_0)$ and $\mathcal{H}^1_{\mathbb{Z}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee$. The desired isomorphism for general $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ comes from the compatability in (4.3.5). To prove that the induced isomorphism of $\mathbb{Q}_p$-local systems takes $\omega_{\mathsf{K}^p,\text{ét}}(\mathbb{T}_0) \otimes 1$ to $\mathbb{T}^{\text{ét}}_{0,p}$, we observe that these constructions admit globalizations over $\mathbf{E}$ in the obvious way, in which case it suffices to check the claim on $\mathbb{C}$-points. But, this then follows from [Mil05, Theorem 7.4]. □

As a result of Proposition 4.10, we see that $\mathbb{T}^{\text{ét}}_{0,p}$ actually lies in the image of the injective map $\mathcal{H}^1_{\mathbb{Z}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee \to \mathcal{H}^1_{\mathbb{Q}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee$. We deduce from the contents of §2.1.1 that $\omega_{\mathsf{K}^p,\text{ét}}$ is the object of $\mathcal{G}$-$\mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})$ associated to the torsor

$$\underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \underline{\mathbb{Z}_p}, \mathbb{T}_0 \otimes 1), (\mathcal{H}^1_{\mathbb{Z}_p}(A_{\mathsf{K}^p}/\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})^\vee, \mathbb{T}^{\text{ét}}_{0,p})\right),$$

which thus is independent of any choices. Similar claims may be verified for $\nu_{\mathsf{K},\text{ét}}$.

We end this section by describing a method, applying ideas from [Lov17b, §4.6–4.7], which will allow us to reduce statements about Shimura data of abelian type to those of Hodge type.

Lemma 4.11. *Let $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ be an unramified Shimura datum of abelian type, and $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$ an adapted unramified Shimura datum of Hodge type. Then, there exists an unramified Shimura datum $(\mathbf{T}, \{h\}, \mathcal{T})$ of special type and an unramified Shimura datum $(\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2)$ of abelian type, both with reflex field $\mathbf{E}_1$, such that*

(1) there exists a morphism of unramified Shimura data

$$\alpha = (\alpha^1, \alpha^2) \colon (\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2) \to (\mathbf{G}_1 \times \mathbf{T}, \mathbf{X}_1 \times \{h\}, \mathcal{G}_1 \times \mathcal{T})$$

such that $\alpha^c \colon \mathcal{G}_2^c \to \mathcal{G}_1 \times \mathcal{T}^c$ is a closed embedding,

(2) there exists a morphism of unramified Shimura data $\beta \colon (\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2) \to (\mathbf{G}, \mathbf{X}, \mathcal{G})$ such that $\beta_{\mathsf{K}^p_2,\mathsf{K}^p} \colon \mathscr{S}_{\mathsf{K}^p_2} \to (\mathscr{S}_{\mathsf{K}^p})_{\mathcal{O}_{E_1}}$ is finite étale.

Proof. Choose a connected component $\mathbf{X}_1^+$ of $\mathbf{X}_1$ and an element h_G of $\mathbf{X}_1^+$. Let $(\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2)$ be the unramifed Shimura datum of abelian type obtained by applying the construction in [Lov17b, §4.6] to $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$ and the choice of $\mathbf{X}_1^+$, and set

$$(\mathbf{T}, \{h\}, \mathcal{T}) := \left(\mathrm{Res}_{\mathbf{E}_1/\mathbb{Q}}\, \mathbb{G}_{m,\mathbf{E}_1}, \{h_E\}, \left(\mathrm{Res}_{\mathcal{O}_{\mathbf{E}_1}/\mathbb{Z}}\, \mathbb{G}_{m,\mathcal{O}_{\mathbf{E}_1}}\right)_{\mathbb{Z}_p}\right),$$

with h_E as in [Lov17b, 4.6.4]. The map α is then constructed from the natural inclusion of $\mathbf{G}_2 = \mathbf{G}_1 \times_{\mathbf{G}_1^{\mathrm{ab}}} \mathbf{T}$ into $\mathbf{G}_1 \times \mathbf{T}$. To prove that α^c is a closed embedding, observe that as $(\mathbf{G}_1, \mathbf{X}_1)$ is of Hodge type that $(\mathbf{Z}_1)_{\mathrm{ac}}$ is trivial, and as $\mathbf{Z}_1 \to \mathbf{G}_1^{\mathrm{ab}}$ is an isogeny that $(\mathbf{G}_1^{\mathrm{ab}})_{\mathrm{ac}}$ is also trivial. From this we deduce that $\mathbf{Z}_2^c = \mathbf{T}_{\mathrm{ac}}$ and so the claim follows. The map β is constructed as in [Lov17b, §4.7.2], the second claim follows from Lemma 4.5. □

4.4. **Prismatic realization functors.** For an unramified Shimura datum $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ of abelian type, and a neat compact open subgroup $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}^p_f)$, we associate the smooth p-adic formal scheme $\widehat{\mathscr{S}}_{\mathsf{K}^p}$, and the open embedding of adic spaces

$$\mathcal{S}_{\mathsf{K}^p} := (\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta \subseteq \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}^p},$$

with quasi-compact source, which is an isomorphism when $\mathscr{S}_{\mathsf{K}^p} \to \mathrm{Spec}(\mathcal{O}_E)$ is proper (see [Hub94, Remark 4.6 (iv)]). The morphisms $t_{\mathsf{K}^p,\mathsf{K}'^p}(g^p)$ induce morphisms on the adic spaces $\mathcal{S}_{\mathsf{K}^p}$

compatible with those maps on the $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}^p}$, and we use similar notational shortenings for them. We may also consider the functors

$$\omega_{\mathsf{K}^p,\mathrm{an}}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \to \mathbf{Loc}_{\mathbb{Z}_p}(\mathcal{S}_{\mathsf{K}^p}), \qquad \Lambda \mapsto \omega_{\mathsf{K}^p,\text{ét}}(\Lambda)^{\mathrm{an}}|_{\mathcal{S}_{\mathsf{K}^p}},$$

which enjoy the same compatabilities for varying level structure and morphisms of unramified Shimura varieties as the $\mathcal{G}^c$-local systems $\omega_{\mathsf{K}^p,\text{ét}}$.

We aim to apply the machinery we have developed to find a 'prismatic model' of this $\mathcal{G}^c$-local system. More precisely, by a *prismatic realization functor* at level K^p, we mean an exact $\mathbb{Z}_p$-linear $\otimes$-functor (which is unique up to unique isomorphism, as $T_{\text{ét}}$ is fully-faithful)

$$\omega_{\mathsf{K}^p,\Delta}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \to \mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\Delta}),$$

together with $\jmath_{\mathsf{K}^p}\colon T_{\text{ét}} \circ \omega_{\mathsf{K}^p,\Delta} \xrightarrow{\sim} \omega_{\mathsf{K}^p,\mathrm{an}}$. If the isomorphisms $\jmath_{\mathsf{K}^p}$ are chosen compatibly in K^p, we call the collection $\{(\omega_{\mathsf{K}^p,\Delta}, \jmath_{\mathsf{K}^p})\}$ a *prismatic canonical model* of $\{\omega_{\mathsf{K}^p,\mathrm{an}}\}$, which is unique up to unique isomorphism. We often omit the data of $\jmath_{\mathsf{K}^p}$ from the notation.

Fix prismatic canonical models $\{\omega_{\mathsf{K}^p,\Delta}\}$ and $\{\omega_{\mathsf{K}^p_1,\Delta}\}$ for unramified Shimura data $(\mathbf{G},\mathbf{X},\mathcal{G})$ and $(\mathbf{G}_1,\mathbf{X}_1,\mathcal{G}_1)$ respectively. If $\alpha\colon(\mathbf{G}_1,\mathbf{X}_1,\mathcal{G}_1)\to(\mathbf{G},\mathbf{X},\mathcal{G})$ is a morphism, then for any ξ in $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$, and neat compact open subgroups $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}^p_f)$ and $\mathsf{K}^p_1 \subseteq \mathbf{G}_2(\mathbb{A}^p_f)$ with $\alpha(\mathsf{K}^p_1)\subseteq \mathsf{K}^p$, one has canonical, compatible in K^p, K^p_1, and ξ, identifications

$$\alpha^*_{\mathsf{K}^p_1,\mathsf{K}^p}(\omega_{\mathsf{K}^p,\Delta}(\xi)_{\mathcal{O}_1}) = \omega_{\mathsf{K}^p_1,\Delta}(\xi\circ\alpha^c). \tag{4.4.1}$$

This follows by appropriately applying $T_{\text{ét}}^{-1}$ and the isomorphisms $\jmath_{\mathsf{K}^p}$ and $\jmath_{\mathsf{K}^p_1}$ to (4.3.5).

Theorem 4.12. *Suppose that $(\mathbf{G},\mathbf{X},\mathcal{G})$ is an unramified Shimura datum of abelian type. Then, for any K^p, the functor $\omega_{\mathsf{K}^p,\mathrm{an}}$ takes values in $\mathbf{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(\mathcal{S}_{\mathsf{K}^p})$. In particular, the collection*

$$T_{\text{ét}}^{-1}\circ\omega_{\mathsf{K}^p,\mathrm{an}} =\colon \omega_{\mathsf{K}^p,\Delta}\colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \to \mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\Delta})$$

forms a prismatic canonical model of $\{\omega_{\mathsf{K}^p,\mathrm{an}}\}$.

Remark 4.13. If $(\mathbf{G},\mathbf{X},\mathcal{G})$ is of special type this theorem was (implicitly) obtained by Daniels in [Dan22], and his construction agrees with ours by the unicity of canonical prismatic models.

We first prove a refined version of this theorem when $(\mathbf{G},\mathbf{X},\mathcal{G})$ is of Hodge type using Theorem 2.27. Choose an integral Hodge embedding $\iota\colon(\mathbf{G},\mathbf{X},\mathcal{G})\to(\mathrm{GSp}(\mathbf{V}_0),\mathfrak{h}^{\pm},\mathrm{GSp}(\Lambda_0))$ and write $\mathcal{A}_{\mathsf{K}^p}\to\mathcal{S}_{\mathsf{K}^p}$ for the generic fiber of $\widehat{\mathscr{A}}_{\mathsf{K}^p}\to\widehat{\mathscr{S}}_{\mathsf{K}^p}$. Define

$$\mathbb{T}^{\mathrm{an}}_{0,p} := (\mathbb{T}^{\text{ét}}_{0,p})^{\mathrm{an}}|_{\mathcal{S}_{\mathsf{K}^p}} \subseteq (\mathcal{H}^1_{\mathbb{Z}_p}(\mathcal{A}_{\mathsf{K}^p}/\mathcal{S}_{\mathsf{K}^p})^{\vee})^{\otimes}.$$

By Proposition 4.10, we have a canonical identification

$$(\omega_{\mathsf{K}^p,\mathrm{an}}(\Lambda_0),\omega_{\mathsf{K}^p,\mathrm{an}}(\mathbb{T}_0)) \xrightarrow{\sim} (\mathcal{H}^1_{\mathbb{Z}_p}(\mathcal{A}_{\mathsf{K}^p}/\mathcal{S}_{\mathsf{K}^p})^{\vee},\mathbb{T}^{\mathrm{an}}_{0,p}).$$

Combining [ALB23, Corollary 4.64] and [GR22, Theorem 1.10 (i)], we deduce that $\mathcal{H}^1_{\mathbb{Z}_p}(\mathcal{A}_{\mathsf{K}^p}/\mathcal{S}_{\mathsf{K}^p})^{\vee}$ has prismatically good reduction with a canonical identification

$$T_{\text{ét}}^{-1}(\mathcal{H}^1_{\mathbb{Z}_p}(\mathcal{A}_{\mathsf{K}^p}/\mathcal{S}_{\mathsf{K}^p})^{\vee}) = \mathcal{H}^1_{\Delta}(\widehat{\mathscr{A}}_{\mathsf{K}^p}/\widehat{\mathscr{S}}_{\mathsf{K}^p})^{\vee},$$

compatible in K^p. Applying $T_{\text{ét}}^{-1}$ to $\mathbb{T}^{\mathrm{an}}_{0,p}$ gives rise to a set $\mathbb{T}^{\Delta}_{0,p}$ of tensors on the object $\mathcal{H}^1_{\Delta}(\widehat{\mathscr{A}}_{\mathsf{K}^p}/\widehat{\mathscr{S}}_{\mathsf{K}^p})^{\vee}$ of $\mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\Delta})$. The following is a direct consequence of Theorem 2.27, and immediately implies Theorem 4.12 for $(\mathbf{G},\mathbf{X},\mathcal{G})$ of Hodge type by Proposition 1.28.

Theorem 4.14. *Suppose that $(\mathbf{G},\mathbf{X},\mathcal{G})$ is an unramified Shimura datum of Hodge type. Then,*

$$\underline{\mathrm{Isom}}\Big((\Lambda_0\otimes_{\mathbb{Z}_p}\mathcal{O}_{(\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\Delta}},\mathbb{T}_0\otimes 1),(\mathcal{H}^1_{\Delta}(\widehat{\mathscr{A}}_{\mathsf{K}^p}/\widehat{\mathscr{S}}_{\mathsf{K}^p})^{\vee},\mathbb{T}^{\Delta}_{0,p})\Big)$$

is a prismatic $\mathcal{G}$-torsor with F-structure on $(\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\Delta}$, compatible in K^p.

To reduce from the abelian type case to the Hodge type case, we first require a simple lemma concerning prismatically good reduction local systems. We use the notation from the beginning of §1.

Lemma 4.15. *Suppose that $\mathfrak{X}_2 \to \mathfrak{X}_1$ is a finite étale cover where $\mathfrak{X}_1 \to \mathrm{Spf}(\mathcal{O}_K)$ is smooth. Then, an object $\mathbb{L}_1$ of $\mathbf{Loc}_{\mathbb{Z}_p}(X_1)$ has prismatically good reduction if and only if $\mathbb{L}_2 := \mathbb{L}_1|_{X_2}$ does.*

Proof. It suffices to prove the if condition. We may assume that $\mathfrak{X}_i = \mathrm{Spf}(R_i)$ where R_i are (framed) small $\mathcal{O}_K$-algebras. That $\mathbb{L}_1$ is crystalline is clear. Let $(\mathcal{V}^i, \varphi_{\mathcal{V}^i})$ denote the object $T_{\mathfrak{X}_i}^{-1}(\mathbb{L}_i)$ of $\mathbf{Vect}^{\mathrm{an},\varphi}((\mathfrak{X}_i)_{\Delta})$. Then, by the flatness of $\mathfrak{X}_2 \to \mathfrak{X}_1$, we have that

$$(j_{(\mathfrak{S}_{R_1},(E))})_* \mathcal{V}^1_{(\mathfrak{S}_{R_1},(E))} \otimes_{\mathfrak{S}_{R_1}} \mathfrak{S}_{R_2} = (j_{(\mathfrak{S}_{R_2},(E))})_* \mathcal{V}^2_{(\mathfrak{S}_{R_2},(E))}.$$

As $\mathbb{L}_2$ has prismatically good reduction the right-hand side is a vector bundle by Proposition 1.24 and thus so is $(j_{(\mathfrak{S}_{R_1},(E))})_* \mathcal{V}^1_{(\mathfrak{S}_{R_1},(E))}$. Thus, $\mathbb{L}_1$ has prismatically good reduction again by Proposition 1.24. □

Proof of Theorem 4.12. We freely use notation from Lemma 4.11. We first prove the claim for $(\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2)$. Choosing a faithful representation $\xi_2' = \xi_1 \otimes \xi_t$ of $\mathcal{G}_1 \times \mathcal{T}^c$, where ξ_1 (resp. ξ_t) is a faithful representation of $\mathcal{G}_1$ (resp. $\mathcal{T}^c$), we obtain the faithful representation $\xi_2 := \xi_2' \circ \alpha^c$ of $\mathcal{G}_2^c$. Choosing neat compact open subgroups $\mathsf{K}_1^p \subseteq \mathbf{G}_1(\mathbb{A}_f^p)$ and $\mathsf{K}_t^p \subseteq \mathbf{T}(\mathbb{A}_f^p)$ such that $\alpha(\mathsf{K}_2^p) \subseteq \mathsf{K}_1^p \times \mathsf{K}_t^p$, we see from (4.3.5) that

$$\omega_{\mathsf{K}_2^p,\mathrm{an}}(\xi_2) = (\alpha^1_{\mathsf{K}_2^p K_{0,2}, \mathsf{K}_1^p \mathsf{K}_{0,1}})^*(\omega_{\mathsf{K}_1^p,\mathrm{an}}(\xi_1)) \otimes_{\underline{\mathbb{Z}_p}} (\alpha^2_{\mathsf{K}_2^p K_{0,2}, \mathsf{K}_t^p \mathsf{K}_{0,t}})^*(\omega_{\mathsf{K}_t^p,\mathrm{an}}(\xi_t)).$$

But, $\omega_{\mathsf{K}^p,\mathrm{an}}(\xi_1)$ has prismatically good reduction by Theorem 4.14. Moreover, as $\widehat{\mathscr{S}_{\mathsf{K}_t^p}}$ is of the form $\coprod \mathrm{Spf}(\mathcal{O}_{E'})$, for connected finite étale $\mathcal{O}_E$-algebras $\mathcal{O}_{E'}$ (e.g. see [Dan22, §4.1 and §4.4]), $\omega_{\mathsf{K}_t^p,\mathrm{an}}(\xi_t)$ has prismatically good reduction by [GR22, Proposition 3.7]. Thus, as having prismatically good reduction is preserved by pullbacks and tensor products, the claim follows from Corollary 2.29.

Now, to prove the claim for $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ it suffices to prove that for each compact open subgroup K^p and each connected component $\mathscr{C}$ of $\mathscr{S}_{\mathsf{K}^p}$ that $\omega_{\mathsf{K}^p,\mathrm{an}}(\xi)|_{\widehat{\mathscr{C}}_\eta}$ has prismatically good reduction. By Lemma 4.6 and Equation (4.3.3), we may assume that there exists some neat compact open subgroup $\mathsf{K}_2^p \subseteq \mathbf{G}_2(\mathbb{A}_f^p)$ such that $\beta(\mathsf{K}_2^p) \subseteq \mathsf{K}^p$ and $\mathscr{C}$ lies in the image of $\beta_{\mathsf{K}_2^p,\mathsf{K}^p}$. As $\beta_{\mathsf{K}_2^p,\mathsf{K}^p}$ is finite étale the claim follows from (4.3.5) and Lemma 4.15. □

4.5. **Potentially crystalline loci and stratifications.** Let K be a complete discrete valuation field with perfect residue field, and let X be a quasi-separated adic space locally of finite type over K, and Σ be either $\mathbb{Z}_p$ or $\mathbb{Q}_p$. For an object $\mathbb{L}$ of $\mathbf{Loc}_\Sigma(X)$ we call a point x of $|X|^{\mathrm{cl}}$ *(potentially) crystalline* for $\mathbb{L}$ if $\mathbb{L}_x$ is a (potentially) crystalline representation of $\Gamma_{k(x)}$. There exists at most one quasi-compact open subset $U \subseteq X$ such that $|U|^{\mathrm{cl}}$ is the set of potentially crystalline points of $\mathbb{L}$ (cf. [Hub93, Corollary 4.3]). In this case we call U the *potentially crystalline locus* of $\mathbb{L}$. For a K-scheme S locally of finite type, and an object $\mathbb{L}$ of $\mathbf{Loc}_\Sigma(S)$, if we speak of the potentially crystalline locus of $\mathbb{L}$ we mean the potentially crystalline locus of $\mathbb{L}^{\mathrm{an}}$. These definitions apply equal well for $\mathcal{G}$-objects (or G-objects) ω in these categories.[26]

Observe that potentially crystalline points satisfy pullback stability. More precisely, for a map of rigid K-spaces $f\colon X' \to X$, a classical point x' of X' is potentially crystalline for $f^*(\mathbb{L})$ if and only if $x = f(x')$ is a potentially crystalline point for $\mathbb{L}$. If $k(x')/k(x)$ is unramified (e.g. $f = \mathfrak{f}_\eta$ for a finite étale model $\mathfrak{f}\colon \mathfrak{X}' \to \mathfrak{X}$), one may replace 'potentially crystalline' by 'crystalline'.

For a Shimura datum $(\mathbf{G}, \mathbf{X})$ of (pre-)abelian type, and a neat compact open subgroup $\mathsf{K} \subseteq \mathbf{G}(\mathbb{A}_f)$, the existence of a potentially crystalline locus $U_\mathsf{K} \subseteq \mathrm{Sh}_\mathsf{K}^{\mathrm{an}}$ for $\nu_{\mathsf{K},\mathrm{\acute{e}t}}$ was established in [IM20, Theorem 5.17] (see [IM20, Remark 2.12] and [LZ17, Theorem 1.2]). If $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is an unramified Shimura datum of abelian type, and $\mathsf{K} = \mathsf{K}_0\mathsf{K}^p$, we abbreviate U_K to U_{K^p} which coincides with the potentially crystalline locus of $\omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}$.

[26]Although as observed in Proposition 2.19, this will coincide with the set of potentially crystalline points for the value of ω on any faithful representation of $\mathcal{G}$ or G.

We now describe U_{K^p} for unramified Shimura data of abelian type, generalizing results of Imai–Mieda in the PEL setting (see [IM20, Corollary 2.11 and Proposition 5.4] and [IM13, §7]).

Proposition 4.16. *Let $(\mathbf{G},\mathbf{X},\mathcal{G})$ be an unramified Shimura datum of abelian type, and $\mathsf{K}^p\subseteq\mathbf{G}(\mathbb{A}_f^p)$ a neat compact open subgroup. Then, $U_{\mathsf{K}^p}=(\widehat{\mathscr{S}_{\mathsf{K}^p}})_\eta$ and all the classical points of U_{K^p} are crystalline for $\omega_{\mathsf{K}^p,\text{ét}}$.*

Proof. We know that all the classical points of $\mathcal{S}_{\mathsf{K}^p}$ are crystalline for $\omega_{\mathsf{K}^p,\text{ét}}$ by Theorem 4.12. Moreover, the full claim holds in the Siegel-type case by [IM20, Theorem 5.17] and its proof.

Assume that $(\mathbf{G},\mathbf{X},\mathcal{G})$ is of Hodge type and choose an integral Hodge embedding $\iota\colon(\mathbf{G},\mathbf{X},\mathcal{G})\hookrightarrow(\mathrm{GSp}(\mathbf{V}_0),\mathfrak{h}^\pm,\mathrm{GSp}(\Lambda_0))$, and a level L^p with $\iota(\mathsf{K}^p)\subseteq\mathsf{L}^p$ and $\iota_{\mathsf{K}_0\mathsf{K}^p,\mathsf{L}_0\mathsf{L}^p}$ is a closed embedding. By the construction of $\mathscr{S}_{\mathsf{K}^p}$ (see [Kis10, Theorem (2.3.8)]), $\mathscr{S}_{\mathsf{K}^p}$ is obtained as the normalization of a closed subscheme of $\mathscr{M}_{\mathsf{L}^p}(\Lambda_0)$, and so finite over $\mathscr{M}_{\mathsf{L}^p}(\Lambda_0)$.[27] So, if x is a classical point of $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}^p}$ not in $\mathcal{S}_{\mathsf{K}^p}$ then the image of x in $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{L}^p}$ is a point outside $\mathcal{S}_{\mathsf{L}^p}$ (see [Hub96, Proposition 1.9.6]). Hence x is not a potentially crystalline point by the Siegel case, and pullback stability.

Assume now that $(\mathbf{G},\mathbf{X},\mathcal{G})$ is of abelian type. We use notation from Lemma 4.11. Let x be a classical point of $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}^p}$ not in $\mathcal{S}_{\mathsf{K}^p}$. By Lemma 4.6 and Equation (4.3.3), we may assume that there is a lift x_2 of x in $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_2^p}$, but not in $\mathcal{S}_{\mathsf{K}_2^p}$, for some neat compact open subgroup $\mathsf{K}_2^p\subset\mathbf{G}_2(\mathbb{A}_f^p)$. The image x_1 of x_2 in $\mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_1^p}$ for an appropriate $\mathsf{K}_1^p\subset\mathbf{G}_1(\mathbb{A}_f^p)$ is a point outside $\mathcal{S}_{\mathsf{K}_1^p}$ because $\mathscr{S}_{\mathsf{K}_2^p}$ is finite over $\mathscr{S}_{\mathsf{K}_1^p}$ (cf. [Hub96, Proposition 1.9.6]). Take a faithful representation ξ_1^{ad} of $\mathbf{G}_1^{\mathrm{ad}}$, inducing faithful representations ξ^{ad} and ξ_2^{ad} of $\mathbf{G}^{\mathrm{ad}}$ and $\mathbf{G}_2^{\mathrm{ad}}$ respectively as $\mathbf{G}^{\mathrm{ad}}\cong\mathbf{G}_2^{\mathrm{ad}}\cong\mathbf{G}_1^{\mathrm{ad}}$. By [IM20, Corollary 2.11 and Proposition 5.4], $\omega_{\mathsf{K}_1^p,\mathrm{an}}(\xi_1^{\mathrm{ad}})_{x_1}$ is not potentially crystalline. Hence $\omega_{\mathsf{K}^p,\mathrm{an}}(\xi^{\mathrm{ad}})_x$ is not potentially crystalline too by pullback stability, as the pullbacks of $\omega_{\mathsf{K}^p,\mathrm{an}}(\xi^{\mathrm{ad}})_x$ and $\omega_{\mathsf{K}_1^p,\mathrm{an}}(\xi_1^{\mathrm{ad}})_{x_1}$ to x_2 are both isomorphic to $\omega_{\mathsf{K}_2^p,\mathrm{an}}(\xi_2^{\mathrm{ad}})_{x_2}$. This implies that x is not potentially crystalline, as desired. $\square$

For neat compact open subgroups $K^p\subseteq\mathbf{G}(\mathbb{A}_f^p)$ and $K_p\subseteq G(\mathbb{Q}_p)$, we may define functions

$$\Sigma_{\mathsf{K}_p\mathsf{K}^p}\colon|U_{\mathsf{K}_p\mathsf{K}^p}|^{\mathrm{cl}}\to B(G^c)$$
$$\left(\text{resp. }\Sigma^\circ_{\mathsf{K}^p}\colon|U_{\mathsf{K}_0\mathsf{K}^p}|^{\mathrm{cl}}\to C(\mathcal{G}^c)\right)$$

(where $C(\mathcal{G}^c)$ is the quotient of $G^c(\breve{\mathbb{Q}}_p)$ by the action of $\mathcal{G}^c(\breve{\mathbb{Z}}_p)$ by σ-conjugacy), associating to x the element of $B(G^c)$ (resp. $C(\mathcal{G}^c)$) associated with the F-isocrystal (resp. F-crystal) with G^c-structure (resp. $\mathcal{G}^c$-structure) given by $\underline{D}_{\mathrm{crys}}\circ(\nu_{\mathsf{K}_p\mathsf{K}^p,\text{ét}})_x$ (resp. $\underline{\mathbb{D}}_{\mathrm{crys}}\circ(\omega_{\mathsf{K}^p,\text{ét}})_x$) (see Example 3.5). These functions are equivariant via the map $\Gamma_E\to\Gamma_k$, when the source (resp. target) is endowed with the natural action of Γ_E (resp. Γ_k). On the other hand, we may define functions

$$\overline{\Sigma}_{\mathsf{K}^p}\colon\mathscr{S}_{\mathsf{K}^p}(\overline{k})\to B(G^c)$$
$$\left(\text{resp. }\overline{\Sigma}^\circ_{\mathsf{K}^p}\colon\mathscr{S}_{\mathsf{K}^p}(\overline{k})\to C(\mathcal{G}^c)\right)$$

in the analogous way using the the $\mathcal{G}$-object in $\mathbf{Vect}^\varphi((\mathscr{S}_{\mathsf{K}^p,k})_{\mathrm{crys}})$ given by $\underline{\mathbb{D}}_{\mathrm{crys}}\circ\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ which is equivariant with respect to the actions of Γ_k.

In the following, we use the notion of an *overconvergent* (also known as *wide*, *partially proper*, or *Berkovich*) open subset of a rigid E-space, as in [FK18, Chapter II, §4.3] .

Proposition 4.17. *The functions $\Sigma_{\mathsf{K}_p\mathsf{K}^p}$ and $\Sigma^\circ_{\mathsf{K}^p}$ are overconvergent locally constant. Moreover,*

$$\Sigma_{\mathsf{K}_0\mathsf{K}^p}=\overline{\Sigma}_{\mathsf{K}^p}\circ\mathrm{sp},\quad\Sigma^\circ_{\mathsf{K}^p}=\overline{\Sigma}^\circ_{\mathsf{K}^p}\circ\mathrm{sp},\tag{4.5.1}$$

where $\mathrm{sp}\colon|(\widehat{\mathscr{S}_{\mathsf{K}^p}})_\eta|^{\mathrm{cl}}=|U_{\mathsf{K}_0\mathsf{K}^p}|^{\mathrm{cl}}\to\mathscr{S}_{\mathsf{K}^p}(\overline{k})$ *is the specialization map.*

[27]More precisely, as a closed subscheme of $\mathscr{M}_{\mathsf{L}^p}(\Lambda_0)$ is finite type over $\mathcal{O}_E$, it is excellent (see [SP, Tag 07QW]), and thus Nagata (see [SP, Tag 07QV]) and so one may apply [SP, Tag 035S].

Proof. The second equality in (4.5.1) follows essentially by construction. To prove the first equality, it suffices to check that for a point x of $|U_{\mathsf{K}_0\mathsf{K}^p}|^{\mathrm{cl}}$ the F-isocrystals with G-structure given by $\underline{D}_{\mathrm{crys}} \circ (\nu_{\mathsf{K}_0\mathsf{K}^p})_x$ and that induced by $\mathbb{D}_{\mathrm{crys}} \circ (\omega_{\mathsf{K}^p,\text{ét}})_x$ agree. It suffices to find an isomorphism between their values at Λ_0 matching the tensors $\mathbb{T}_0$. But, this can be reduced to the second equality in (4.5.1) considering (4.3.3).

The claim concerning overconvergent local constancy for $\Sigma_{\mathsf{K}_0\mathsf{K}^p}$ and $\Sigma^\circ_{\mathsf{K}^p}$ follows from the equations in (4.5.1) as the the tube open subsets $\mathrm{sp}^{-1}(x)^\circ$, for x in $\mathscr{S}_{\mathsf{K}^p}(\overline{k})$, are overconvergent open and contain every classical point of $(\widehat{\mathscr{S}}_{\mathsf{K}^p})_\eta$ (see [ALY22, Proposition 2.13]). To prove that $\Sigma_{\mathsf{K}_p\mathsf{K}^p}$ is overconvergent locally constant for all K_p, observe that for $\mathsf{K}'_p \subseteq \mathsf{K}_p$ we have that

$$\Sigma_{\mathsf{K}'_p\mathsf{K}^p} = \Sigma_{\mathsf{K}_p\mathsf{K}^p} \circ \pi_{\mathsf{K}'_p\mathsf{K}^p,\mathsf{K}_p\mathsf{K}^p}.$$

Using this, and that $\pi_{\mathsf{K}'_p\mathsf{K}^p,\mathsf{K}_p\mathsf{K}^p}$ is finite étale and so preserves overconvergent opens under both preimage and image (cf. [Hub96, p. 427 (a)]), one reduces to the previous case $\mathsf{K}_p = \mathsf{K}_0$. □

4.6. **Comparison with work of Lovering.** In this subsection we compare our work to that in [Lov17a], and derive several consequences about Shimura varieties of abelian type. Throughout this section we fix an unramified Shimura datum $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ of abelian type.

4.6.1. *Comparison result.* In [Lov17a], Lovering constructs a so-called *crystalline canonical model* $\{\omega_{\mathsf{K}^p,\mathrm{crys}}\}$ of the system $\{\omega_{\mathsf{K}^p,\mathrm{an}}\}$ for an unramified Shimura datum of abelian type. More precisely, he constructs exact $\mathbb{Z}_p$-linear $\otimes$-functors

$$\omega_{\mathsf{K}^p,\mathrm{crys}} \colon \mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \to \mathbf{VectF}^{\varphi,\mathrm{div}}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\mathrm{crys}})$$

compatible in K^p, together with compatible identifications of filtered objects of $\mathcal{G}^c$-$\mathbf{MIC}(\mathcal{S}_{\mathsf{K}^p})$:

$$i_{\mathsf{K}^p} \colon D_{\mathrm{dR}} \circ \omega_{\mathsf{K}^p,\mathrm{an}}[1/p] \xrightarrow{\sim} \omega_{\mathsf{K}^p,\mathrm{crys}}[1/p].$$

Moreover, he shows that for all finite unramified E'/E, and points $x \colon \mathrm{Spf}(\mathcal{O}_{E'}) \to \widehat{\mathscr{S}}_{\mathsf{K}^p}$ that:

(ICM1) for all ξ in $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$, the morphism of isocrystals on $\mathrm{Spa}(E')$

$$i_{\mathsf{K}^p,x} \colon (D_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathrm{an}}[1/p])(\xi)_x \xrightarrow{\sim} \omega_{\mathsf{K}^p,\mathrm{crys}}[1/p](\xi)_x,$$

is Frobenius equivariant,[28]

(ICM2) for all ξ in $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$, the morphism $i_{\mathsf{K}^p,x}$ matches the lattice $\omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)_x$ with

$$M/uM \hookrightarrow (D_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathrm{an}}[1/p])(\xi)_x,$$

where $M = \phi^*\mathfrak{M}(\omega_{\mathsf{K}^p,\mathrm{an}}(\xi)_x)$, and this embedding is as in [Kis10, Theorem (1.2.1)].

As explained in [Lov17a, Proposition 3.1.6], these conditions uniquely characterize $\{\omega_{\mathsf{K}^p,\mathrm{crys}}\}$.

Theorem 4.18. *There is an identification $\mathbb{D}_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathbb{\Delta}} \xrightarrow{\sim} \omega_{\mathsf{K}^p,\mathrm{crys}}$ compatible in K^p.*

To prove this, we will need the fact that $\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ takes values in $\mathbf{Vect}^{\varphi,\mathrm{lff}}((\widehat{\mathscr{S}}_{\mathsf{K}^p})_{\mathbb{\Delta}})$. This is true, see Corollary 5.20, but we delay the proof until later because the methodology is separate than what is being considered here.

Proof of Theorem 4.18. It suffices to show that $\{\mathbb{D}_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathbb{\Delta}}\}$ can be given the structure of a crystalline canonical model. Using Corollary 5.20, the fact that it is an exact tensor functor valued in $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$ follows from Proposition 3.36. Furthermore, by Proposition 3.26, there are isomorphisms in $\mathbf{IsocF}^{\varphi}(\widehat{\mathscr{S}}_{\mathsf{K}^p})$

$$(\mathbb{D}_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathbb{\Delta}})[1/p] \xrightarrow{\sim} D_{\mathrm{crys}} \circ (T_{\text{ét}} \circ \omega_{\mathsf{K}^p,\mathbb{\Delta}})[1/p] \xrightarrow{\jmath_{\mathsf{K}^p}} D_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathrm{an}}[1/p], \tag{4.6.1}$$

and we denote the inverses by $\jmath^{\mathrm{crys}}_{\mathsf{K}^p}$, which are compatible in K^p. As $\jmath^{\mathrm{crys}}_{\mathsf{K}^p}$ is an isomorphism of filtered F-isocrystals, condition **(ICM1)** is automatic. Condition **(ICM2)** follows from the compatability of $T_{\text{ét}}$ with pullbacks, and Example 3.5. □

[28] Here we are implicitly using the fact that for a $\mathbb{Z}_p$-local system $\mathbb{L}$ on a smooth rigid space X with a smooth formal model over an unramified base, the underlying vector bundles $D_{\mathrm{dR}}(\mathbb{L})$ and $D_{\mathrm{crys}}(\mathbb{L})$ on $(X, \mathcal{O}_X)$ are the same.

We can provide an explication of this result when $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is of Hodge type. Fix an integral Hodge embedding $\iota\colon (\mathbf{G}, \mathbf{X}, \mathcal{G}) \to (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm}, \mathrm{GSp}(\Lambda_0))$ and a tensor package $(\Lambda_0, \mathbb{T}_0)$ with $\mathrm{Fix}(\mathbb{T}_0) = \mathcal{G}$. By [Kim18b, §3.1.2] and [Kis10, Corollary 2.3.9], one may construct from $\mathbb{T}_0$ tensors

$$\mathbb{T}_{0,p}^{\mathrm{dR}} \subseteq (\mathcal{H}^1_{\mathrm{dR}}(\mathscr{A}_{\mathsf{K}^p}/\mathscr{S}_{\mathsf{K}^p})^\vee)^\otimes,$$

which are compatible in K^p. By pulling back $\mathbb{T}_{0,p}^{\mathrm{dR}}$, we obtain a set $\widehat{\mathbb{T}}_{0,p}^{\mathrm{dR}}$ of tensors on $\mathcal{H}^1_{\mathrm{dR}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$. Using the canonical isomorphism from [BO78, Theorem 7.23 and Summary 7.26.3],

$$\mathcal{H}^1_{\mathrm{dR}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee_{\widehat{\mathscr{S}_{\mathsf{K}^p}}},$$

of vector bundles with connection, we obtain tensors $\mathbb{T}_{0,p}^{\mathrm{crys}}$ in $\mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$. By [Kim18b, Proposition 3.3.7] these are tensors in $\mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$ considered as an object of $\mathbf{VectF}^{\varphi}(\widehat{\mathscr{S}_{\mathsf{K}^p}})$.

Proposition 4.19. *There is an isomorphism*

$$\mathbb{D}_{\mathrm{crys}}(\omega_{\mathsf{K}^p,\mathbbl{\Delta}}(\Lambda_0)) \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$$

in $\mathbf{VectF}^{\varphi,\mathrm{div}}(\widehat{\mathscr{S}_{\mathsf{K}^p}})$*, compatible in* K^p *and carrying* $\mathbb{D}_{\mathrm{crys}}(\omega_{\mathsf{K}^p,\mathbbl{\Delta}}(\mathbb{T}_0))$ *to* $\mathbb{T}_{0,p}^{\mathrm{crys}}$.

Proof. By Theorem 4.14, there is an isomorphism $\omega_{\mathsf{K}^p,\mathbbl{\Delta}}(\Lambda_0) \xrightarrow{\sim} \mathcal{H}^1_{\mathbbl{\Delta}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$ in $\mathbf{Vect}^{\varphi}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_{\mathbbl{\Delta}})$, compatible in K^p, and carrying $\omega_{\mathsf{K}^p,\mathbbl{\Delta}}(\mathbb{T}_0)$ to $\mathbb{T}_{0,p}^{\mathbbl{\Delta}}$. Thus, it suffices to construct an isomorphism

$$\mathbb{D}_{\mathrm{crys}}\left(\mathcal{H}^1_{\mathbbl{\Delta}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})\right)^\vee \xrightarrow{\sim} \mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee,$$

in $\mathbf{VectF}^{\varphi}(\widehat{\mathscr{S}_{\mathsf{K}^p}})$ carrying $\mathbb{D}_{\mathrm{crys}}(\mathbb{T}_{0,p}^{\mathbbl{\Delta}})$ to $\mathbb{T}_{0,p}^{\mathrm{crys}}$. That there is an isomorphism in $\mathbf{VectF}^{\varphi}(\widehat{\mathscr{S}_{\mathsf{K}^p}})$ follows from Proposition 3.50 (see [ALB23, Theorem 4.6.2] and [BBM82, (3.3.7.3)]).

Thus, it suffices to show that this isomorphism carries $\mathbb{D}_{\mathrm{crys}}(\mathbb{T}_{0,p}^{\mathbbl{\Delta}})$ to $\mathbb{T}_{0,p}^{\mathrm{crys}}$. As $\mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$ is a vector bundle on $\widehat{\mathscr{S}_{\mathsf{K}^p}}$, there is an injection

$$\Gamma\left(\widehat{\mathscr{S}_{\mathsf{K}^p}}, \mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee\right) \to \Gamma\left(\mathcal{S}_{\mathsf{K}^p}, \mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee_\eta\right),$$

and so it suffices to show that the images of these two sets of tensors agree. But, by Proposition 3.26 the image of $\mathbb{D}_{\mathrm{crys}}(\mathbb{T}_{0,p}^{\mathbbl{\Delta}})$ may be identified with $D_{\mathrm{crys}}(T_{\mathrm{\acute{e}t}}(\mathbb{T}_{0,p}^{\mathbbl{\Delta}})) = D_{\mathrm{crys}}(\mathbb{T}_{0,p}^{\mathrm{\acute{e}t}})$. The claimed matching is then given by [Kim18b, Proposition 3.3.7]. □

By Theorem 4.14, $\omega_{\mathsf{K}^p,\mathbbl{\Delta}}$ is associated with the prismatic $\mathcal{G}$-torsor with F-structure

$$\underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{(\widehat{\mathscr{S}_{\mathsf{K}^p}})_{\mathbbl{\Delta}}}, \mathbb{T}_0 \otimes 1), (\mathcal{H}^1_{\mathbbl{\Delta}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee, \mathbb{T}_{0,p}^{\mathbbl{\Delta}})\right),$$

compatibly in K^p. Thus, by Proposition 4.19, we may identify $\mathbb{D}_{\mathrm{crys}} \circ \omega_{\mathsf{K}^p,\mathbbl{\Delta}}$ with

$$\underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathscr{S}_{\mathsf{K}^p}}/\mathcal{O}}, \mathbb{T}_0 \otimes 1), (\mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee, \mathbb{T}_{0,p}^{\mathrm{crys}})\right),$$

with Frobenius and Rees structure (see [Lov17a, §2.4]) inherited from $\mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}})^\vee$, compatibly in K^p. But, this is Lovering's construction of $\omega_{\mathsf{K}^p,\mathrm{crys}}$ in the Hodge type case.

4.6.2. *Cohomological consequences.* To obtain cohomological implications of Theorem 4.18, it is useful to recall that there is a group-theoretic description of when a Shimura variety is proper.

Proposition 4.20. *For a neat compact open subgroup* $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$*, the following are equivalent:*

(1) $\mathbf{G}^{\mathrm{ad}}$ *is* $\mathbb{Q}$*-anisotropic,*
(2) $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p} \to \mathrm{Spec}(E)$ *is proper,*
(3) $\mathscr{S}_{\mathsf{K}^p} \to \mathrm{Spec}(\mathcal{O}_E)$ *is proper.*

Proof. The equivalence of of the first two conditions is classical (e.g. see [Pau04, Lemma 3.1.5]). The equivalence of the first and third conditions is [MP19, Corollary 4.1.7] when $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is of Hodge type, and one quickly reduces to this case using Lemma 4.5. □

So then, combining Corollary 3.47 and Corollary 5.20 reproves [Lov17a, Theorem 3.6.1].

Proposition 4.21 ([Lov17a, Theorem 3.6.1])**.** *Suppose that* $\mathbf{G}^{\mathrm{der}}$ *is* $\mathbb{Q}$*-anisotropic. Then, for any object* ξ *of* $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$*, the Galois representation* $H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi)[1/p])$ *of* E *is crystalline, and the morphism from* (3.3.3) *constitutes a canonical isomorphism of filtered* F*-isocrystals*

$$D_{\mathrm{crys}}\left(H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi)[1/p])\right) \xrightarrow{\sim} H^i_{\mathrm{crys}}\left(((\mathscr{S}_{\mathsf{K}^p})_{\breve{\mathbb{Z}}_p}/\breve{\mathbb{Z}}_p)_{\mathrm{crys}}, \omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)[1/p]\right). \quad (4.6.2)$$

If the $\boldsymbol{\mu}^c_h$*-weights of* $\xi[1/p]$ *are at most* $p-2-i$*, then the isomorphism in* (4.6.2) *sends the lattice* $H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi))$ *to the lattice* $H^i_{\mathrm{crys}}\left(((\mathscr{S}_{\mathsf{K}^p})_{\breve{\mathbb{Z}}_p}/\breve{\mathbb{Z}}_p)_{\mathrm{crys}}, \omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)\right)$*.*

In words, this says that if the $\boldsymbol{\mu}^c_h$-weights of $\xi[1/p]$ are at most $p-2-i$, then $H^i_{\mathrm{\acute{e}t}}((\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p})_{\overline{\mathbb{Q}}_p}, \omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\xi))$ is a Fontaine–Laffaille lattice in a crystalline Galois representation with associated strongly divisible F-crystal given by $H^i_{\mathrm{crys}}\left(((\mathscr{S}_{\mathsf{K}^p})_{\breve{\mathbb{Z}}_p}/\breve{\mathbb{Z}}_p)_{\mathrm{crys}}, \omega_{\mathsf{K}^p,\mathrm{crys}}(\xi)\right)$.

Another immediate application of the existence of prismatic F-crystals is given in the recent paper [GL23]. For the notation used in the following statement see [GL23, Definitions 2.10, 2.14, and 2.17].

Proposition 4.22. *Suppose that* $\mathbf{G}^{\mathrm{der}}$ *is* $\mathbb{Q}$*-anisotropic and let* $f\colon \widehat{\mathscr{S}}_{\mathsf{K}^p} \to \mathrm{Spec}(\mathcal{O}_E)$ *be the structure map and* d *its relative dimension. Fix an object* ξ *of* $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$ *and let* a *and* b *denote the maximum and minimum* $\boldsymbol{\mu}^c_h$*-height of* $\xi[1/p]$ *respectively. Then, for each* $i \geqslant 0$*, the following statements are true.*

(1) One has that $R^i f_{\Delta,*}\omega_{\mathsf{K}^p,\Delta}$ *is a coherent prismatic* F*-crystal on* $\mathcal{O}_E$ *whose* $\mathcal{I}_\Delta$*-torsion-free quotient has Frobenius height in* $[a+\max\{0,i-d\}, b+\min\{i,d\}]$*.*

(2) There exists a map (a 'Verschiebung operator')

$$\psi_i\colon \mathcal{I}_\Delta^{(i+b)\otimes} \otimes_{\mathcal{O}_\Delta} R^i f_{\Delta,*}\omega_{\mathsf{K}^p,\Delta}(\xi) \to \phi^* R^i f_{\Delta,*}\omega_{\mathsf{K}^p,\Delta}(\xi)$$

which is inverse to the Frobenius operator on $R^i f_{\Delta,*}$ *up to multiplication by* $\mathcal{I}_\Delta^{(i+b)\otimes}$*.*

4.6.3. *Comparison of stratifications.* In [SZ22], Newton stratifications and central leaves are defined on the special fiber of $\mathscr{S}_{\mathsf{K}^p}$, extending all previously known cases (see the references in op. cit.). This gives functions

$$\Upsilon_{\mathsf{K}^p}\colon \mathscr{S}_{\mathsf{K}^p}(\bar{k}) \to B(G^c), \qquad \Upsilon^\circ_{\mathsf{K}^p}\colon \mathscr{S}_{\mathsf{K}^p}(\bar{k}) \to C(\mathcal{G}^c).$$

equivariant with respect to the actions of Γ_k. By the results of [SZ22, §5.4.2 and §5.4.5], $\Upsilon_{\mathsf{K}_p\mathsf{K}^p}$ agrees with the function to $B(G^c)$ defined using $\omega_{\mathsf{K}^p,\mathrm{crys}}$, and agrees with the function $C(\mathcal{G}^c)$ defined using $\omega_{\mathsf{K}^p,\mathrm{crys}}$ when $(\mathbf{G},\mathbf{X})$ is of Hodge type or $Z(\mathbf{G})$ is connected. Combining this with Proposition 4.17 and Theorem 4.18 then gives the following corollary.

Corollary 4.23. *Suppose that* $(\mathbf{G},\mathbf{X},\mathcal{G})$ *is an unramified Shimura datum of abelian type (resp. of Hodge type or of abelian type and* $Z(\mathbf{G})$ *is connected). Then, for any neat compact open subgroup* $\mathsf{K}_p \subseteq \mathsf{K}_0$ *and neat compact open subgroup* $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}^p_f)$ *we have that*

$$\Sigma_{\mathsf{K}_p\mathsf{K}^p} = \Upsilon_{\mathsf{K}^p} \circ \mathrm{sp} \circ \pi_{\mathsf{K}_p\mathsf{K}^p,\mathsf{K}_0\mathsf{K}^p}, \qquad \left(\text{resp. } \Sigma^\circ_{\mathsf{K}^p} = \Upsilon^\circ_{\mathsf{K}^p} \circ \mathrm{sp}\right).$$

Remark 4.24. While some of the rational (i.e., $B(G^c)$ related) results in Proposition 4.17 and Corollary 4.23 could have been proven using results in [Lov17a], the integral (i.e., $C(\mathcal{G}^c)$ related) results could not, as Lovering is only able to establish matching between the lattices $\omega_{\mathsf{K}^p,\mathrm{\acute{e}t}}(\Lambda)$ and $\omega_{\mathsf{K}^p,\mathrm{crys}}(\Lambda)$ for low Hodge–Tate weights.

5. A prismatic characterization of integral canonical models

In this section we formulate and prove a prismatic characterization of integral canonical models for unramified Shimura data of abelian type. Throughout this section we assume that p is odd.

5.1. **Shtukas and prismatic F-crystals.** In this subsection we recall the notion of a $\mathcal{G}$-shtuka on a (formal) scheme and discuss the relationship to prismatic $\mathcal{G}$-torsors with F-structure. Our notation and conventions concerning shtukas, their accompanying adic spaces (e.g. the spaces $\mathcal{Y}_I(S)$), and v-sheaves are as in [PR22, §2].

Fix k to be a perfect extension of $\mathbb{F}_p$, and set $W := W(k)$ and $E_0 = \mathrm{Frac}(W)$. Fix E to be a totally ramified finite extension of E_0 with uniformizer π, and fix an algebraic closure $\overline{E}$ of E which induces an algebraic closure $\overline{k}$ of k, and let C denote the p-adic completion of $\overline{E}$. Write $\breve{W} := W(\overline{k})$. Fix a parahoric group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ with generic fiber denoted G, and a conjugacy class $\boldsymbol{\mu}$ of cocharacters of $G_{\overline{E}}$ with field of definition E over E_0. Extending the notation from §2.1.5, for a finite type $\mathcal{O}_E$-scheme (resp. E-scheme) $\mathscr{X}$, write $\mathscr{X}^{\mathrm{ad}}$ for $\mathscr{X} \times_{\mathrm{Spec}(\mathcal{O}_E)} \mathrm{Spa}(\mathcal{O}_E)$ (resp. $\mathscr{X} \times_{\mathrm{Spec}(E)} \mathrm{Spa}(E)$) as in [Hub94, Proposition 3.8], an adic space over $\mathrm{Spa}(\mathcal{O}_E)$ (resp. $\mathrm{Spa}(E)$).[29]

5.1.1. *$\mathcal{G}$-shtukas.* Let $\mathbf{Perf}_k$ be the category of affinoid perfectoid spaces $\mathrm{Spa}(R, R^+)$ with R a k-algebra, endowed with the v-topology. If $\mathscr{X}$ is a pre-adic space over $\mathcal{O}_E$ we have the v-sheaf

$$\mathscr{X}^{\lozenge}\colon \mathbf{Perf}_k \to \mathbf{Set}, \qquad S \mapsto \left\{ (S^\sharp, f) : \begin{array}{ll} (1) & S^\sharp \text{ is an untilt of } S, \\ (2) & f\colon S^\sharp \to \mathscr{X} \end{array} \right\},$$

(see [SW20, Lecture 18]). Denote by $\mathbf{Perf}_{\mathscr{X}}$ the slice category of $\mathbf{Perf}_k$ over $\mathscr{X}^{\lozenge}$, consisting of morphisms $\alpha\colon S \to \mathscr{X}^{\lozenge}$, endowed with the v-topology. We often conflate the pair (S, α) with the pair $(S^\sharp, f)$. A morphism $a\colon \mathscr{X}_1 \to \mathscr{X}$ over $\mathcal{O}_E$ gives a continuous functor $a\colon \mathbf{Perf}_{\mathscr{X}_1} \to \mathbf{Perf}_{\mathscr{X}}$.

Consider the v-stack $p_{\mathscr{X}}\colon \mathcal{G}\text{-}\mathbf{Sht} \to \mathbf{Perf}_{\mathscr{X}}$ (resp. $q_{\mathscr{X}}\colon \mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}} \to \mathbf{Perf}_{\mathscr{X}}$), whose fiber over $(S^\sharp, f)$ is the groupoid of $\mathcal{G}$-shtukas over S with one leg at $S^\sharp$ (resp. those bounded by $\boldsymbol{\mu}$ at $S^\sharp$) in the sense of [PR22, Definition 2.2.1] (resp. [PR22, Definition 2.4.3]). Following [PR22, Definition 2.3.1], a *$\mathcal{G}$-shtuka (resp. $\mathcal{G}$-shtuka bounded by $\boldsymbol{\mu}$)* over $\mathscr{X}$ is a Cartesian section $s = (\mathscr{P}, \varphi_{\mathscr{P}})$ of $p_{\mathscr{X}}$ (resp. $q_{\mathscr{X}}$) (see [SP, Tag 07IV]). We write $\mathscr{P}(S^\sharp, f)$, for the value of $\mathscr{P}$ at $(S^\sharp, f)$, and abuse notation by writing just $\varphi_{\mathscr{P}}$ for its Frobenius.

Write $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X})$ (resp. $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X})$) for the category of $\mathcal{G}$-shtukas (resp. $\mathcal{G}$-shtukas bounded by $\boldsymbol{\mu}$) over $\mathscr{X}$. For a morphism $a\colon \mathscr{X}_1 \to \mathscr{X}$ over $\mathcal{O}_E$, we obtain functors $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X}) \to \mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X}_1)$ and $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X}) \to \mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X}_1)$ sending s to $s \circ a$. Both $\mathcal{G}\text{-}\mathbf{Sht}(-)$ and $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(-)$ form stacks on the category of pre-adic spaces over $\mathcal{O}_E$ with (topological) open covers.

Recall that $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X})(S^\sharp, f)$ satisfies a Tannakian formalism (cf. [SW20, Appendix to Lecture 19]). Namely, it is equivalent to the category of pairs (ω, φ), where ω is an object of $\mathcal{G}\text{-}\mathbf{Vect}(S \dot{\times} \mathbb{Z}_p)$ and φ is a *Frobenius isomorphism*

$$\varphi\colon \phi^*\omega|_{S\dot{\times}\mathbb{Z}_p - S^\sharp} \to \omega|_{S\dot{\times}\mathbb{Z}_p - S^\sharp}$$

in $\mathcal{G}\text{-}\mathbf{Vect}(S \dot{\times} \mathbb{Z}_p - S^\sharp)$ which is meromorphic along $S^\sharp$ when evaluated at every object of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ (when this notation and terminology is given the obvious meaning). This identification is functorial in $(S^\sharp, f)$ and thus gives $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X})$ a Tannakian formalism (suitably interpreted). In particular, when $\mathcal{G} = \mathrm{GL}_{n,\mathbb{Z}_p}$ there is a natural equivalence between $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X})$ and $\mathbf{Sht}_n(\mathscr{X})$ which associates to $(S^\sharp, f)$ the groupoid $\mathbf{Sht}_n(S^\sharp, f)$ of height n shtukas over S with one leg at $S^\sharp$. We shall make these identifications freely in the sequel.

5.1.2. *$\mathcal{G}$-shtukas and gluing triples.* If $\mathscr{X}$ is a p-adic formal scheme, or a finite type scheme over $\mathcal{O}_E$ or E, we shorten the notation $(\mathscr{X}^{\mathrm{ad}})^{\lozenge}$, $\mathbf{Perf}_{\mathscr{X}^{\mathrm{ad}}}$, $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X}^{\mathrm{ad}})$, and $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X}^{\mathrm{ad}})$, to $\mathscr{X}^{\lozenge}$, $\mathbf{Perf}_{\mathscr{X}}$, $\mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X})$, and $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X})$, respectively.

Definition 5.1. The category $\mathbf{Tri}(\mathcal{O}_E)$ of *gluing triples* over $\mathcal{O}_E$ has

- objects of the form $(X, \mathfrak{X}, j)$ with X a separated locally of finite type E-scheme, $\mathfrak{X}$ a separated locally of finite type flat formal $\mathcal{O}_E$-scheme, and $j\colon \mathfrak{X}_\eta \to X^{\mathrm{an}}$ an open embedding,

[29]If $\mathscr{X}$ is finite type over E, then $\mathscr{X}^{\mathrm{ad}} = \mathscr{X}^{\mathrm{an}}$, and we mostly use the latter notation.

- morphisms $(f,g)\colon (X_1,\mathfrak{X}_1,j_1)\to (X_2,\mathfrak{X}_2,j_2)$ with $f\colon X_1\to X_2$ a morphism of E-schemes and $g\colon \mathfrak{X}_1\to\mathfrak{X}_2$ a morphism of formal $\mathcal{O}_E$-schemes, such that $f^{\mathrm{an}}\circ j_1 = j_2\circ g_\eta$.

If $\mathscr{X}$ is a separated locally of finite type flat $\mathcal{O}_E$-scheme, one may functorially associate a gluing triple $(\mathscr{X}_E,\widehat{\mathscr{X}},j_{\mathscr{X}})$, where $j_{\mathscr{X}}\colon \widehat{\mathscr{X}_\eta}\to\mathscr{X}_E^{\mathrm{an}}$ is the tautological open embedding (see [Hub96, §1.9]). This in particular gives a functor

$$\mathtt{t}\colon \begin{Bmatrix}\text{Locally of finite type}\\ \text{separated flat } \mathcal{O}_E\text{-schemes}\end{Bmatrix}\to \mathbf{Tri}(\mathcal{O}_E),\qquad \mathscr{X}\mapsto \mathtt{t}(\mathscr{X})=(\mathscr{X}_E,\widehat{\mathscr{X}},j_{\mathscr{X}}). \tag{5.1.1}$$

We use $j_{\mathscr{X}}$ to consider $\widehat{\mathscr{X}_\eta}$ as an open adic subspace of $\mathscr{X}_E^{\mathrm{an}}$ without comment in the sequel.

Proposition 5.2. *The functor* $\mathtt{t}$ *is fully faithful.*

Proof. By a standard devissage, we reduce ourselves to the following. Suppose that A is a finite type flat $\mathcal{O}_E$-algebra. Then, the preimage of $\widehat{A}$ under $A_E\to\widehat{A}_E$ is A. Indeed, writing an element of A_E as $\frac{a}{\pi^k}$ with a in A, by assumption a maps into $\pi^k\widehat{A}$ and thus a lies in the preimage of $\pi^k\widehat{A}$ which is the kernel of $A\to\widehat{A}\to\widehat{A}/\pi^k\widehat{A}=A/\pi^kA$ and so π^kA. Thus $\frac{a}{\pi^k}$ is in A as desired. ☐

Remark 5.3. There is an explicit 'gluing' functor which is the quasi-inverse to $\mathtt{t}$ on its essential image, thus justifying the name 'gluing triple'. In future work of the final author and P. Achinger, it is shown (over more general bases) that results of Artin may be used to upgrade Proposition 5.2 to an equivalence between gluing triples of algebraic spaces and separated flat locally of finite type algebraic spaces. This formalizes the idea that separated flat locally of finite type algebraic spaces over $\mathcal{O}_E$ are precisely the objects obtained by gluing an algebraic space X over E to a formal algebraic space $\mathfrak{X}$ over $\mathcal{O}_E$ along some rigid open subset of X^{an}.

Fix a gluing triple $(X,\mathfrak{X},j)$ over $\mathcal{O}_E$. We denote by $\mathcal{G}\text{-}\mathbf{Sht}(X,\mathfrak{X},j)$ (resp. $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(X,\mathfrak{X},j)$) the category of *$\mathcal{G}$-shtukas (resp. $\mathcal{G}$-shtukas bounded by $\boldsymbol{\mu}$)* over $(X,\mathfrak{X},j)$, consisting of triples $(s_X,s_{\mathfrak{X}},\gamma)$ where s_X and $s_{\mathfrak{X}}$ are $\mathcal{G}$-shtukas (resp. $\mathcal{G}$-shtukas bounded by $\boldsymbol{\mu}$) on X and $\mathfrak{X}$ respectively, and $\gamma\colon j^*(s_X)\xrightarrow{\sim} s_{\mathfrak{X}}|_{\mathfrak{X}_\eta}$ is an isomorphism, with the obvious notion of morphisms.

Fix $\mathscr{X}$ to be separated locally of finite type flat $\mathcal{O}_E$-scheme. Then, there are natural functors

$$\mathtt{t}\colon \mathcal{G}\text{-}\mathbf{Sht}(\mathscr{X})\to\mathcal{G}\text{-}\mathbf{Sht}(\mathtt{t}(\mathscr{X})),\qquad \mathtt{t}\colon \mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X})\to\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathtt{t}(\mathscr{X})), \tag{5.1.2}$$

functorial in $\mathscr{X}$. In fact, the functors in (5.1.2) are equivalences as $\mathscr{X}^{\mathrm{ad}}$ is the gluing of $\mathscr{X}_E^{\mathrm{an}}$ and $\widehat{\mathscr{X}}^{\mathrm{ad}}$ along the open embedding of adic spaces $j_{\mathscr{X}}\colon\widehat{\mathscr{X}_\eta}\to\mathscr{X}_E^{\mathrm{an}}$.

5.1.3. *$\mathcal{G}$-shtukas and prismatic $\mathcal{G}$-torsors with F-structure.* We would now like to clarify the relationship between $\mathcal{G}$-shtukas and prismatic $\mathcal{G}$-torsors with F-structure.

Shtukas associated to $\mathcal{G}(\mathbb{Z}_p)$-local systems. Let X be a locally Noetherian adic space over E. Then, constructed in [PR22, §2.5.1–§2.5.2] is an equivalence of categories

$$\Phi_X\colon \mathcal{G}\text{-}\mathbf{Sht}(X)\xrightarrow{\sim}\left\{(\mathbb{P},H)\colon \begin{matrix}(1)\ \mathbb{P}\text{ is an object of }\mathbf{Tors}_{\mathcal{G}(\mathbb{Z}_p)}(X),\\ (2)\ \text{ a } \underline{\mathcal{G}(\mathbb{Z}_p)}\text{-equvariant map } H\colon\mathbb{P}^\lozenge\to\mathrm{Gr}_{G,\mathrm{Spd}(E)}.\end{matrix}\right\}. \tag{5.1.3}$$

Here $\mathbb{P}^\lozenge=\varprojlim(\mathbb{P}/K_n)^\lozenge$ is the diamond associated to $\mathbb{P}$ as in §2.1.5, and $\mathrm{Gr}_{G,\mathrm{Spd}(E)}$ is the $\mathrm{B}_{\mathrm{dR}}^+$-Grassmannian associated to G as in [FS21, §III.3], and H is a morphism of v-sheaves over $\mathrm{Spd}(E)$. The morphisms in the target category are the obvious ones.

To explain this functor, fix an object of $S=\mathrm{Spa}(R,R^+)$ of $\mathbf{Perf}_k$ as well as an untilt $S^\sharp=\mathrm{Spa}(R^\sharp,R^{\sharp+})$ over E with $(\xi):=\ker(\theta\colon W(R^+)\to R^{\sharp+})$. Also, for a closed subset Z of a topological space Y and a sheaf $\mathcal{F}$ on Y, we write $\Gamma^\dagger(Z,\mathcal{F})$ as shorthand for $\varinjlim_{Z\subseteq U}\mathcal{F}(U)$.

As in [KL15, Definitions 4.2.2 and 5.1.1], consider the *integral Robba ring (over S)*

$$\widetilde{\mathcal{R}}_S^{\mathrm{int}}:=\varinjlim_{r\to 0^+}\mathcal{O}(\mathcal{Y}_{[0,r]}(S))=\Gamma^\dagger(S,\mathcal{O}_{S\dot{\times}\mathbb{Z}_p}).$$

For this last object, we are considering S as a closed Cartier divisor of $S\dot{\times}\mathbb{Z}_p$ in the usual way (see [FS21, Proposition II.1.4]). We observe that $\widetilde{\mathcal{R}}^{\mathrm{int}}_S$ carries a natural Frobenius morphism, compatible with that on $S\dot{\times}\mathbb{Z}_p$. As $S^\sharp$ does not intersect S we see that we have a functor

$$\mathbf{Sht}_n(S^\sharp) \to \mathbf{Mod}^\varphi(\widetilde{\mathcal{R}}^{\mathrm{int}}_S), \qquad (\mathcal{P}, \varphi_{\mathcal{P}}) \mapsto (\Gamma^\dagger(S,\mathcal{P}), \varphi_{\mathcal{P}}),$$

where the target is the category of étale φ-modules over $\widetilde{\mathcal{R}}^{\mathrm{int}}_S$ (see [KL15, Definition 6.1.1]).

Denote by $\mathbf{Sht}_{n,\mathrm{free}}(S^\sharp)$ the full subgroupoid of $\mathbf{Sht}_n(S^\sharp)$ consisting of shtukas $(\mathcal{P},\varphi_{\mathcal{P}})$ with the property that the associated object of $\mathbf{Mod}^\varphi(\widetilde{\mathcal{R}}^{\mathrm{int}}_S)$ is free (which is called 'trivial' in loc. cit.). On the other hand, let $\mathrm{B}^+_{\mathrm{dR}}$-$\mathbf{Pair}_n(S^\sharp)$ be the groupoid of pairs (T,Ξ), where T a finite free $\mathbb{Z}_p$-module of rank n and, Ξ is a $\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)$-lattice of $T\otimes_{\mathbb{Z}_p}\mathrm{B}_{\mathrm{dR}}(S^\sharp)$, with the obvious notion of (iso)morphisms. There is a natural morphism of groupoids

$$\Phi_{S^\sharp}\colon \mathbf{Sht}_{\mathrm{n,free}}(S^\sharp) \to \mathrm{B}^+_{\mathrm{dR}}\text{-}\mathbf{Pair}_n(S^\sharp), \qquad (\mathcal{P},\varphi_{\mathcal{P}}) \mapsto (T_{\mathcal{P}}, \Xi_{\mathcal{P}}).$$

Here $T_{\mathcal{P}} := \Gamma^\dagger(S,\mathcal{P})^{\varphi=1}$, which is free of rank n. Let $\mathcal{P}_{\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)}$ denote $\Gamma(\mathcal{Y}_{[0,1]},\mathcal{P})^\wedge_\xi$, and set $\mathcal{P}_{\mathrm{B}_{\mathrm{dR}}(S^\sharp)} := \mathcal{P}_{\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)}[1/\xi]$. Then there is a natural isomorphism

$$T_{\mathcal{P}}\otimes_{\mathbb{Z}_p}\mathrm{B}^+_{\mathrm{dR}}(S^\sharp) \xrightarrow{\sim} \mathcal{P}_{\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)}\ (=\Gamma^\dagger(S^\sharp,\mathcal{P})^\wedge_\xi), \tag{5.1.4}$$

obtained by extending an element of $T_{\mathcal{P}}$, which is a priori an element of $\Gamma(\mathcal{Y}_{[0,r]}(S),\mathcal{P})$ for some $r>0$, to $\Gamma(\mathcal{Y}_{[0,1]},\mathcal{P})$ via $\varphi_{\mathcal{P}}$ (cf. [SW20, Corollary 12.4.1]). Set $\Xi_{\mathcal{P}}$ to be the $\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)$-lattice in $T_{\mathcal{P}}\otimes_{\mathbb{Z}_p}\mathrm{B}_{\mathrm{dR}}(S^\sharp)$ corresponding under (5.1.4) to $\varphi_{\mathcal{P}}((\phi^*\mathcal{P})_{\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)})$ where

$$\varphi_{\mathcal{P}}\colon (\phi^*\mathcal{P})_{\mathrm{B}_{\mathrm{dR}}(S^\sharp)} \to \mathcal{P}_{\mathrm{B}_{\mathrm{dR}}(S^\sharp)},$$

is the map induced by the Frobenius structure of $\mathcal{P}$.

If an object $(\mathcal{P},\varphi_{\mathcal{P}})$ of $\mathbf{Sht}_{n,\mathrm{free}}(S^\sharp)$ is obtained by restriction from an object (M,φ_M) of $\mathbf{Vect}^\varphi(W(R^+),(\xi))$ then we can describe $\Phi_{S^\sharp}(\mathcal{P},\varphi_{\mathcal{P}})$ more concretely. Namely, we have the equality $T_{\mathcal{P}} = (M\otimes_{W(R^+)}\widetilde{\mathcal{R}}^{\mathrm{int}}_S)^{\varphi=1}$, and $\Xi_{\mathcal{P}}$ corresponds under (5.1.4) to the image of the $\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)$-lattice $\phi^*M\otimes_{W(R^+)}\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)$ under

$$\varphi_M\otimes 1\colon \phi^*M\otimes_{W(R^+)}\mathrm{B}_{\mathrm{dR}}(S^\sharp) \to M\otimes_{W(R^+)}\mathrm{B}_{\mathrm{dR}}(S^\sharp).$$

In any case, we have the following result, which can be proven exactly in the same way as [SW20, Proposition 12.4.6], as mentioned in the proof of [PR22, Proposition 2.5.1].

Lemma 5.4 (cf. [SW20, Proposition 12.4.6] and [PR22, Proposition 2.5.1])**.** *The functor*

$$\Phi_{S^\sharp}\colon \mathbf{Sht}_{n,\mathrm{free}}(S^\sharp) \to \mathrm{B}^+_{\mathrm{dR}}\text{-}\mathbf{Pair}_n(S^\sharp), \qquad (\mathcal{P},\varphi_{\mathcal{P}}) \mapsto (T_{\mathcal{P}},\Xi_{\mathcal{P}})$$

is an equivalence.

When $\mathcal{G} = \mathrm{GL}_{n,\mathbb{Z}_p}$, the equivalence Φ_X is then obtained from the observations that (a) $\Phi_{S^\sharp}$ is functorial in $(S^\sharp, f)$ in $\mathbf{Perf}_{\mathscr{X}}$, and (b) the source and target are the global sections of the stackification of $\mathbf{Sht}_{n,\mathrm{free}}(S^\sharp,f)$ and $\mathrm{B}^+_{\mathrm{dR}}$-$\mathbf{Pair}_n(S^\sharp,f)$ for the pro-étale topology, respectively.[30] The case for general $\mathcal{G}$ is then obtained by applying the Tannakian formalism.

Let $\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}$ and $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$ be as in [SW20, Definition 19.2.2]. Observe that if $\boldsymbol{\mu}$ is minuscule, then $\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)} = \mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$. The following lemma is just an unraveling of the definitions.

Lemma 5.5. *If $(\mathscr{P},\varphi_{\mathscr{P}})$ is an object of $\mathcal{G}$-$\mathbf{Sht}(X)$, and $\Phi_X(\mathscr{P},\varphi_{\mathscr{P}}) = (\mathbb{P},H)$, then $(\mathscr{P},\varphi_{\mathscr{P}})$ is an object of $\mathcal{G}$-$\mathbf{Sht}_{\boldsymbol{\mu}^{-1}}(X)$ if and only if H factorizes through $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$.*

[30] The fact that every object of $\mathbf{Sht}_n(S^\sharp)$ is pro-étale locally in $\mathbf{Sht}_{n,\mathrm{free}}(S^\sharp)$ follows from [KL15, Theorem 8.5.3] as any local system can be trivialized pro-étale locally.

Proof. Let us begin by observing that as $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$ is a closed subdiamond of $\mathrm{Gr}_{G,\mathrm{Spd}(E)}$ (see [SW20, Proposition 19.2.3]). Thus, H factorizes through $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$ if and only if it does so at the level of points. Moreover, as $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$ is closed in $\mathrm{Gr}_{G,\mathrm{Spd}(E)}$ it is partially proper (see [SW20, Lemma 19.1.4]), and so we further see that H factorizes through $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}$ if and only if it does so for points of the form $S^\sharp=\mathrm{Spa}(C^\sharp,C^{\sharp\circ})$ with C an algebraically closed perfectoid field. Tracing through the definitions, this means that for trivializations $G_{\mathrm{B}^+_{\mathrm{dR}}(C^{\sharp+})}\xrightarrow{\sim}\mathscr{P}_{\mathrm{B}^+_{\mathrm{dR}}(C^{\sharp+})}$ and $G_{\mathrm{B}^+_{\mathrm{dR}}(C^{\sharp+})}\xrightarrow{\sim}\phi^*\mathscr{P}_{\mathrm{B}^+_{\mathrm{dR}}(C^{\sharp+})}$, that the relative position of $\mathscr{P}$ and $\varphi_{\mathscr{P}}(\phi^*\mathscr{P})$ defines an element of $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu},\mathrm{Spd}(E)}(C^\sharp,C^{\sharp\circ})$. On the other hand, by definition, $(\mathscr{P},\varphi_{\mathscr{P}})$ lies in $\mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}^{-1}}(X)$ if and only if for all such $S^\sharp$ and all such trivializations, the relative position of $\varphi_{\mathscr{P}}(\phi^*\mathcal{P})$ and $\mathscr{P}$ defines an element of $\mathrm{Gr}_{G,\leqslant\boldsymbol{\mu}^{-1},\mathrm{Spd}(E)}(C^\sharp,C^{\sharp\circ})$. These are clearly equivalent. □

Shtukas associated to de Rham local systems. Let X be a smooth rigid E-space. In [PR22, §2.6.1–§2.6.2] there is constructed a functor

$$U_{\mathrm{sht}}\colon \mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)\to\mathcal{G}\text{-}\mathbf{Sht}(X),$$

which we now recall. Thanks to above discussion, it suffices to construct a functor

$$\Phi_X\circ U_{\mathrm{sht}}\colon \mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)\to\left\{(\mathbb{P},H)\colon \begin{array}{l}(1)\ \mathbb{P}\text{ is an object of }\mathbf{Tors}_{\mathcal{G}(\mathbb{Z}_p)}(X),\\ (2)\ \ \text{a }\underline{\mathcal{G}(\mathbb{Z}_p)}\text{-equvariant map }H\colon\mathbb{P}^\lozenge\to\mathrm{Gr}_{G,\mathrm{Spd}(E)}.\end{array}\right\}$$

Further, by employing the Tannakian formalism, we may further assume that $\mathcal{G}=\mathrm{GL}_{n,\mathbb{Z}_p}$. In this case, an object of $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)$ is nothing but a de Rham local system $\mathbb{L}$ on X. Then one defines $\Phi_X(U_{\mathrm{sht}}(\mathbb{L}))$ to be $(\mathbb{P}_{\mathbb{L}},H_{\mathbb{L}})$. Here, $\mathbb{P}_{\mathbb{L}}=\underline{\mathrm{Isom}}(\underline{\mathbb{Z}}_p^n,\mathbb{L})$ is the $\underline{\mathrm{GL}_n(\mathbb{Z}_p)}$-torsor as in §2.1.2. Recall that by definition of being de Rham, there is a canonical $\mathrm{B}^+_{\mathrm{dR}}$-equivariant isomorphism

$$c_{\mathrm{Sch}}\colon\mathbb{L}\otimes_{\underline{\mathbb{Z}}_p}\mathrm{B}^+_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{Fil}^0(D_{\mathrm{dR}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathrm{B}_{\mathrm{dR}})^{\nabla=0}.$$

This induces a canonical B_{dR}-equivariant isomorphism

$$c_{\mathrm{Sch}}\otimes 1\colon\mathbb{L}\otimes_{\underline{\mathbb{Z}}_p}\mathrm{B}_{\mathrm{dR}}\xrightarrow{\sim}(D_{\mathrm{dR}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathrm{B}_{\mathrm{dR}})^{\nabla=0}.$$

So, from an isomorphism $a\colon\underline{\mathbb{Z}}_p^n\xrightarrow{\sim}\mathbb{L}_{S^\sharp}$, where $S^\sharp$ is an untilt over X of some S in $\mathbf{Perf}_k$, we obtain an isomorphism of $\mathrm{B}_{\mathrm{dR}}(S^\sharp)$-modules

$$(c_{\mathrm{Sch}}\otimes 1)\circ a\colon\underline{\mathbb{Z}_p}^n(S^\sharp)\otimes_{\underline{\mathbb{Z}_p}(S^\sharp)}\mathrm{B}_{\mathrm{dR}}(S^\sharp)\xrightarrow{\sim}(D_{\mathrm{dR}}(\mathbb{L})(S^\sharp)\otimes_{S^\sharp}\mathcal{O}\mathrm{B}_{\mathrm{dR}}(S^\sharp))^{\nabla=0}.$$

We then set

$$H_{\mathbb{L}}(a):=((c_{\mathrm{Sch}}\otimes 1)\circ a)^{-1}((D_{\mathrm{dR}}(\mathbb{L})(S^\sharp)\otimes_{S^\sharp}\mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(S^\sharp))^{\nabla=0}),$$

a $\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)$-lattice in $\underline{\mathbb{Z}_p}^n(S^\sharp)\otimes_{\underline{\mathbb{Z}_p}(S^\sharp)}\mathrm{B}_{\mathrm{dR}}(S^\sharp)$.

Proposition 5.6. *Let ω be an object of $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)$ and write $\Phi_X(U_{\mathrm{sht}}(\omega))=(\mathbb{P},H)$. Then, we have that ω belongs to $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}$ if and only if H factorizes through $\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}$.*

Proof. Observe that $\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}$ is an open subdiamond of a closed subdiamond of $\mathrm{Gr}_{G,\mathrm{Spd}(E)}$ (see [SW20, Proposition 19.2.3]). So, if $\Phi_X(U_{\mathrm{sht}}(\omega))=(\mathbb{P},H)$, then H factorizes through $\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}$ if and only if $|H|$ factorizes through $|\,\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}|$ (see [Sch22, Proposition 12.15]). But, as $|\mathbb{P}^\lozenge|=|\mathbb{P}|$ (see [Sch22, Lemma 15.6]), and the projection map $|\mathbb{P}|\to|X|$ is open (see [Sch13, Lemma 3.10 (iv)]), so that the preimage of a dense set is dense, we see that H factorizes through $\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}$ if and only if it does so after restriction to each classical point x. Thus, we may assume that $X=\mathrm{Spa}(E)$. By the Tannakian formalism, we are reduced to the case when $\mathcal{G}=\mathrm{GL}_{n,\mathbb{Z}_p}$, and we identify ω with its value $\mathbb{L}$ at the tautological representation. Set T to be $\mathbb{L}(C,\mathcal{O}_C)$ with its natural Galois action. By definition, H factorizes through $\mathrm{Gr}_{\mathrm{GL}_n,\boldsymbol{\mu},\mathrm{Spd}(E)}(C,\mathcal{O}_C)$ if and only if we can find trivializations $\mathrm{B}^+_{\mathrm{dR}}(C)^n\xrightarrow{\sim}T\otimes_{\mathbb{Z}_p}\mathrm{B}^+_{\mathrm{dR}}(C)$ and

$$(\mathrm{B}^+_{\mathrm{dR}}(C))^n\xrightarrow{\sim}D_{\mathrm{dR}}(T)\otimes_E\mathrm{B}^+_{\mathrm{dR}}(C)$$

such that through the filtered isomorphism

$$c_{\mathrm{Sch}}\colon D_{\mathrm{dR}}(T)\otimes_E \mathrm{B}_{\mathrm{dR}}(C)\xrightarrow{\sim} T\otimes_{\mathbb{Z}_p}\mathrm{B}_{\mathrm{dR}}(C)$$

the induced automorphism of $\mathrm{B}_{\mathrm{dR}}(C)^n$ given by the multiplication of an element of the double coset $\mathrm{GL}_n(\mathrm{B}^+_{\mathrm{dR}}(C))\boldsymbol{\mu}(\xi)\,\mathrm{GL}_n(\mathrm{B}^+_{\mathrm{dR}}(C))$. Choose an element μ of $\boldsymbol{\mu}$, and write $\mu(\xi)=(\xi^{r_1},\ldots,\xi^{r_n})$. Then this condition holds if and only if there exists a $\mathrm{B}^+_{\mathrm{dR}}(C)$-basis $(e_\nu)_{\nu=1}^n$ of $D_{\mathrm{dR}}(T)\otimes_E\mathrm{B}^+_{\mathrm{dR}}(C)$ such that the filtration Fil^r on $D_{\mathrm{dR}}(T)\otimes_E\mathrm{B}_{\mathrm{dR}}(C)$ is given by $\mathrm{Fil}^r=\sum_{\nu=1}^n\xi^{r-r_\nu}\mathrm{B}^+_{\mathrm{dR}}(C)\cdot e_\nu$. The proof of Lemma 3.30 shows that this is equivalent to the existence of a filtered basis $(e_\nu,r_\nu)_{\nu=1}^n$ of $D_{\mathrm{dR}}(T)\otimes_E C$, i.e., that $\mathbb{L}$ belongs to $\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}(\mathrm{Spa}(E))$. □

The shtuka realization functor. Let $\mathfrak{X}$ be a smooth formal $\mathcal{O}_K$-scheme. We now wish to explicate a construction made in [PR22, §4.4] and [Dan22, §3.1]. Namely, the definition of a *shtuka realization functor*

$$T_{\mathrm{sht}}\colon \mathbf{Tors}^{\mathrm{an},\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta})\to\mathcal{G}\text{-}\mathbf{Sht}(\mathfrak{X}),\qquad(\mathcal{A},\varphi_{\mathcal{A}})\mapsto T_{\mathrm{sht}}(\mathcal{A},\varphi_{\mathcal{A}})=(\mathcal{A}_{\mathrm{sht}},\varphi_{\mathcal{A}_{\mathrm{sht}}}).$$

Intuitively this is obtained by just ‘restricting to perfect prisms’.

As in [Gle22, Definition 4.6], call a pair $(\mathrm{Spa}(R^\sharp,R^{\sharp+}),f)$ in $\mathbf{Perf}_{\mathfrak{X}}$ *formalizing* if $f=g^{\mathrm{ad}}\circ i_R$ where g is some morphism $\mathrm{Spf}(R^{\sharp+})\to\mathfrak{X}$, and $i_R\colon \mathrm{Spa}(R^\sharp,R^{\sharp+})\to\mathrm{Spa}(R^{\sharp+})$ is the canonical map. By [Gle22, Proposition 4.17], such a g is unique and evidently the subcategory $\mathbf{Perf}^+_{\mathfrak{X}}$ of formalizing pairs is a basis of $\mathbf{Perf}_{\mathfrak{X}}$ with the analytic (i.e., topological open cover) topology, as can be seen from an affine open cover $\mathfrak{X}$. We call g the *formalization* of f.

For an object $(S^\sharp,f)$ of $\mathbf{Perf}^+_{\mathfrak{X}}$, with $S^\sharp=\mathrm{Spa}(R^\sharp,R^{\sharp+})$ and g a formalization of f, observe that the triple $(\mathrm{A}_{\mathrm{inf}}(R^{\sharp+}),(\xi),g)$ defines an object of $\mathfrak{X}_{\Delta}$. So, $\mathcal{A}(\mathrm{A}_{\mathrm{inf}}(R^{\sharp+}),(\xi),g)$ defines an object of $\mathbf{Tors}^{\mathrm{an},\varphi}_{\mathcal{G}}(\mathrm{A}_{\mathrm{inf}}(R^{\sharp+}),(\xi))$. Pulling this back along the map of locally ringed spaces $\mathcal{Y}_{[0,\infty)}(S)\to\mathrm{Spec}(W(R^+))-V(p,\xi)$, gives a $\mathcal{G}$-shtuka $\mathcal{A}_{\mathrm{sht}}(S^\sharp,f)$ over S with one leg at $S^\sharp$ (cf. [Dan22, §2.1]). It is clear that this construction defines a sheaf on $\mathbf{Perf}^+_{\mathfrak{X}}$ for the analytic topology, and thus extends uniquely to give a section of $p_{\mathfrak{X}}$, and thus an object of $\mathcal{G}$-$\mathbf{Sht}(\mathfrak{X})$.

Comparison isomorphism. Let $\mathfrak{X}$ be a smooth formal $\mathcal{O}_K$ and let X be its generic fiber. Recall (see [GR22, Corollary 2.37] and [TT19, Propositions 3.21 and 3.22]) that there is a containment $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)\subseteq\mathbf{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)$. Thus, associated to an object $(\mathcal{A},\varphi_{\mathcal{A}})$ of $\mathbf{Tors}^{\mathrm{an},\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$, one may build two ostensibly different shtukas on X: the shtukas $T_{\mathrm{sht}}(\mathcal{A},\varphi_{\mathcal{A}})_\eta$ and $U_{\mathrm{sht}}(T_{\text{ét}}(\mathcal{A},\varphi_{\mathcal{A}}))$.

Proposition 5.7. *For an object $(\mathcal{A},\varphi_{\mathcal{A}})$ of $\mathbf{Tors}^{\mathrm{an},\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$, there is a natural identification*

$$\varrho_{(\mathcal{A},\varphi_{\mathcal{A}})}\colon T_{\mathrm{sht}}(\mathcal{A},\varphi_{\mathcal{A}})_\eta\xrightarrow{\sim}U_{\mathrm{sht}}(T_{\text{ét}}(\mathcal{A},\varphi_{\mathcal{A}})).$$

Proof. By the Tannakian formalism, we are reduced to the case when $\mathcal{G}=\mathrm{GL}_{n,\mathbb{Z}_p}$, in which case we identify $(\mathcal{A},\varphi_{\mathcal{A}})$ with an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ and write $\mathbb{L}=T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})$. Consider an object $S=\mathrm{Spa}(R,R^+)$ of $\mathbf{Perf}_k$ and let $(S^\sharp,f)$ be an element of $\mathbf{Perf}^+_{\mathfrak{X}}$ where $S^\sharp\to\mathrm{Spa}(\mathcal{O}_E)$ factorizes over $\mathrm{Spa}(E)$. Write $S^\sharp=\mathrm{Spa}(R^\sharp,R^{\sharp+})$. We must show that the shtuka given by $(\mathcal{E},\varphi_{\mathcal{E}})(\mathrm{A}_{\mathrm{inf}}(R^{\sharp+}),(\xi))|_{\mathcal{Y}_{[0,\infty)}(S)}$ and $U_{\mathrm{sht}}(\mathbb{L})(S^\sharp,f)$ are isomorphic functorially in $(S^\sharp,f)$.

By [GR22, Lemma 4.10], it suffices to work only with $(S^\sharp,f)$ where $S^\sharp$ belongs to the category X^w_{qrsp} from [GR22, Definition 4.9]. In particular, we may assume that both of these shtukas over S lie in $\mathbf{Sht}_{n,\mathrm{free}}(S^\sharp)$ and so by Lemma 5.4 it suffices to show that they have the same value under $\Phi_{S^\sharp}$. For both $(\mathcal{E},\varphi_{\mathcal{E}})(\mathrm{A}_{\mathrm{inf}}(R^{\sharp+}),(\xi))|_{\mathcal{Y}_{[0,\infty)}(S)}$ and $U_{\mathrm{sht}}(\mathbb{L})(S^\sharp,f)$, the underlying free $\mathbb{Z}_p$-module is (by definition) $T=\mathbb{L}(S^\sharp)$. We denote by (T,Ξ_{GR}) the $\mathrm{B}^+_{\mathrm{dR}}$-pair corresponding to the former, and by (T,Ξ_{dR}) the one attached to the latter.

The former lattice Ξ_{GR} is described as follows. By the description of $\Phi_{S^\sharp}$ (before Lemma 5.4), we have $\Xi_{\mathrm{GR}}=\varphi_{\mathcal{E}}((\phi^*\mathcal{E})_{\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)})$. Observe that we have a commutative diagram of isomorphisms as in the proof of [GR22, Theorem 4.8] (see [GR22, Remark 4.12])

$$\begin{tikzcd}
\phi^*\mathcal{E}(\mathrm{A}_{\mathrm{inf}}(S),(\xi))_{\mathrm{B}_{\mathrm{dR}}(S^\sharp)} \arrow[rr, "\varphi_{\mathcal{E}}"] \arrow[d, "(\text{cf. }(2.3.4))"'] & & \mathcal{E}(\mathrm{A}_{\mathrm{inf}}(S),(\xi))_{\mathrm{B}_{\mathrm{dR}}(S^\sharp)} \\
D_{\mathrm{crys}}(\mathbb{L})(\mathrm{A}_{\mathrm{crys}}(R^{\sharp+}) \twoheadrightarrow R^{\sharp+}) \otimes_{\mathrm{A}_{\mathrm{crys}}(R^{\sharp+})} \mathrm{B}_{\mathrm{dR}}(S^\sharp) \arrow[rr, "c_{\mathrm{Fal}}^{-1}\otimes 1"'] & & T \otimes_{\mathbb{Z}_p} \mathrm{B}_{\mathrm{dR}}(S^\sharp) \arrow[u, "(5.1.4)"']
\end{tikzcd}$$

Thus, we have

$$\Xi_{\mathrm{GR}} = (c_{\mathrm{Fal}} \otimes 1)^{-1}\left(D_{\mathrm{crys}}(\mathbb{L})(\mathrm{A}_{\mathrm{crys}}(R^{\sharp+}) \twoheadrightarrow R^{\sharp+}) \otimes_{\mathrm{A}_{\mathrm{crys}}(R^{\sharp+})} \mathrm{B}^+_{\mathrm{dR}}(S^\sharp)\right).$$

On the other hand, recall that the lattice Ξ_{dR} is given by

$$\Xi_{\mathrm{dR}} = (c_{\mathrm{Sch}} \otimes 1)^{-1}(D_{\mathrm{dR}}(\mathbb{L})(S^\sharp) \otimes_{S^\sharp} \mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(S^\sharp))^{\nabla=0}).$$

Thus, considering the isomorphism

$$\theta^{+,\nabla}_{\mathrm{dR}} \colon D_{\mathrm{crys}}(\mathbb{L})(\mathrm{A}_{\mathrm{crys}}(R^{\sharp+}) \twoheadrightarrow R^{\sharp+}) \otimes_{\mathrm{A}_{\mathrm{crys}}(R^{\sharp+})} \mathrm{B}^+_{\mathrm{dR}}(S^\sharp) \xrightarrow{\sim} (D_{\mathrm{dR}}(\mathbb{L})(S^\sharp) \otimes_{S^\sharp} \mathcal{O}\mathrm{B}^+_{\mathrm{dR}}(S^\sharp))^{\nabla=0}$$

from (2.3.3), the assertion follows from Lemma 2.18. □

Prismatic $\mathcal{G}$-torsors with F-structure bounded by μ. Temporarily fix the following data:

- k is a perfect field of characteristic p,
- $W := W(k)$,
- $\mathfrak{Y}$ is a quasi-syntomic p-adic formal W-scheme,
- $\mu\colon \mathbb{G}_{m,W} \to \mathcal{G}_W$ is a minuscule cocharacter.

Let (A, I) be an object of $\mathfrak{Y}_{\Delta}$. Define $\mathbf{Tors}^{\varphi,\mu}_{\mathcal{G}}(A, I)$ to be the full subcategory of $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(A, I)$ consisting of $(\mathcal{A}, \varphi_{\mathcal{A}})$ such that there exists a p-adic étale cover $A \to A'$ such that IA' is principal, and there exists a trivialization $\mathcal{A} \xrightarrow{\sim} \mathcal{G}$ after restriction to the slice site over (A', IA') such that under this trivialization $\varphi_{\mathcal{A}}$ corresponds to left multiplication by an element of $\mathcal{G}(A')\mu(d)\mathcal{G}(A')$ for some (equiv. for any) generator d of IA'. Set

$$\mathbf{Tors}^{\varphi,\mu}_{\mathcal{G}}(\mathfrak{Y}_{\Delta}) := \underset{(A,I)\in\mathfrak{Y}_{\Delta}}{2\text{-}\lim}\ \mathbf{Tors}^{\varphi,\mu}_{\mathcal{G}}(A, I).$$

We give the objects of the category the following name.

Definition 5.8. An object of $\mathbf{Tors}^{\varphi,\mu}_{\mathcal{G}}(\mathfrak{Y}_{\Delta})$ is called a *prismatic $\mathcal{G}$-torsors with F-structure bounded by μ* on $\mathfrak{Y}$.

Assume that $\mathcal{O}_E$ is absolutely unramified. Suppose further that $\mathfrak{X}$ is a smooth formal $\mathcal{O}_E$-scheme, with generic fiber X, and let $\mu\colon \mathbb{G}_{m,\mathcal{O}_E} \to \mathcal{G}_{\mathcal{O}_E}$ be a minuscule cocharacter with $\mu_{\overline{\mathbb{Q}}_p}$ an element of $\boldsymbol{\mu}$. Let us define $\mathbf{Tors}^{\varphi}_{\mathcal{G},\boldsymbol{\mu}}(\mathfrak{X}_{\Delta})$ to be the full subcategory of $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$ consisting of those $(\mathcal{A}, \varphi_{\mathcal{A}})$ such that $T_{\text{ét}}(\mathcal{A}, \varphi_{\mathcal{A}})$ corresponds to an element of $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{crys},\boldsymbol{\mu}}_{\mathbb{Z}_p}(X)$. To understand the relationship between $\mathbf{Tors}^{\varphi}_{\mathcal{G},\boldsymbol{\mu}}(\mathfrak{X}_{\Delta})$ and $\mathbf{Tors}^{\varphi,\mu^{-1}}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$, we make the following observation.

Proposition 5.9. *Let $\{\mathrm{Spf}(R_i)\}$ be an open cover $\mathfrak{X}$ with each R_i small. Then, an object $(\mathcal{A}, \varphi_{\mathcal{A}})$ of $\mathbf{Tors}^{\varphi}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$ lies in $\mathbf{Tors}^{\varphi}_{\mathcal{G},\boldsymbol{\mu}}(\mathfrak{X}_{\Delta})$ if and only if the following condition holds: for each i there exists an étale cover $(\widetilde{R}_i[1/p], \widetilde{R}_i) \to (S^\sharp, S^{\sharp+})$ in $\mathbf{Perf}_E$ such that $\varphi_{\mathcal{A}}$ on $\mathcal{A}(\mathrm{A}_{\mathrm{inf}}(S^{\sharp+}), \xi_{S^{\sharp+}})$ lies in $G(\mathrm{B}^+_{\mathrm{dR}}(S^\sharp))\mu^{-1}(\xi_{S^{\sharp+}})G(\mathrm{B}^+_{\mathrm{dR}}(S^\sharp))$.*

Proof. Observe that by Lemma 1.14 and Lemma 2.10, $\{\mathrm{Spa}(\widetilde{R}_i[1/p], \widetilde{R}_i) \to X^\lozenge\}$ (see §5.1.1 for notation) is a v-cover, where $(\widetilde{R}_i[1/p], \widetilde{R}_i)$ is a perfectoid Huber pair over E (cf. [BMS18, Lemma 3.21]). Note that $H\colon \mathbb{P} \to \mathrm{Gr}_{G,\mathrm{Spd}(E)}$, where $(\mathbb{P}, H) = \Phi_X(T_{\mathrm{sht}}(\mathcal{A}, \varphi_{\mathcal{A}}))$, factorizing through

$\mathrm{Gr}_{G,\boldsymbol{\mu},\mathrm{Spd}(E)}$ can be checked on a v-cover. The claim is then clear since as $\boldsymbol{\mu}$ is minuscule, one has that $\mathrm{Gr}_{G,\boldsymbol{\mu}^{-1},\mathrm{Spd}(E)}$ is the étale sheafification of

$$(S^\sharp, S^{\sharp+}) \mapsto G(\mathrm{B}^+_{\mathrm{dR}}(S^\sharp))\mu^{-1}(\xi_{S^{\sharp+}})G(\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)) \subseteq G(\mathrm{B}_{\mathrm{dR}}(S^\sharp))/G(\mathrm{B}^+_{\mathrm{dR}}(S^\sharp)),$$

a presheaf on $\mathbf{Perf}_E$. □

Corollary 5.10. *There is a containment* $\mathbf{Tors}^{\varphi,\mu^{-1}}_{\mathcal{G}}(\mathfrak{X}_{\Delta}) \subseteq \mathbf{Tors}^{\varphi}_{\mathcal{G},\boldsymbol{\mu}}(\mathfrak{X}_{\Delta})$ *and so, in particular, for* $(\mathcal{E},\varphi_{\mathcal{E}})$ *in* $\mathbf{Tors}^{\varphi,\mu^{-1}}_{\mathcal{G}}(\mathfrak{X}_{\Delta})$ *one has that* $T_{\text{ét}}(\mathcal{E},\varphi_{\mathcal{E}})$ *corresponds to an element of* $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{crys},\boldsymbol{\mu}}_{\mathbb{Z}_p}(X)$.

Remark 5.11. The assumption above that $\mathcal{O}_E$ is absolutely unramified is not conceptually necessary, but would require working with $\mathcal{O}_E$-prisms as in [Ito23b, §2].

The category of local systems with prismatic models. Suppose now that $\mathscr{X}$ is a separated locally of finite type flat $\mathcal{O}_E$-scheme with $\mathscr{X}_E \to \mathrm{Spec}(E)$ smooth.

Definition 5.12. A $\mathcal{G}(\mathbb{Z}_p)$-$\boldsymbol{\mu}$-*local system with prismatic model* on $\mathscr{X}$ is a triple $(\omega_{\text{ét}}, \omega_{\Delta}, \iota)$ where

- $\omega_{\text{ét}}$ is an object of $\mathcal{G}\text{-}\mathbf{Loc}^{\mathrm{dR},\boldsymbol{\mu}}_{\mathbb{Z}_p}(\mathscr{X}_E)$,
- ω_{Δ} is an object of $\mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{an},\varphi}(\widehat{\mathscr{X}}_{\Delta})$,
- and $\iota\colon T_{\text{ét}} \circ \omega_{\Delta} \xrightarrow{\sim} \omega^{\mathrm{an}}_{\text{ét}}|_{\widehat{\mathscr{X}}_\eta}$ is an isomorphism.

We call the pair (ω_{Δ}, ι) a *prismatic model* of $\omega_{\text{ét}}$, but usually surpress ι from the notation.

A morphism of $\mathcal{G}(\mathbb{Z}_p)$-$\boldsymbol{\mu}$-local systems with prismatic models

$$(f,g)\colon (\omega'_{\text{ét}}, \omega'_{\Delta}, \iota') \to (\omega_{\text{ét}}, \omega_{\Delta}, \iota)$$

consists of an isomorphism $f\colon \omega'_{\text{ét}} \xrightarrow{\sim} \omega_{\text{ét}}$ as well as an isomorphism $g\colon \omega'_{\Delta} \xrightarrow{\sim} \omega_{\Delta}$ satisfying $\iota \circ f^{\mathrm{an}}|_{\widehat{\mathscr{X}}_\eta} = T_{\text{ét}}(g) \circ \iota'$. Denote by $\mathcal{G}\text{-}\mathbf{M}^{\boldsymbol{\mu}}_{\Delta}(\mathscr{X})$ the groupoid of $\mathcal{G}(\mathbb{Z}_p)$-$\boldsymbol{\mu}$-local systems with prismatic models on $\mathscr{X}$. For a map $\mathscr{X}' \to \mathscr{X}$ there is a pullback functor $\mathcal{G}\text{-}\mathbf{M}^{\boldsymbol{\mu}}_{\Delta}(\mathscr{X}) \to \mathcal{G}\text{-}\mathbf{M}^{\boldsymbol{\mu}}_{\Delta}(\mathscr{X}')$, and $\mathcal{G}\text{-}\mathbf{M}^{\boldsymbol{\mu}}_{\Delta}$ forms a stack over the category of such $\mathscr{X}$ endowed with the Zariski topology.

Using Proposition 5.7, there is a natural functor

$$\mathcal{G}\text{-}\mathbf{M}^{\boldsymbol{\mu}}_{\Delta}(\mathscr{X}) \to \mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathsf{t}(\mathscr{X})), \qquad (\omega_{\text{ét}}, \omega_{\Delta}, \iota) \mapsto (U_{\mathrm{sht}}(\omega^{\mathrm{an}}_{\text{ét}}), T_{\mathrm{sht}}(\omega_{\Delta}), U_{\mathrm{sht}}(\iota) \circ \varrho_{\omega_{\Delta}}).$$

Thus, from the discussion at the end of §5.1.2, we obtain a *shtuka realization functor*,

$$T_{\mathrm{sht}}\colon \mathcal{G}\text{-}\mathbf{M}^{\boldsymbol{\mu}}_{\Delta}(\mathscr{X}) \to \mathcal{G}\text{-}\mathbf{Sht}_{\boldsymbol{\mu}}(\mathscr{X}).$$

The functor T_{sht} is compatible with pullbacks in the obvious sense.

5.2. **The universal deformation spaces of Ito.** To formulate our prismatic characterization of Shimura varieties, we need a prismatic version of the completions of integral moduli spaces of $\mathcal{G}$-shtukas at a point. This is furnished by a construction of Ito which we now recall.

Fix k to be a perfect field of characteristic p, and set $W = W(k)$. We also fix a reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$, and a minuscule cocharacter $\mu\colon \mathbb{G}_{m,W} \to \mathcal{G}_W$. Let $\mathcal{C}^{\mathrm{reg}}_W$ denote the category of complete regular local rings R equipped with a local ring map $W \to R$ such that $k \to R/\mathfrak{m}_R$ is an isomorphism, with morphisms being local W-algebra maps. We consider objects $\mathcal{C}^{\mathrm{reg}}_W$ as p-adic topological rings.

5.2.1. $\mathcal{G}$-μ-*displays.* Let R be a quasi-syntomic W-algebra. Following [Ito23b], for an object (A, I) of R_{Δ}, and a generator d of I, we define the sheaf

$$\mathcal{G}_{\mu,(A,I)}\colon \mathrm{Spec}(A)_{\text{ét}} \to \mathbf{Grp}, \qquad B \mapsto \left\{g \in \mathcal{G}(B) : \mu(d)g\mu(d)^{-1} \in \mathcal{G}(B) \subseteq \mathcal{G}(B[1/d])\right\},$$

which does not depend on d. For d generating I, define an action of $\mathcal{G}_{\mu,(A,I)}$ on $\mathcal{G}_d := \mathcal{G}$ by

$$\mathcal{G}_d \times \mathcal{G}_{\mu,(A,I)} \to \mathcal{G}_d, \qquad (x,g) \mapsto g^{-1}x\phi(\mu(d)g\mu(d)^{-1}).$$

For another generator d' of I there exists a unique unit u of A with $d = ud'$ and the morphism $\mathcal{G}_d \to \mathcal{G}_{d'}$ given by sending x to $x\phi(\mu(u))$ is a $\mathcal{G}_{\mu,(A,I)}$-equivariant isomorphism of sheaves. Thus, $\mathcal{G}_{\Delta,(A,I)} := \varprojlim \mathcal{G}_d$, is a sheaf of sets carrying a canonical action of $\mathcal{G}_{\mu,(A,I)}$.

As in [Ito23b], a *$\mathcal{G}$-μ-display* on (A, I) is a pair $(\mathcal{Q}_{(A,I)}, \alpha_{\mathcal{Q}_{(A,I)}})$ where $\mathcal{Q}_{(A,I)}$ is a $\mathcal{G}_{\mu,(A,I)}$-torsor and $\alpha_{\mathcal{Q}_{(A,I)}} : \mathcal{Q} \to \mathcal{G}_{\Delta,(A,I)}$ is a $\mathcal{G}_{\mu,(A,I)}$-equivariant map of sheaves. There is an evident notion of morphism of $\mathcal{G}$-μ-displays on (A, I), and we denote by $\mathcal{G}\text{-}\mathbf{Disp}_\mu(A, I)$ the category of prismatic $\mathcal{G}$-μ-displays on (A, I). For a morphism $(A, I) \to (B, J)$, where both I and J are principal, there is an obvious pullback morphism $\mathcal{G}\text{-}\mathbf{Disp}_\mu(A, I) \to \mathcal{G}\text{-}\mathbf{Disp}_\mu(B, J)$ and Ito defines a *prismatic $\mathcal{G}$-μ-display* on R to be an object $(\mathcal{Q}, \alpha_\mathcal{Q})$ of the category

$$\mathcal{G}\text{-}\mathbf{Disp}_\mu(R_\Delta) := \underset{(A,I)\in R_\Delta}{2\text{-}\lim}\ \mathcal{G}\text{-}\mathbf{Disp}_\mu(A, I),$$

which makes sense as every object (A, I) of R_Δ has a cover $(A, I) \to (B, J)$ where J is principal.

5.2.2. *Relationship to prismatic $\mathcal{G}$-torsors with F-structure.* We now show that $\mathcal{G}$-μ-displays are precisely the prismatic $\mathcal{G}$-torsors with F-structure bounded by μ. More precisely, we construct an equivalence

$$\mathcal{G}\text{-}\mathbf{Disp}_\mu(A, (d)) \xrightarrow{\sim} \mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(A, (d)),$$

functorial in an object $(A, (d))$ of R_Δ.

To this end, we first construct an equivalence between the category of banal (see [Ito23b, §5.1]) $\mathcal{G}$-μ-displays on $(A, (d))$ and the full-subcategory of $\mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(A, (d))$ consisting of those $(\mathcal{A}, \varphi_\mathcal{A})$ with $\mathcal{A}$ trivializable. Let $(\mathcal{G}_{\mu,(A,I)}, X : \mathcal{G}_{\mu,(A,I)} \to \mathcal{G}_{\Delta,(A,I)})$ be a banal $\mathcal{G}$-μ-display on (A, I). Choose a generator d of I, and write $X_d \in \mathcal{G}(A)$ for the d-component of $X(1)$. Define $P_{X,d}$ to be the trivial $\mathcal{G}_A$-torsor and $\varphi_{P_{X,d}}$ to be the composition

$$\phi^*\mathcal{G}_{A[1/d]} = \mathcal{G}_{A[1/d]} \xrightarrow{\sim} \mathcal{G}_{A[1/d]} \xrightarrow{\sim} \mathcal{G}_{A[1/d]},$$

where the first map is left multiplication by X_d and the second map is left multiplication by $\mu(d)$. Note that for another generator d' of I with $d = ud'$ with $u \in A$ a unit, the left multiplication by $\mu(u)$ defines an isomorphism $(P_{X,d'}, \varphi_{P_{X,d'}}) \xrightarrow{\sim} (P_{X,d}, \varphi_{P_X,d})$. We define (P_X, φ_{P_X}) to be the inverse limit $\varprojlim_d (P_{X,d}, \varphi_{P_{X,d}})$ with transition maps given by $\mu(u)$.

The pair (P_X, φ_{P_X}) is then seen to be an object of $\mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(A, (d))$ whose underlying $\mathcal{G}$-torsor is trivalizable. This defines a functor as for an element g of $\mathcal{G}_{\mu,(A,(d))}$, we have an induced morphism $(\mu(d)g\mu(d)^{-1})_d : P_{X\cdot g} \to P_X$, which is functorial and preserves the Frobenius structures by construction. This functor is clearly fully faithful.

Stackifying this association gives us a functor

$$\mathcal{G}\text{-}\mathbf{Disp}_\mu(A, (d)) \to \mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(A, (d)),$$

which is seen to be compatible in $\mathcal{G}$ and $(A, (d))$. Thus, for a quasi-syntomic W-algebra R, passing to the limit defines a functor

$$\mathcal{G}\text{-}\mathbf{Disp}_\mu(R_\Delta) \to \mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(R_\Delta), \tag{5.2.1}$$

functorial in $\mathcal{G}$ and R.

Proposition 5.13. *The functor (5.2.1) defines an equivalence of categories*

$$\mathcal{G}\text{-}\mathbf{Disp}_\mu(R_\Delta) \xrightarrow{\sim} \mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(R_\Delta), \qquad (\mathcal{Q}, \alpha_\mathcal{Q}) \mapsto (\mathcal{Q}, \alpha_Q)_\phi,$$

functorial in $\mathcal{G}$ and R.

Proof. As we have already observed, this functor is fully-faithful, it remains to show that it is essentially surjective. As both the source and target are stacks on $\mathfrak{X}_\Delta$, this fully faithfulness allows us to reduce ourselves to showing that the functor is essentially surjective on banal objects over some $(A, (d))$. Let (P, φ_P) be an object of $\mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(A, (d))$ with P trivializable. Then by definition, φ_P is defined by $Y\mu(d)X$ for some X and Y in $\mathcal{G}(A)$. But left multiplication by Y then defines an isomorphism $P_{X\phi(Y),d} \to P$ in $\mathbf{Tors}_\mathcal{G}^{\varphi,\mu}(A, (d))$, from where the claim follows. $\square$

Remark 5.14. In [Ito23b, Definition 5.5.4], Ito defines the notion of a ϕ-$\mathcal{G}$-torsor on a prism (A, I). This differs from our definition of a $\mathcal{G}$-torsor with F-structure on (A, I), as Ito inverts $\phi(I)$, and not I. That said, when (A, I) is a perfect prism, there is a natural equivalence between these two notions, and under this translation our equivalence in Proposition 5.13 agrees with that in [Ito23b, Proposition 5.6.2].

5.2.3. *Universal deformation spaces.* Denote by $\mathcal{U}_{\mu^{-1}}$ the unipotent group scheme over W associated to μ^{-1} via the dynamic method (see [Con14, Theorem 4.1.7]). Set $R_{\mathcal{G},\mu} := \mathcal{O}(\widehat{\mathcal{U}}_{\mu^{-1}})$, which is a p-adically complete ring non-canonically isomorphic to $W[\![t_1, \ldots, t_d]\!]$ for some d (see [Ito23b, Lemma 4.2.6]). If $f\colon \mathcal{G}_1 \to \mathcal{G}$ is a morphism of reductive groups over $\mathbb{Z}_p$ mapping μ_1 to μ then we obtain an induced continuous morphism of W-algebras $R_{\mathcal{G},\mu} \to R_{\mathcal{G}_1,\mu_1}$. Furthermore, if the map $\mathcal{G}_1^{\mathrm{ad}} \to \mathcal{G}^{\mathrm{ad}}$ induced by f is an isomorphism, then the map $R_{\mathcal{G},\mu} \to R_{\mathcal{G}_1,\mu_1}$ is an isomorphism.[31]

Fix an element b of $\mathcal{G}(W)\mu(p)\mathcal{G}(W)$. The pair $(\mathcal{G}_W, \varphi_b)$, where φ_b corresponds to right multiplication by b, defines an element of $\mathbf{Tors}_{\mathcal{G}}^{\varphi,\mu}(W,(p))$. As in [Ito23a, Definition 1.1.1], for an object R of $\mathcal{C}_W^{\mathrm{reg}}$, a *deformation* of $(\mathcal{G}_W, \varphi_b)$ over R is a pair $((\mathcal{A}, \varphi_{\mathcal{A}}), \gamma)$ where $(\mathcal{A}, \varphi_{\mathcal{A}})$ is an object of $\mathbf{Tors}_{\mathcal{G}}^{\varphi,\mu}(R_{\triangle})$, and $\gamma\colon (\mathcal{A}, \varphi_{\mathcal{A}})|_{k_{\triangle}} \xrightarrow{\sim} (\mathcal{G}_W, \varphi_b)$ is an isomorphism. In [Ito23a, Theorem 1.1.2] (using Proposition 5.13), Ito shows that there is a *universal deformation* $((\mathcal{A}_b^{\mathrm{univ}}, \varphi_{\mathcal{A}_b^{\mathrm{univ}}}), \gamma^{\mathrm{univ}})$ over $R_{\mathcal{G},\mu}$, which means that for any other deformation $((\mathcal{A}, \varphi_{\mathcal{A}}), \gamma)$ over an object R of $\mathcal{C}_W^{\mathrm{reg}}$, there exists a unique morphism $h\colon R_{\mathcal{G},\mu} \to R$ such that $h^*((\mathcal{A}_b^{\mathrm{univ}}, \varphi_{\mathcal{A}_b^{\mathrm{univ}}}), \gamma^{\mathrm{univ}})$ is isomorphic to $((\mathcal{A}, \varphi_{\mathcal{A}}), \gamma)$. Define ω_b^{univ} to be the object of $\mathcal{G}$-$\mathbf{Vect}^{\varphi}(R_{\triangle})$ associated to $(\mathcal{A}_b^{\mathrm{univ}}, \varphi_{\mathcal{A}_b^{\mathrm{univ}}})$.

Lemma 5.15. *For a morphism of reductive groups $f\colon \mathcal{G}_1 \to \mathcal{G}$ sending μ_1 to μ, and b_1 to b, the induced map $f\colon \mathrm{Spf}(R_{\mathcal{G}_1,\mu_1}) \to \mathrm{Spf}(R_{\mathcal{G},\mu})$ satisfies the following, with ξ an object of $\mathbf{Rep}_{\mathbb{Z}_p}(\mathcal{G})$:*

$$(\mathcal{A}_{b_1}^{\mathrm{univ}}, \varphi_{\mathcal{A}_{b_1}^{\mathrm{univ}}}) \times^{\mathcal{G}_{1,\triangle}} \mathcal{G}_{\triangle} \xrightarrow{\sim} f^*(\mathcal{A}_b^{\mathrm{univ}}, \varphi_{\mathcal{A}_b^{\mathrm{univ}}}), \qquad f^*(\omega_b^{\mathrm{univ}}(\xi)) = \omega_{b_1}^{\mathrm{univ}}(\xi \circ f).$$

Proof. It suffices to show the first isomorphism. By [Ito23b, Theorem 6.1.3], it suffices to construct an isomorphism after evaluating at $(R_{\mathcal{G}_1,\mu_1}[\![t]\!], (p-t))$. By the construction in the proof of [Ito23a, Theorem 4.4.2], both evaluations are isomorphic to $\mathcal{Q}_{f(X_1^{\mathrm{univ}})}$, where X_1^{univ} is X^{univ} in the proof of [Ito23a, Theorem 4.4.2], when applied to $(\mathcal{G}_W, \varphi_{b_1})$, $\mathcal{G}_1$ and μ_1. ☐

If g is in $\mathcal{G}(W)$ then conjugation by g^{-1} induces an isomorphism $c_g\colon R_{\mathcal{G},\mu} \to R_{\mathcal{G},g\mu g^{-1}}$ and $c_g^*(\omega_b^{\mathrm{univ}}) = \omega_{gbg^{-1}}^{\mathrm{univ}}$. If b' is another element of $\mathcal{G}(W[1/p])$ and g is an element of $\mathcal{G}(W)$ such that $b' = gb\phi(g)^{-1}$ then left multiplication by g induces an isomorphism $(\mathcal{G}_W, \varphi_b) \to (\mathcal{G}_W, \varphi_{b'})$ and thus there is an isomorphism $\omega_b^{\mathrm{univ}} \xrightarrow{\sim} \omega_{b'}^{\mathrm{univ}}$. Thus, the pair $(R_{\mathcal{G},\mu}, \omega_b^{\mathrm{univ}})$ only depends, up to isomorphism, on the $\mathcal{G}(W)$-conjugacy class of μ, and the σ-$\mathcal{G}(W)$-conjugacy class of b.

5.2.4. *Comparison to Faltings universal deformations with Tate tensors.* We now wish to use the functor $\mathbb{D}_{\mathrm{crys}}$ to compare the work of Ito to that of Faltings on universal deformations of p-divisible groups with Tate tensors. We freely use terminology/notation from §3.4.

We use the notation of the previous subsection, but now assume that k is algebraically closed. We also write $\omega_{b,\mathrm{crys}}^{\mathrm{univ}}$ for the composition $\mathbb{D}_{\mathrm{crys}} \circ \omega_b^{\mathrm{univ}}$, which is a $\mathcal{G}$-object in $\mathbf{VectF}^{\varphi}((R_{\mathcal{G},\mu})_{\mathrm{crys}})$.

Universal deformation of p-divisible groups. Fix a p-divisible group H_0 over k, and choose any lift $\widetilde{H}_0$ of H_0 over W. Set $M_0 := \underline{\mathbb{D}}(H_0)(W \twoheadrightarrow k)$, and let b in $\mathrm{GL}(M_0[1/p])$ correspond to the Frobenius on M_0. From $\widetilde{H}_0$ we obtain the Hodge filtration (with level in $[0, 1]$) given by

$$\mathrm{Fil}^1_{\widetilde{H}_0,\mathrm{Hodge}} \subseteq \mathbb{D}(\widetilde{H}_0)(\mathrm{id}\colon W \to W) = \mathbb{D}(H_0)(W \twoheadrightarrow k).$$

[31]Indeed, we have a homomorphism of group W-schemes $U_{\mu_1^{-1}} \to U_{\mu^{-1}}$, and so it suffices to show it's an isomorphism of schemes. By loc. cit., it suffices to show that the map $\mathrm{Lie}(U_{\mu_1^{-1}}) \to \mathrm{Lie}(U_{\mu^{-1}})$ is an isomorphism of schemes. But, this is obvious since, with notation as in [Ito23b, Lemma 4.2.6], the subspace $\bigoplus_{n\geqslant 1} \mathfrak{g}_n$ only depends on the pair $(\mathcal{G}, \mu)$ up to passage to the adjoint groups.

From the Cartan decomposition, we know that b lies in $\mathcal{G}(W)\mu_0^{-1}(p)\mathcal{G}(W)$ for a cocharacter $\mu_0\colon \mathbb{G}_{m,W}\to \mathrm{GL}(M_0)$ uniquely determined up to conjugacy. We take the unique cocharacter μ_0 such that $\mathrm{Fil}^1_{\widetilde{H}_0,\mathrm{Hodge}}$ is induced by μ_0 in the sense of [Kim18a, Definition 2.2.1].

Let us choose an isomorphism $R_{\mathrm{GL}(M_0),\mu_0}\xrightarrow{\sim} W[\![t_1,\ldots,t_d]\!]$ (for some $d\geqslant 0$) and equip this with the usual Frobenius ϕ_0 (i.e., ϕ_0 is the usual map on W and $\phi_0(t_i)=t_i^p$). From the above considerations we obtain the following data on $R_{\mathrm{GL}(M_0),\mu_0}$:

$$\mathbf{M}_b^{\mathrm{univ}}:=R_{\mathrm{GL}(M_0),\mu_0}\otimes_W M_0,\quad \mathrm{Fil}^1\,\mathbf{M}_b^{\mathrm{univ}}:=1\otimes \mathrm{Fil}^1 M_0,\qquad \varphi_{\mathbf{M}_b^{\mathrm{univ}}}:=u_t^{-1}\circ(1\otimes b),$$

where u_t corresponds to the tautological element of $\widehat{U}_{\mu^{-1}}(R_{\mathrm{GL}(M_0),\mu_0})$, and $\varphi_{\mathbf{M}_b^{\mathrm{univ}}}$ is considered as a map $\phi_0^*\mathbf{M}_b^{\mathrm{univ}}\to\mathbf{M}_b^{\mathrm{univ}}$. As explained in [Moo98, §4.5], Faltings produced a (unique) $\varphi_{\mathbf{M}_b^{\mathrm{univ}}}$-horizontal integrable connection $\nabla_{\mathbf{M}_b^{\mathrm{univ}}}$ on $\mathbf{M}_b^{\mathrm{univ}}$ so that the quadruple $(\mathbf{M}_b^{\mathrm{univ}},\varphi_{\mathbf{M}_b^{\mathrm{univ}}},\mathrm{Fil}^\bullet_{\mathbf{M}_b^{\mathrm{univ}}},\nabla_{\mathbf{M}_b^{\mathrm{univ}}})$ is a filtered Dieudonné crystal on $R_{\mathrm{GL}(M_0),\mu_0}$ (see also [Kim18a, §3.3]). It thus corresponds to a p-divisible group H_b^{univ} over $R_{\mathrm{GL}(M),\mu_0}$.

As explained in loc. cit., this notation is not misleading as H_b^{univ} is a universal deformation of H_0 (in the sense that it pro-represents the functor in [Kim18a, Definition 3.1]).

Proposition 5.16. *There is a natural isomorphism*

$$\omega_{b,\mathrm{crys}}^{\mathrm{univ}}(M_0)(R_{\mathrm{GL}(M_0),\mu_0})\xrightarrow{\sim}(\mathbf{M}_b^{\mathrm{univ}},\varphi_{\mathbf{M}_b^{\mathrm{univ}}},\mathrm{Fil}^\bullet_{\mathbf{M}_b^{\mathrm{univ}}}).$$

Proof. This follows from Proposition 3.50 and and [Ito23a, Theorem 6.2.1]. □

Deformations with Tate tensors. Fix a triple $(\mathcal{G},b,\mu)$ as in §5.2.3. We assume that $(\mathcal{G},b,\mu)$ is of *Hodge type*: there exists a faithful representation $\iota\colon\mathcal{G}\to\mathrm{GL}(\Lambda_0)$ such that $\iota\circ\mu$ has only weights 0 and 1. Set $(b_0,\mu_0):=(\iota(b),\iota\circ\mu)^\vee$. Further fix isomorphisms

$$R_{\mathrm{GL}(\Lambda_0^\vee),\mu_0}\xrightarrow{\sim}W[\![t_1\ldots,t_d]\!],\qquad R_{\mathcal{G},\mu}\xrightarrow{\sim}W[\![s_1,\ldots,s_k]\!]$$

such that the natural map $R_{\mathrm{GL}(\Lambda_0^\vee),\mu_0}\to R_{\mathcal{G},\mu}$ is Frobenius equivariant when the source and target are given the (usual) Frobenii induced by these isomorphisms.[32] Finally, fix a tensor package (in the sense in §A.5) $(\Lambda_0,\mathbb{T}_0)$ with $\mathcal{G}=\mathrm{Fix}(\mathbb{T}_0)$.

As explained in [Kim18a, §2.5] associated to $(\mathcal{G},b,\mu)$ and ι is a p-divisible group H_{b_0} over k together with an identification $\mathbb{D}(H_{b_0})(W)=\Lambda_0^\vee$ where Frobenius acts by b_0. Moreover, under this identification the set $\mathbb{T}_0$ is a set of tensors on $\mathbb{D}(H_{b_0})$ as an F-crystal. Set

$$(\mathbf{M}_b^{\mathrm{univ}},\varphi_{\mathbf{M}_b^{\mathrm{univ}}},\mathrm{Fil}^1_{\mathbf{M}_b^{\mathrm{univ}}}):=(\mathbf{M}_{b_0}^{\mathrm{univ}},\varphi_{\mathbf{M}_{b_0}^{\mathrm{univ}}},\mathrm{Fil}^1_{\mathbf{M}_{b_0}^{\mathrm{univ}}})\otimes_{R_{\mathrm{GL}(\Lambda_0^\vee),\mu_0}}R_{\mathcal{G},\mu},$$

and let H_b^{univ} be the pullback of $H_{b_0}^{\mathrm{univ}}$ to $R_{\mathcal{G},\mu}$. Then,

$$\mathbb{D}(H_b^{\mathrm{univ}})(R_{\mathcal{G},\mu})=(\mathbf{M}_b^{\mathrm{univ}},\varphi_{\mathbf{M}_b^{\mathrm{univ}}},\mathrm{Fil}^1_{\mathbf{M}_b^{\mathrm{univ}}},\nabla_{\mathbf{M}_b^{\mathrm{univ}}}),$$

for some connection $\nabla_{\mathbf{M}_b^{\mathrm{univ}}}$. Observe that $\mathbb{T}_0$ naturally defines a set of tensors on $\mathbf{M}_{b_0}^{\mathrm{univ}}$ and (by base change) on $\mathbf{M}_b^{\mathrm{univ}}$, which we denote $\mathbb{T}_0^{\mathrm{Fal},\prime}$ and $\mathbb{T}_0^{\mathrm{Fal}}$, respectively. The tensors $\mathbb{T}_0^{\mathrm{Fal}}$ on $\mathbb{D}(H_b^{\mathrm{univ}})$ are Frobenius equivariant and lie in the 0^{th}-part of the filtration (see [Kim18a, §3.5]).

By work of Faltings (see [Kim18a, Theorem 3.6]) H_b^{univ} satisfies a universality property. Suppose that $R_0=W[\![u_1,\ldots,u_r]\!]$ for some r, and X is a p-divisible group over R_0 deforming H_{b_0}. By the universality of $H_{b_0}^{\mathrm{univ}}$, there exists a unique map $f_X\colon R_{\mathrm{GL}(\Lambda_0^\vee),\mu_0}\to R_0$ such that $f_X^*(H_{b_0}^{\mathrm{univ}})$ is isomorphic (as a deformation) to X. Then, f_X factorizes through $R_{\mathrm{GL}(\Lambda_0^\vee),\mu_0}\to R_{\mathcal{G},\mu}$ if and only if there exists tensors $\{t_\alpha\}$ on $\mathbb{D}(X)(R_0)$ lifting those on $\mathbb{T}_0$ on $\mathbb{D}(H_{b_0})$, and which are Frobenus equivariant and lie in the 0^{th}-part of the filtration. In this case $\{t_\alpha\}=f_X^*(\mathbb{T}_0^{\mathrm{Fal},\prime})$.

[32] This is possible, for instance, by the discussion in [Ito23b, §4.2], which shows that $U_{-\mu}$ is isomorphic to $\mathrm{Lie}(U_{-\mu})$ as W-schemes, and $\mathrm{Lie}(U_{-\mu})$ is a direct summand of $\mathrm{Lie}(\mathrm{GL}(\Lambda_0^\vee))$.

Consider now the obvious morphism $f\colon (\mathcal{G}, b, \mu) \to (\mathrm{GL}(\Lambda_0^\vee), b_0, \mu_0)$. Combining Lemma 5.15 and Proposition 5.16, there is a canonical identification

$$\begin{aligned} \omega_{b,\mathrm{crys}}^{\mathrm{univ}}(\Lambda_0^\vee)(R_{\mathcal{G},\mu}) &\xrightarrow{\sim} f^*(\mathbb{D}(H_{b_0}^{\mathrm{univ}}))(R_{\mathcal{G},\mu}) \\ &= \mathbb{D}(H_b^{\mathrm{univ}})(R_{\mathcal{G},\mu}) \\ &= (\mathbf{M}_b^{\mathrm{univ}}, \varphi_{\mathbf{M}_b^{\mathrm{univ}}}, \mathrm{Fil}^1_{\mathbf{M}_b^{\mathrm{univ}}}), \end{aligned}$$

of naive filtered F-crystals.

Proposition 5.17. *The isomorphism of naive filtered F-crystals on $R_{\mathcal{G},\mu}$*

$$\omega_{b,\mathrm{crys}}^{\mathrm{univ}}(\Lambda_0^\vee)(R_{\mathcal{G},\mu}) \to (\mathbf{M}_b^{\mathrm{univ}}, \varphi_{\mathbf{M}_b^{\mathrm{univ}}}, \mathrm{Fil}^1_{\mathbf{M}_b^{\mathrm{univ}}})$$

carries $\omega_{b,\mathrm{crys}}^{\mathrm{univ}}(\mathbb{T}_0)(R_{\mathcal{G},\mu})$ to $\mathbb{T}_0^{\mathrm{Fal}}$.

Proof. Under this identification $\omega_{b,\mathrm{crys}}^{\mathrm{univ}}(\mathbb{T}_0)(R_{\mathcal{G},\mu})$ constitutes a Frobenius-equivariant tensors lying in the 0^{th}-part of the of the filtration and lifting those on $\mathbb{D}(H_{b_0})$. Thus, by the universality statement from above, $\omega_{b,\mathrm{crys}}^{\mathrm{univ}}(\mathbb{T}_0)(R_{\mathcal{G},\mu})$ must be equal to $f^*(\mathbb{T}_0^{\mathrm{Fal},\prime}) = \mathbb{T}_0^{\mathrm{Fal}}$. ☐

5.3. **Comparison to Shimura varieties.** We now show that for the integral canonical model $\mathscr{S}_{\mathsf{K}^p}$, the prismatic realization functor $\omega_{\mathbb{\Delta}}$ recovers Ito's universal prismatic $\mathcal{G}^c$-torsor with F-structure at the completion of $\mathscr{S}_{\mathsf{K}^p}$ at each point of $\mathscr{S}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$. This may be seen as a prismatic refinement of [Kis17, (1.3.9) Proposition], in the general abelian type setting.

We adopt notation and conventions as in §4. In particular, suppose that $(\mathbf{G}, \mathbf{X}, \mathcal{G})$ is an unramified Shimura datum of abelian type, and x is a point of $\mathscr{S}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$. As in §4.5 we have the associated element $\boldsymbol{b}_{x,\mathrm{crys}} := \overline{\Sigma}^\circ_{\mathsf{K}^p}(x)$ in $C(\mathcal{G}^c)$. Addtionally, recall that we have the associated conjugacy class μ_h^c of cocharacters $\mathbb{G}_{m,\breve{\mathbb{Z}}_p} \to \mathcal{G}^c_{\breve{\mathbb{Z}}_p}$. Choose an element μ_h^c in μ_h^c.

Lemma 5.18. *The element $\boldsymbol{b}_{x,\mathrm{crys}}$ lies in the image of the map*

$$\mathcal{G}^c(\breve{\mathbb{Z}}_p)\sigma(\mu_h^c(p))\mathcal{G}^c(\breve{\mathbb{Z}}_p) \to C(\mathcal{G}^c).$$

Proof. In the Hodge type case, this follows from [Kim18b, Lemma 3.3.14]. In the special type case, we may assume that the torus in the Shimura datum is cuspidal following the argument given on [Dan22, pp. 25]. From there the claim follows from [KSZ21, Proposition 4.3.14 and Corollary 4.4.12].

For the abelian type case, we take $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$, $(\mathbf{T}, \{h\}, \mathcal{T})$ and $(\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2)$ as in Lemma 4.11. The question is reduced to the case of $(\mathbf{G}_2, \mathbf{X}_2, \mathcal{G}_2)$ by functoriality. For the reduction to the case of $(\mathbf{G}_1 \times \mathbf{T}, \mathbf{X}_1 \times \{h\}, \mathcal{G}_1 \times \mathcal{T})$, it suffices to show the injectivity of

$$\mathcal{G}_2^c(\breve{\mathbb{Z}}_p)\backslash\mathcal{G}_2^c(\breve{\mathbb{Q}}_p)/\mathcal{G}_2^c(\breve{\mathbb{Z}}_p) \to \mathcal{G}_3^c(\breve{\mathbb{Z}}_p)\backslash\mathcal{G}_3^c(\breve{\mathbb{Q}}_p)/\mathcal{G}_3^c(\breve{\mathbb{Z}}_p),$$

where we put $\mathcal{G}_3 = \mathcal{G}_1 \times \mathcal{T}$. We take a Borel pair $\mathcal{T}_1 \subset \mathcal{B}_1 \subset \mathcal{G}_1$. This gives Borel pairs $\mathcal{T}_2 \subset \mathcal{B}_2 \subset \mathcal{G}_2^c$ and $\mathcal{T}_3 \subset \mathcal{B}_3 \subset \mathcal{G}_3^c$ by the constructions of $\mathcal{G}_2^c$ and $\mathcal{G}_3^c$. Then the injectivity of the above map of double cosets follows from the Cartan decomposition since the unipotent radials of $\mathcal{B}_2$ and $\mathcal{B}_3$ are same. ☐

Choose an element b_x in $\mathcal{G}^c(\breve{\mathbb{Z}}_p)\mu_h^c(p)\mathcal{G}^c(\breve{\mathbb{Z}}_p)$ such that $\sigma(b_x)$ maps to $\boldsymbol{b}_{x,\mathrm{crys}}$ in $C(\mathcal{G}^c)$.

Theorem 5.19. *There exists an isomorphism $i_x\colon R_{\mathcal{G}^c,\mu_h^c} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}$ such that*

$$i_x^*(\omega_{b_x}^{\mathrm{univ}}) \cong (\omega_{\mathsf{K}^p,\mathbb{\Delta}})|_{\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}}.$$

Note while the pair $(R_{\mathcal{G}^c,\mu_h^c}, \omega_{b_x}^{\mathrm{univ}})$ depends on the choice of b_x and μ_h^c, the isomorphism type of the pair $(R_{\mathcal{G}^c,\mu_h^c}, \omega_{b_x}^{\mathrm{univ}})$ does not, and therefore neither does the statement of Theorem 5.19.

Proof of Theorem 5.19. We perform a standard devissage to the Hodge and special type cases.

Hodge type case. Let $(\mathbf{G}, \mathbf{X}, \mathcal{G}) \hookrightarrow (\mathrm{GSp}(\mathbf{V}_0), \mathfrak{h}^{\pm}, \mathrm{GSp}(\Lambda_0))$ be an integral Hodge embedding, and $\mathscr{A}_{\mathsf{K}^p} \to \mathscr{S}_{\mathsf{K}^p}$ the associated abelian scheme. Let $\mathbb{T}_{0,p}^{\mathrm{crys}}$ be the tensors on the filtered F-crystal

$$\mathcal{H}^1_{\mathrm{crys}}(\widehat{\mathscr{A}_{\mathsf{K}^p}}/\widehat{\mathscr{S}_{\mathsf{K}^p}}) = \mathbb{D}(\mathscr{A}_{\mathsf{K}^p}[p^\infty])$$

(see [BBM82, (3.3.7.2)] for this identification), as in §4.6. The triple $(\mathcal{G}, b_x, \mu_h)$ is of Hodge type relative to the embedding $\iota\colon \mathcal{G} \to \mathrm{GL}(\Lambda_0)$. By work of Kisin, there exists an isomorphism $i_x\colon R_{\mathcal{G},\mu_h} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}$ together with an isomorphism $i_x^*(H^{\mathrm{univ}}_{b_x}, \mathbb{T}_0^{\mathrm{Fal}}) \cong (\mathscr{A}_{\mathsf{K}^p}[p^\infty], \mathbb{T}^{\mathrm{crys}}_{0,p})_{\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}}$ (see [Kim18b, Proposition 4.1.6]). We claim that $i_x^*(\omega^{\mathrm{univ}}_{b_x})$ is isomorphic to $\omega := (\omega_{\mathsf{K}^p,\mathbb{\Delta}})|_{\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}}$. We consider the following isomorphisms

$$\omega(\Lambda_0^\vee) \xrightarrow{\sim} \mathcal{M}_{\mathbb{\Delta}}(\mathscr{A}[p^\infty]_{\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}}) \xrightarrow{\sim} i_x^*\mathcal{M}_{\mathbb{\Delta}}(H^{\mathrm{univ}}_{b_x}) \xleftarrow{\sim} i_x^*(\omega^{\mathrm{univ}}_{b_x})(\Lambda_0^\vee),$$

where the first isomorphism is from Theorem 4.14, the second is the one obtained by applying $\mathcal{M}_{\mathbb{\Delta}}$ to the above isomorphism of Kisin, and the third one is obtained as follows. By [Ito23a, Theorem 6.2.1] we have a canonical isomorphism $\mathcal{M}_{\mathbb{\Delta}}(H^{\mathrm{univ}}_{\iota(b_x)^\vee}) \xleftarrow{\sim} \omega^{\mathrm{univ}}_{\iota(b_x)^\vee}(\Lambda_0^\vee)$. Then, by Lemma 5.15, we get the third isomorphism as the restriction of this isomorphism along

$$R_{\mathrm{GL}(\Lambda_0^\vee),\iota(\mu_h)^\vee} \to R_{\mathcal{G},\mu_h} \xrightarrow{i_x} \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}.$$

By Proposition 1.28 it suffices to show the above composite carries $\omega(\mathbb{T}_0)$ to $i_x^*(\omega^{\mathrm{univ}}_{b_x})(\mathbb{T}_0)$

It further suffices to check that $\mathbb{D}_{\mathrm{crys}}(\omega(\mathbb{T}_0))(R_{\mathcal{G},\mu_h})$ is matched to $\mathbb{D}_{\mathrm{crys}}(i_x^*(\omega^{\mathrm{univ}}_{b_x})(\mathbb{T}_0))(R_{\mathcal{G},\mu_h})$ as

$$\mathbf{Vect}((R_{\mathcal{G},\mu_h})_{\mathbb{\Delta}}) \to \mathbf{Vect}(R_{\mathcal{G},\mu_h}), \quad \mathcal{E} \mapsto \mathcal{E}^{\mathrm{crys}}(R_{\mathcal{G},\mu_h})$$

is faithful as follows from [dJ95a, Corollary 2.2.3] and the second equivalence in (2.3.2). But, this matching follows by combining Proposition 4.19 and Proposition 5.17.

Special type case. Let us write $(\mathbf{T}, \mathbf{X}, \mathcal{T})$ for the unramified Shimura datum. In this case $\mathscr{S}_{\mathsf{K}^p}$ is a disjoint union of schemes of the form $\mathrm{Spec}(\mathcal{O}_{E'})$, for a finite unramified extension E' of E (see [Dan22, Equation (4.2) and Remark 4.1]), and so there is a tautological identification $R_{\mathcal{G}^c,\mu_h^c} = \breve{\mathbb{Z}}_p = \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}$. We show that under this identification that the prismatic $\mathcal{T}^c$-torsors with F-structure are matched.

By Remark 4.13, Lemma 5.15, and the argument given in [Dan22, pp. 24–26] we are reduced to showing the following. Let $\mathcal{T} = \mathrm{Res}_{\mathcal{O}_{E'}/\mathbb{Z}_p} \mathbb{G}_{m,\mathcal{O}_{E'}}$, $T := \mathcal{T}_{\mathbb{Q}_p}$, and μ the $\mathbb{Q}_p^{\mathrm{ur}}$-cocharacter of T with weights $(1, 0, \ldots, 0)$. Then, $b_0 = (p^{-1}, 1, \ldots, 1)$ represents the unique class in the image of $\mathcal{T}(\breve{W})\mu^{-1}(p)\mathcal{T}(\breve{W})$. Then, we must show that $T_{\text{ét}} \circ \omega^{\mathrm{univ}}_{b_0}$ restricted to the inertia subgroup $\Gamma_{E',0}$ agrees with the Lubin–Tate character $\alpha_0\colon \Gamma_{E',0} \to \mathcal{T}(\mathbb{Z}_p)$ (see the discussion before [Dan22, Proposition 4.8]), or equivalently that their compositions with embedding $\iota\colon \mathcal{T}(\mathbb{Z}_p) \to \mathrm{GL}(\mathcal{O}_{E'})$ are equal. But, as $(\mathcal{T}, b_0, \mu^{-1})$ is the Lubin–Tate triple, we know by [Ito23a, Theorem 6.2.1] that this $\omega^{\mathrm{univ}}_{b_0}(\iota)$ is $\mathcal{M}_{\mathbb{\Delta}}(X_{\mathrm{LT}})$, if X_{LT} is the p-divisible group with $\mathcal{O}_E$-structure over $\mathrm{Spf}(\breve{W})$ coming from Lubin–Tate theory. Thus, the composition of the character $\Gamma_{E',0} \to \mathcal{T}(\mathbb{Z}_p) \to \mathrm{GL}(\mathcal{O}_E)$ corresponding to $T_{\text{ét}} \circ \omega^{\mathrm{univ}}_{b_0}$ is $T_{\text{ét}}(\mathcal{M}_{\mathbb{\Delta}}(X_{\mathrm{LT}})) = T_p(X_{\mathrm{LT}})$ (see [DLMS22, Proposition 3.34]). But, this is the composition of α_0 with $\mathcal{T}(\mathbb{Z}_p) \to \mathrm{GL}(\mathcal{O}_E)$ (see the proof of [Dan22, Proposition 4.8]).

Abelian type case. Let $(\mathbf{G}_1, \mathbf{X}_1, \mathcal{G}_1)$ be an unramified Shimura datum of Hodge type adapted to $(\mathbf{G}, \mathbf{X}, \mathcal{G})$. Consider the morphism of Shimura data obtained in Lemma 4.11. Then, as the map $\alpha^c\colon (\mathcal{G}_2^c)^{\mathrm{der}} \to (\mathcal{G}_1 \times \mathcal{T}^c)^{\mathrm{der}}$ is an isogeny, it induces an isomorphism

$$R_{\mathcal{G}_2,\mu^c_{h,2}} \xrightarrow{\sim} R_{\mathcal{G}_1 \times \mathcal{T}^c, \mu^c_{h,1} \times \mu^c_{h,\mathcal{T}}}.$$

Moreover, for the same reason, for any x_2 in $\mathscr{S}_{\mathsf{K}_p^2}(\overline{\mathbb{F}}_p)$ we obtain an induced isomorphism

$$\alpha^c\colon \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}_2^p},x_2} \to \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}_1^p \times \mathsf{K}_{\mathcal{T}}^p},(x_1,x_{\mathcal{T}})},$$

as $\alpha_{\mathsf{K}_2^p,\mathsf{K}^p}$ is finite étale by Lemma 4.5. Thus, we may use Lemma 5.15, together with the claims in the case of Hodge and special type, to deduce the existence of an isomorphism

$i_{x_2}\colon R_{\mathcal{G}_2^c,\mu_{h,2}^c} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}_2^p},x_2}$ such that the prismatic $\mathcal{G}$-torsors with F-structure $\omega_{\mathsf{K}_2^p,\mathbb{\Delta}}|_{\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}_2^p},x_2}}$ and $i_{x_2}^*(\omega_{b_{x_2}}^{\mathrm{univ}})$ for $\mathcal{G}_2^c$ agree when pushed forward along $\alpha^c\colon \mathcal{G}_2^c \to \mathcal{G}_1 \times \mathcal{T}^c$. This implies they are isomorphic by [Lov17b, Lemma 4.7.5]. Finally, as the morphism $\beta\colon \mathcal{G}_2^{\mathrm{der}} \to \mathcal{G}^{\mathrm{der}}$ is an isogeny, it again induces isomorphisms $R_{\mathcal{G}_2^c,\mu_{h,2}^c} \to R_{\mathcal{G}^c,\mu_h^c}$ and $\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}_2^p},x_2} \to \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}$, where x is the image of x_2. If $i_x\colon R_{\mathcal{G}^c,\mu_h^c} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}$ is the resulting isomorphism, then there are isomorphisms between $\omega_{\mathsf{K}^p,\mathbb{\Delta}}|_{\widehat{\mathcal{O}}_{\mathscr{S}_{\mathsf{K}^p},x}}$ and $i_x^*(\omega_{b_x}^{\mathrm{univ}})$. While arbitrary x may not be the image of such an x_2, we may reduce to this case by applying Lemma 4.6 and the compatibility in (4.3.3). ☐

We observe an important corollary of the above proof.

Corollary 5.20. *The prismatic realization functor $\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ belongs to $\mathbf{Tors}_{\mathcal{G}^c}^{\varphi,\mu_h^c}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_{\mathbb{\Delta}})$. In particular, $\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ takes values in $\mathbf{Vect}^{\varphi,\mathrm{lff}}((\widehat{\mathscr{S}_{\mathsf{K}^p}})_{\mathbb{\Delta}})$.*

Proof. It suffices to show that for each small open subset $\mathrm{Spf}(R)$ of $\widehat{\mathscr{S}_{\mathsf{K}^p}}$, and for every point x of the special fiber of $\mathrm{Spf}(R)$, there exists a p-adically etale neighborhood $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$ such that the Frobenius for $\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ over $\mathfrak{S}_S$ is in the double coset $\mathcal{G}^c(\mathfrak{S}_S)\mu_h^c(p)\mathcal{G}^c(\mathfrak{S}_S)$.

By moving to an étale neighborhood if necessary, we may assume without loss of generality that the underlying $\mathcal{G}^c$-torsor is trivial on $\mathrm{Spf}(R)$. Write g for the element of $\mathcal{G}^c(\mathfrak{S}_R[1/E])$ corresponding to the Frobenius for $\omega_{\mathsf{K}^p,\mathbb{\Delta}}$ on $\mathfrak{S}_R = R[\![t]\!]$. Consider the functor

$$F\colon \mathbf{Alg}_{R[\![t]\!]} \to \mathbf{Set}, \quad A \mapsto \{(h,h') \in \mathcal{G}^c(A) \times \mathcal{G}^c(A) : hgh' = \mu_h^c(p) \in \mathcal{G}^c(A[1/E])\}.$$

Let y be the point of $\mathrm{Spec}(R[\![t]\!])$ equal to (x,t), with the obvious meaning. Observe that we have the equality $\widehat{\mathcal{O}}_{\mathrm{Spec}(R[\![t]\!]),y} = \widehat{\mathcal{O}}_{\mathrm{Spf}(R),x}[\![t]\!]$. Thus, as a result of Theorem 5.19, we have that $F(\widehat{\mathcal{O}}_{\mathrm{Spec}(R[\![t]\!]),y})$ is non-empty. The claim then follows from Artin approximation.

More precisely, first note that $R[\![t]\!]$ is excellent (see Proposition 1.11). Moreover, the functor F is clearly limit-preserving as $\mathcal{G}^c$ is. Thus, by Artin approximation for an excellent base (see [AHR23, Theorem 3.4]), there exists some affine etale neighborhood $\mathrm{Spec}(B) \to \mathrm{Spec}(R[\![t]\!])$ containing y in its image and with $F(\mathrm{Spec}(B))$ non-empty. Let A be the (p,t)-adic completion of B, so that $\mathrm{Spf}(A) \to \mathrm{Spf}(R[\![t]\!])$ is a (p,t)-adically etale neighborhood of y. Set $S = A/tA$. Then, $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$ is a p-adically etale map, and there is a unique deformation (by the topological invariance of the etale site of a formal scheme) to a (p,t)-adically etale map over $\mathrm{Spf}(R[\![t]\!])$ and, in fact, it must be $\mathrm{Spf}(S[\![t]\!])$. Thus, in fact, $A = S[\![t]\!]$ where $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$ is a p-adically etale neighborhood of x. Observe then that, by set-up, $F(S[\![t]\!])$ is non-empty, but this means precisely that the Frobenius is in the double coset of $\mu_h^c(p)$ over S as desired. ☐

5.4. **A prismatic characterization of integral canonical models.** We now formulate and prove a prismatic characterization of the integral canonical models of an abelian type Shimura variety at hyperspecial level. Throughout this section we fix notation and conventions as in §4, and in particular fix $(\mathbf{G},\mathbf{X},\mathcal{G})$ to be an unramified Shimura datum of abelian type.

5.4.1. *Characterization of integral canonical models.* Throughout this subsection let us fix a neat compact open subgroup $\mathsf{K}^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$. Recall from §4.5 that there exists a potentially crystalline locus $U_{\mathsf{K}^p} \subseteq \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}^{\mathrm{an}}$ of $\nu_{\mathsf{K}_0\mathsf{K}^p,\text{ét}}$ or, equivalently $\omega_{\mathsf{K}^p,\text{ét}}$.

Consider a smooth formal $\mathcal{O}_E$-model $\mathfrak{X}_{\mathsf{K}^p}$ of U_{K^p}, and a prismatic model ζ_{K^p} of $\omega_{\mathsf{K}^p,\mathrm{an}}$, i.e., an object of $\mathcal{G}^c\text{-}\mathbf{Vect}^{\varphi}((\mathfrak{X}_{\mathsf{K}^p})_{\mathbb{\Delta}})$ with $T_{\text{ét}} \circ \zeta_{\mathsf{K}^p}$ isomorphic to $\omega_{\mathsf{K}^p,\mathrm{an}}$. For x in $\mathfrak{X}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$, there is an element $\boldsymbol{b}_{x,\mathrm{crys}}$ in $C(\mathcal{G}^c)$ associated to the F-crystal with $\mathcal{G}^c$-structure given by $\underline{\mathbb{D}}_{\mathrm{crys}} \circ (\zeta_{\mathsf{K}^p})_x$. Fix μ_h^c in $\boldsymbol{\mu}_h^c$. Then, we have the following property of $\boldsymbol{b}_{x,\mathrm{crys}}$.

Lemma 5.21. *The element $\boldsymbol{b}_{x,\mathrm{crys}}$ lies in the image of the map*

$$\mathcal{G}^c(\breve{\mathbb{Z}}_p)\sigma(\mu_h^c(p))\mathcal{G}^c(\breve{\mathbb{Z}}_p) \to C(\mathcal{G}^c).$$

Proof. As $\mathfrak{X}$ is smooth over $\mathcal{O}_E$, we know that the specialization map $\mathrm{sp}\colon |U_{\mathsf{K}^p}|^{\mathrm{cl}} \to \mathfrak{X}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ is surjective (see [Bos14, §8.3, Proposition 8]). Let y be a point of $|U_{\mathsf{K}^p}|^{\mathrm{cl}}$ such that $\mathrm{sp}(y) = x$. Then, arguing as in Proposition 4.17 shows that $\boldsymbol{b}_{x,\mathrm{crys}}$ is equal to the element associated to the isocrystal with $\mathcal{G}$-structure associated to $\underline{\mathbb{D}}_{\mathrm{crys}} \circ (\omega_{\mathsf{K}^p,\text{ét}})_y$. But, the claim then follows from Proposition 4.17 and Lemma 5.18. □

Choose an element b_x in $\mathcal{G}^c(\breve{\mathbb{Z}}_p)\mu_h^c(p)\mathcal{G}^c(\breve{\mathbb{Z}}_p)$ such that $\sigma(b_x)$ maps to $\boldsymbol{b}_{x,\mathrm{crys}}$ in $C(\mathcal{G}^c)$.

Definition 5.22. An *integral canonical model* of U_{K^p} is a smooth and separated formal $\mathcal{O}_E$-model $\mathfrak{X}_{\mathsf{K}^p}$ such that there exists a prismatic model ζ_{K^p} of $\omega_{\mathsf{K}^p,\mathrm{an}}$ with the following property: for each K^p and each x in $\mathfrak{X}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ there exists an isomorphism

$$\Theta_x^{\Delta}\colon R_{\mathcal{G}^c,\mu_h^c} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathfrak{X}_{\mathsf{K}^p},x}$$

such that $(\Theta_x^{\Delta})^*(\omega_{b_x}^{\mathrm{univ}})$ is isomorphic to the pullback of ζ_{K^p} to $\widehat{\mathcal{O}}_{\mathfrak{X}_{\mathsf{K}^p},x}$.

Remark 5.23.

(1) This condition is (in contrast to the notion of integral canonical model of Shimura varieties) about an individual level and not the system as the level varies.
(2) One may relax the smoothness condition on $\mathfrak{X}_{\mathsf{K}^p}$ to the following: $\mathfrak{X}_{\mathsf{K}^p}$ is a flat separated locally of finite type formal $\mathcal{O}_E$-scheme. In this case each $\mathfrak{X}_{\mathsf{K}^p}$ is automatically smooth as it is flat and for each point x of $\mathfrak{X}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ one has that $\widehat{\mathcal{O}}_{\mathfrak{X}_{\mathsf{K}^p},x}$ is regular.
(3) While the choices μ_h^c and b_x within their respective equivalence classes were arbitrary, the definition of an integral canonical model of U_{K^p} is independent of such choices.

We now aim to show that the unique integral canonical model of U_{K^p} is $\widehat{\mathscr{S}_{\mathsf{K}^p}}$. Our method of proof relies on providing a slight technical extension of the fully-faithfulness portion of Theorem 2.20 to certain semi-stable formal schemes, mimicking the argument in [DLMS22, Theorem 3.28].

To state this let us fix a complete discrete valuation ring $\mathcal{O}_K$ with fraction field K and perfect residue field k. Set W to be $W(k)$, and fix a uniformizer ϖ of K.

Proposition 5.24. *Set R to be $\mathcal{O}_K[\![x_1, \ldots, x_d]\!]/(x_1x_2\cdots x_m - \varpi)$, where d and m are integers with $1 \leqslant m \leqslant d$. Then the étale realization functor*

$$T_{\text{ét}}\colon \mathbf{Vect}^{\varphi}(R_{\Delta}) \to \mathbf{Loc}_{\mathbb{Z}_p}(R[1/p])$$

is fully faithful.

As in [Ito23b], we consider the following Breuil–Kisin type prism. Let $\mathfrak{S}_R$ be the ring $W[\![x_1, \ldots, x_d]\!]$ equipped with a Frobenius lift ϕ determined by $\phi(x_i) = x_i^p$, and E in $\mathfrak{S}_R$ be the polynomial $E_\varpi(x_1x_2\cdots x_m)$, where E_ϖ is the minimal polynomial of ϖ relative to $\mathrm{Frac}(W)$. Then the pair $(\mathfrak{S}_R, (E))$ defines an object of R_{Δ}.

Lemma 5.25. *The object $(\mathfrak{S}_R, (E))$ covers the final object $*$ of $\mathbf{Sh}(R_{\Delta})$.*

Proof. Similarly to Proposition 1.15, the assertion follows from Proposition 1.10 (using [ALB23, Proposition 5.8] in place of Lemma 1.14). □

In particular, we can regard a prismatic F-crystal on R_{Δ} as a finite free Breuil–Kisin module equipped with a descent datum. More precisely, we let $\mathfrak{S}_R^{(2)}$ be $(\mathfrak{S}_R\widehat{\otimes}_{\mathbb{Z}_p}\mathfrak{S}_R)\left\{\frac{J}{E}\right\}_{\delta}^{\wedge}$, where J is the kernel of the composition

$$\mathfrak{S}_R\widehat{\otimes}_{\mathbb{Z}_p}\mathfrak{S}_R \to \mathfrak{S}_R \to \mathfrak{S}_R/(E) \xrightarrow{\sim} R.$$

As in [DLMS22, Example 3.4], it represents the self-product of $\mathfrak{S}_R$ over $*$ in $\mathbf{Sh}(R_{\Delta})$ (similarly for the triple self-product).

For $i = 1, \ldots, d$, let ϵ_i denote the product of x_j for $1 \leqslant j \leqslant d$ excluding i. Denote the ring $R[1/\epsilon_i]_p^{\wedge}$ by R_i, which is a base ring in the sense in subsection 1.1.5. Using the map $\mathcal{O}_K\langle x_j^{\pm 1}; j \neq i\rangle \to R_i$ given by sending x_j to x_j as a formal framing, we obtain the relative

Breuil–Kisin ring $\mathfrak{S}_{R_i}$ which we denote by $\mathfrak{S}_i$. Then we have a morphism $(\mathfrak{S}_R, (E)) \to (\mathfrak{S}_i, (E_\varpi))$ sending x_i to $\frac{u}{\epsilon_i}$ and x_j to x_j for $j \neq i$.

We let $\mathcal{O}_\mathcal{E}$ (resp. $\mathcal{O}_{\mathcal{E},i}$) be the p-adic completion of $\mathfrak{S}_R[1/E]$ (resp. $\mathfrak{S}_i[1/E_\varpi]$). We use the following lemma to reduce Proposition 5.24 to the case of R_i.

Lemma 5.26. *Let $\mathfrak{S}_{(i)}$ denote the intersection $\mathfrak{S}_i \cap \mathcal{O}_\mathcal{E}$ in the ring $\mathcal{O}_{\mathcal{E},i}$. Then the inclusion $\mathfrak{S}_R \subseteq \bigcap_{i=1}^d \mathfrak{S}_{(i)}$ in $\mathcal{O}_\mathcal{E}$ is an equality.*

Proof. We put $\mathfrak{S}' := \bigcap_{i=1}^d \mathfrak{S}_{(i)}$. Since $\mathfrak{S}_R$ and $\mathfrak{S}'$ are p-adically complete, it suffices to show that the containment $\mathfrak{S}_R \subseteq \mathfrak{S}'$ is an equality modulo p. We consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{S}_R/(p) = k[\![x_1, \ldots, x_d]\!] \arrow[r] \arrow[d] & (k[\![x_j; j \neq i]\!][1/x_j; j \neq i])\,[\![u]\!] = \mathfrak{S}_i/(p) \arrow[d] \\
\mathcal{O}_\mathcal{E}/(p) = k[\![x_1, \ldots, x_d]\!][1/x_1 \cdots x_m] \arrow[r] & (k[\![x_j; j \neq i]\!][1/x_j; j \neq i])\,[\![u]\!][1/u] = \mathcal{O}_{\mathcal{E},i}/(p),
\end{tikzcd}$$

in which all the maps are injective. Setting $\overline{\mathfrak{S}}_{(i)}$ to be the intersection of $\mathfrak{S}_i/(p)$ and $\mathcal{O}_\mathcal{E}/(p)$ in $\mathcal{O}_{\mathcal{E},i}/(p)$, we have $\mathfrak{S}_R/(p) = \bigcap_{i=1}^d \overline{\mathfrak{S}}_{(i)}$. We claim that the induced surjection $\mathfrak{S}'/(p) \to \bigcap_{i=1}^d \overline{\mathfrak{S}}_{(i)}$ is an isomorphism. This is equivalent to the equality

$$p\mathcal{O}_\mathcal{E} \cap \bigcap_i \mathfrak{S}_{(i)} = p \cdot \bigcap_i \mathfrak{S}_{(i)}.$$

We note that the right hand side is equal to $\bigcap_i (p\mathfrak{S}_{(i)})$ as p is a nonzerodivisor in $\mathcal{O}_\mathcal{E}$. Hence, it suffices to show the equality $p\mathcal{O}_\mathcal{E} \cap \mathfrak{S}_{(i)} = p\mathfrak{S}_{(i)}$ for all i. But this follows from the injectivity of the right vertical map in the above diagram. $\square$

Proof of Proposition 5.24. By the proof of Lemma 5.25, and the fact that restriction to a cover is faithful, the faithfulness portion of the claim is reduced to the case of a perfectoid base, which is clear. Thus, it suffices to check fullness. Let $\mathcal{F}$ and $\mathcal{F}'$ be two objects of $\mathbf{Vect}^\varphi(R_{\Delta})$ and let $T_{\text{ét}}(\mathcal{F}) \to T_{\text{ét}}(\mathcal{F})$ be a morphism in $\mathbf{Loc}_{\mathbb{Z}_p}(R[1/p])$, which, by [BS23, Corollary 3.7], corresponds to a morphism $\mathcal{F}[1/\mathcal{I}_{\Delta}]^\wedge_p \to \mathcal{F}'[1/\mathcal{I}_{\Delta}]^\wedge_p$ of prismatic Laurent F-crystals on R_{Δ} (cf. subsection 2.2). Let $\mathfrak{M}$ and $\mathfrak{M}'$ (resp. $\mathcal{M}$ and $\mathcal{M}'$) denote the evaluation of $\mathcal{F}$ and $\mathcal{F}'$ (resp. $\mathcal{F}[1/\mathcal{I}_{\Delta}]^\wedge_p$ and $\mathcal{F}[1/\mathcal{I}_{\Delta}]^\wedge_p$) at the prism $(\mathfrak{S}_R, (E))$.

Since the étale realization functor

$$\mathbf{Vect}^\varphi(R_{i,\Delta}) \to \mathbf{Vect}(R_{i,\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^\wedge_p) \xrightarrow{\sim} \mathbf{Loc}_{\mathbb{Z}_p}(R_i[1/p])$$

is fully faithful by [DLMS22, Theorem 3.28 (1)], the restriction of $\mathcal{F}[1/\mathcal{I}_{\Delta}]^\wedge_p \to \mathcal{F}'[1/\mathcal{I}_{\Delta}]^\wedge_p$ to $R_{i,\Delta}$ induces a morphism $\mathcal{F}|_{R_{i,\Delta}} \to \mathcal{F}'|_{R_{i,\Delta}}$. In particular, the map

$$\mathcal{M}_i := \mathcal{M} \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{E},i} \to \mathcal{M}'_i := \mathcal{M}' \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{E},i}$$

sends $\mathfrak{M}_i := \mathfrak{M} \otimes_\mathfrak{S} \mathfrak{S}_i$ into $\mathfrak{M}'_i := \mathfrak{M}' \otimes_\mathfrak{S} \mathfrak{S}_i$. Then, by Lemma 5.26, we get that the map $\mathcal{M} \to \mathcal{M}'$ sends $\mathfrak{M}$ into $\mathfrak{M}'$.

Since the prism $(\mathfrak{S}_R, (E))$ is a cover of the final object by Lemma 5.25, it suffices to show that the map $\mathfrak{M} \to \mathfrak{M}'$ is compatible with the descent data for $\mathcal{F}$ and $\mathcal{F}'$. To see this, we observe that the natural map $\mathfrak{S}_R^{(2)} \to \mathfrak{S}_R^{(2)}[1/E]^\wedge_p$ is injective. Indeed, it can be checked after passing modulo p, where it is reduced to showing that E is a nonzerodivisor in $\mathfrak{S}_R^{(2)}/(p)$, which follows from the flatness of $\mathfrak{S}_R \to \mathfrak{S}_R^{(2)}$, cf. [DLMS22, Lemma 3.5]). Thus, the assertion follows from the compatibility of the map $\mathcal{M} \to \mathcal{M}'$ with the descent data for $\mathcal{F}[1/\mathcal{I}_{\Delta}]^\wedge_p$ and $\mathcal{F}'[1/\mathcal{I}_{\Delta}]^\wedge_p$. $\square$

Given this we are now ready to prove our uniqueness claim concerning integral canonical models of U_{K^p}. We roughly follow the strategy employed in [Pap23, Theorem 7.1.7], with some key differences (see Remark 5.32 below).

Theorem 5.27. *The unique integral canonical model of U_{K^p} is $\widehat{\mathscr{S}}_{\mathsf{K}^p}$.*

Proof. That $\widehat{\mathscr{S}_{\mathsf{K}^p}}$ is an integral canonical model of U_{K^p} follows from combining Theorem 4.12, Proposition 4.16, Proposition 5.19. Thus, it suffices to show that if $\mathfrak{X}_{\mathsf{K}^p}$ and $\mathfrak{X}'_{\mathsf{K}^p}$ are two integral canonical models of U_{K^p}, then they are isomorphic.

Denote by $\mathfrak{X}''_{\mathsf{K}^p}$ the normalization of $\mathfrak{X}_{\mathsf{K}^p} \times_{\mathrm{Spf}(\mathcal{O}_E)} \mathfrak{X}'_{\mathsf{K}^p}$ in U_{K^p}. More precisely, we set $\mathfrak{X}''_{\mathsf{K}^p}$ to be the relative formal spectrum $\underline{\mathrm{Spf}}(\mathcal{A}) \to \mathfrak{X}_{\mathsf{K}^p} \times_{\mathrm{Spf}(\mathcal{O}_E)} \mathfrak{X}'_{\mathsf{K}^p}$, where $\mathcal{A}$ is the integral closure of $\mathcal{O}_{\mathfrak{X}_{\mathsf{K}^p} \times_{\mathrm{Spf}(\mathcal{O}_E)} \mathfrak{X}'_{\mathsf{K}^p}}$ in $s_*(\mathcal{O}^+_{U_{\mathsf{K}^p}})$, where $s\colon (U_{\mathsf{K}^p}, \mathcal{O}^+_{\mathsf{U}_{\mathsf{K}^p}}) \to \mathfrak{X}_{\mathsf{K}^p} \times_{\mathrm{Spf}(\mathcal{O}_E)} \mathfrak{X}'_{\mathsf{K}^p}$ is the composition of the following map of locally ringed spaces

$$(U_{\mathsf{K}^p}, \mathcal{O}^+_{U_{\mathsf{K}^p}}) \xrightarrow{\Delta} (U_{\mathsf{K}^p} \times_{\mathrm{Spa}(E)} U_{\mathsf{K}^p}, \mathcal{O}^+_{U_{\mathsf{K}^p} \times_{\mathrm{Spa}(E)} U_{\mathsf{K}^p}}) \xrightarrow{\mathrm{sp}} \mathfrak{X}_{\mathsf{K}^p} \times_{\mathrm{Spf}(\mathcal{O}_E)} \mathfrak{X}'_{\mathsf{K}^p}.$$

As E is a discrete valuation field, and therefore the local rings of each formal scheme and rigid space are excellent (cf. [Con99, §1.1]), these normalizations are finite over their original base and so topologically of finite type and normal (cf. [SP, Tag 0AVK] and [SP, Tag 035L]). Let $\pi\colon \mathfrak{X}''_{\mathsf{K}^p} \to \mathfrak{X}_{\mathsf{K}^p}$ and $\pi'\colon \mathfrak{X}''_{\mathsf{K}^p} \to \mathfrak{X}'_{\mathsf{K}^p}$ be the natural projection maps. We show that π and π' are isomorphisms.

To prove this, fix a point x'' in $\mathfrak{X}''_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ and let x and x' be their images in $\mathfrak{X}_{\mathsf{K}^p}$ and $\mathfrak{X}'_{\mathsf{K}^p}$ respectively. Let $\boldsymbol{b}_{x,\mathrm{crys}}$ and $\boldsymbol{b}_{x',\mathrm{crys}}$ be as in the definition of an integral canonical model. Observe that $\boldsymbol{b}_{x,\mathrm{crys}}$ actually equals $\boldsymbol{b}_{x',\mathrm{crys}}$. Indeed, from the diagram of isomorphisms

$$\mathrm{Spec}(k(x)) \xleftarrow{\sim} \mathrm{Spec}(k(x'')) \xrightarrow{\sim} \mathrm{Spec}(k(x')),$$

and the identification of $T_{\text{ét}} \circ \zeta_{\mathsf{K}^p}$ and $T_{\text{ét}} \circ \zeta'_{\mathsf{K}^p}$ with $\omega_{\mathsf{K}^p,\mathrm{an}}$, we obtain an isomorphism

$$T_{\text{ét}} \circ (\zeta_{\mathsf{K}^p})_{x''} \cong T_{\text{ét}} \circ (\zeta'_{\mathsf{K}^p})_{x''},$$

from where the claim follows (for instance) by Theorem 2.20. Let us denote this common class of $\mathbf{b}_x$, $\mathbf{b}_{x'}$ by $\mathbf{b}_{x'',\mathrm{crys}}$, and choose an element $b_{x''}$ of $\mathcal{G}^c(\breve{\mathbb{Z}}_p)\mu_h^c(p)\mathcal{G}^c(\breve{\mathbb{Z}}_p)$ such that $\sigma(b_{x''})$ maps to $\mathbf{b}_{x'',\mathrm{crys}}$.

Let $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{O}''$ be the complete local rings of x, x', and x'' of their respective formal schemes. By excellence each of these complete local rings is normal and formally of finite type over $\mathcal{O}_E$ (see [SP, Tag 0C23]). Choose isomorphisms $\Theta_x^{\mathbb{A}}$ and $\Theta_{x'}^{\mathbb{A}}$ as in the definition of integral canonical model. We claim that the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{Spf}(\mathcal{O}'') \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\
\mathrm{Spf}(\mathcal{O}) \arrow[dr, "\Theta_x^{\mathbb{A}}"'] & & \mathrm{Spf}(\mathcal{O}') \arrow[dl, "\Theta_{x'}^{\mathbb{A}}"] \\
 & \mathrm{Spf}(R_{\mathcal{G}^c,\mu_h^c}). &
\end{tikzcd} \tag{5.4.1}$$

To prove this, we first make the following observation.

Claim: There exists an epimorphism of formal schemes of the form $\mathrm{Spf}(R) \to \mathrm{Spf}(\mathcal{O}'')$, where

$$R = \mathcal{O}_K[\![t_1, \ldots, t_n, x_1, \ldots, x_m]\!]/(x_1 \cdots x_m - \varpi),$$

with notation as in Proposition 5.24.

Proof. Let $\mathrm{Spf}(A)$ be an affine open neighborhood of x'' in $\mathfrak{X}''_{\mathsf{K}^p}$. As $\mathrm{Spf}(A)_\eta$ is an open subset of the smooth rigid space U_{K^p}, it is smooth, and so by Elkik's algebraization theorem (see [Elk73, Théorème 7]) there exists some finite type smooth morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(\mathcal{O}_E)$ such that A is isomorphic to the p-adic completion of B over $\mathcal{O}_E$. Let $f\colon Y \to \mathrm{Spec}(B)$ be a strictly semi-stable over $\mathcal{O}_K$ (for some finite extension K of E) alteration of $\mathrm{Spec}(B)$ as in [dJ96, Theorem 6.5], and choose any closed point y of of Y mapping to x''. Then $\widehat{\mathcal{O}}_{Y,y}$ is isomorphic to

$$\mathcal{O}_K[\![x_1, \ldots, x_d]\!]/(x_1 \cdots x_m - \varpi)$$

over $\mathcal{O}_K$ for some integers d and m with $1 \leq m \leq d$. We then claim that the induced map $f\colon \mathrm{Spf}(\widehat{\mathcal{O}}_{Y,y}) \to \mathrm{Spf}(\mathcal{O}'')$ is an epimorphism, from where the conclusion will follow. But, the map $f\colon \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(B_x)$ is dominant by assumption, and thus induces an injection $B_x \to \mathcal{O}_{Y,y}$.

Note that as both the source and target are regular local rings we may deduce from [GD71, I, Corollaire 3.9.8] that $\mathcal{O}'' \to \widehat{\mathcal{O}}_{Y,y}$ is an injection. Since $\widehat{\mathcal{O}}_{Y,y}$ and $\mathcal{O}''$ are complete local rings, the claim follows easily. $\square$

To prove that Equation (5.4.1) commutes, it thus suffices to show that the outer square of the following diagram commutes

$$\begin{tikzcd}
& \mathrm{Spf}(R) \arrow[d] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "f'"] & \\
& \mathrm{Spf}(\mathcal{O}'') \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\
\mathrm{Spf}(\mathcal{O}) \arrow[dr, "\Theta^{\Delta}_{x}"'] & & \mathrm{Spf}(\mathcal{O}') \arrow[dl, "\Theta^{\Delta}_{x'}"] \\
& \mathrm{Spf}(R_{\mathcal{G}^c,\mu_h^c}), &
\end{tikzcd}$$

where f and f' are defined to make the triangle diagrams they sit in commute. But, observe that as R is a complete regular local ring, it suffices by the universality condition of $\mathrm{Spf}(R_{\mathcal{G}^c,\mu_h^c})$, and our definition of integral canonical models, to show that $f^*(\zeta_{\mathsf{K}^p})$ is isomorphic to $(f')^*(\zeta'_{\mathsf{K}^p})$. But, by setup we know that

$$T_{\text{ét}} \circ f^*(\zeta_{\mathsf{K}^p}) \cong T_{\text{ét}} \circ (f')^*(\zeta'_{\mathsf{K}^p}),$$

and so the claim follows from Proposition 5.24.

Given the commutativity of (5.4.1), we can now argue as in [Pap23, Proposition 6.3.1 (b)] to show that $\pi\colon \mathrm{Spf}(\mathcal{O}'') \to \mathrm{Spf}(\mathcal{O})$ and $\pi'\colon \mathrm{Spf}(\mathcal{O}'') \to \mathrm{Spf}(\mathcal{O}')$ are isomorphisms. Indeed, first note that, the commutativity of (5.4.1) is equivalent to the natural map $\pi \otimes \pi'\colon \mathcal{O}\widehat{\otimes}_{\mathcal{O}_{\breve{E}}}\mathcal{O}' \to \mathcal{O}''$ factorizing through the map

$$\mathcal{O}\widehat{\otimes}_{\mathcal{O}_{\breve{E}}}\mathcal{O}' \xrightarrow{a\otimes \mathrm{id}_{\mathcal{O}'}} \mathcal{O}'\widehat{\otimes}_{\mathcal{O}'_{\breve{E}}}\mathcal{O}' \xrightarrow{\Delta} \mathcal{O}',$$

where a denotes the isomorphism $(\Theta^{\Delta}_{x'})^{-1} \circ \Theta^{\Delta}_{x}$. Let R denote the image of $\pi \otimes \pi'$. Then, we see that this factorization gives rise to a surjection $\mathcal{O}' \to R$.

We next claim that $\mathcal{O}''$ has the same Krull dimension as $\mathcal{O}'$. To see this, observe that $\dim(\mathcal{O}'') = \dim(\mathcal{O}_{\mathfrak{X}'',x''})$ and $\dim(\mathcal{O}') = \dim(\mathcal{O}_{\mathfrak{X}',x'})$. As these are closed points on integral formal schemes of finite type over $\mathcal{O}_K$, they have the same dimension as $\mathfrak{X}''$ and $\mathfrak{X}'$, respectively. But, $\dim(\mathfrak{X}'')$ and $\dim(\mathfrak{X}')$ each decrease by 1 when passing to the rigid generic fiber, but these rigid generic fibers are isomorphic.

On the other hand, the dimension of R is equal to the dimension of $\mathcal{O}''$, as $R \to \mathcal{O}''$ is an integral embedding (see [SP, Tag 00OK]). Thus, combining these two claims we deduce that the dimension of R and $\mathcal{O}'$ are the same. Thus, the surjection $\mathcal{O}' \to R$ must be an isomorphism, being a surjection of integral domains of the same finite dimension.

We then get a finite map $\mathcal{O}' \xrightarrow{\sim} R \to \mathcal{O}''$. We claim that this map is an isomorphism. This follows from taking $A = \mathcal{O}_{\mathfrak{X}',x'}$ and $B = \mathcal{O}_{\mathfrak{X}'',x''}$ in the following lemma.

Lemma 5.28. *Let $(A,\mathfrak{m})$ and $(B,\mathfrak{n})$ be normal local Noetherian rings flat over $\mathbb{Z}_{(p)}$. Suppose that $(A,\mathfrak{m}) \to (B,\mathfrak{n})$*

(1) $\widehat{A}_{\mathfrak{m}} \to \widehat{B}_{\mathfrak{n}}$ is finite,

(2) $A[1/p] \to B[1/p]$ is an isomorphism.

Then, $\widehat{A}_{\mathfrak{m}} \to \widehat{B}_{\mathfrak{n}}$ is an isomorphism.

Proof. As $\widehat{A}_{\mathfrak{m}} \to \widehat{B}_{\mathfrak{n}}$ is a finite map between normal domains, it suffices to show that the map $\widehat{A}_{\mathfrak{m}}[1/p] \to \widehat{B}_{\mathfrak{n}}[1/p]$ is an isomorphism. Let us begin by observing that the map $A \to B$ is automatically injective as the source and target are both $\mathbb{Z}_{(p)}$-flat and the map $A[1/p] \to B[1/p]$ is injective. As $(A,\mathfrak{m})$ is a normal domain, we deduce from [GD71, I, Corollaire 3.9.8] that

$\widehat{A}_{\mathfrak{m}} \to \widehat{B}_{\mathfrak{n}}$ is injective, and thus that $\widehat{A}_{\mathfrak{m}}[1/p] \to \widehat{B}_{\mathfrak{n}}[1/p]$ is injective. Thus, it suffices to show that $\widehat{A}_{\mathfrak{m}}[1/p] \to \widehat{B}_{\mathfrak{n}}[1/p]$ is surjective. But, observe that as $A[1/p] \to B[1/p]$ is an isomorphism that the map $\widehat{A}_{\mathfrak{m}}[1/p] \to (\widehat{A}_{\mathfrak{m}} \otimes_A B)[1/p]$ is an isomorphism. As one has a factorization

$$\widehat{A}_{\mathfrak{m}}[1/p] \to (\widehat{A}_{\mathfrak{m}} \otimes_A B)[1/p] \to \widehat{B}_{\mathfrak{n}}[1/p],$$

with the second map being the obvious one, it suffices to show that the map $\widehat{A}_{\mathfrak{m}} \otimes_A B \to \widehat{B}_{\mathfrak{n}}$ is surjective. But, by Nakayama's lemma, using the fact that $\widehat{B}_{\mathfrak{n}}$ is a finite $\widehat{A}_{\mathfrak{m}}$-module, it suffices to show this surjectivity modulo $\mathfrak{m}$. But, as $\widehat{B}_{\mathfrak{n}}$ is a finite $\widehat{A}_{\mathfrak{m}}$-module, its topology agrees with the $\mathfrak{m}$-adic one. Thus, one has that $B/\mathfrak{m}B$ is naturally equal to $\widehat{B}_{\mathfrak{n}}/\mathfrak{m}\widehat{B}_{\mathfrak{n}}$, and thus the surjectivity of $\widehat{A}_{\mathfrak{m}} \otimes_A B \to \widehat{B}_{\mathfrak{n}}$ modulo $\mathfrak{m}$ is clear. ☐

From the above we deduce that the map $\pi' \colon \mathrm{Spf}(\mathcal{O}'') \to \mathrm{Spf}(\mathcal{O}')$ is an isomorphism and, by symmetry, the same holds for π. We are then done by Lemma 5.29 below. ☐

Lemma 5.29. *Let $\alpha \colon \mathfrak{Y}_1 \to \mathfrak{Y}_2$ be a morphism of finite type flat formal $\mathcal{O}_E$-schemes such that*

(a) α_η is an isomorphism of rigid E-spaces,

(b) for every point y_1 of $\mathfrak{Y}_1(\overline{\mathbb{F}}_p)$ with $y_2 = \alpha(y_1)$ the induced map $\widehat{\mathcal{O}}_{\mathfrak{Y}_2,y} \to \widehat{\mathcal{O}}_{\mathfrak{Y}_1,y_1}$ is an isomorphism.

Then, α is an isomorphism.

Proof. For each $n \geqslant 0$ let $\alpha_n \colon \mathfrak{Y}_{1,n} \to \mathfrak{Y}_{2,n}$ denote the reduction of α modulo p^{n+1}. It suffices to show that α_n is an isomorphism for all n. Indeed, we first observe that as the flat locus of each α_n is open, and contains every closed point of the scheme $\mathfrak{Y}_{1,n}$, which are evidently dense by consideration of the variety $\mathfrak{Y}_{1,0}$, we deduce that α_n is flat. Thus, to prove that it's étale, it suffices to prove this claim for α_0 (see [SP, Tag 06AG]). But, in this case the fact that (b) implies α is étale is classical. To prove that α_0, and thus each α_n (see loc. cit.), is an isomorphism it suffices to show that the fiber over each $\overline{\mathbb{F}}_p$-point y_2 of $\mathfrak{Y}_2(\overline{\mathbb{F}}_p)$ is a singleton (see [SP, Tag 02LC]). But, as $\mathfrak{Y}_2$ is a finite type and flat over $\mathcal{O}_E$, the specialization map $\mathrm{sp} \colon |(\mathfrak{Y}_2)_\eta|^{\mathrm{cl}} \to \mathfrak{Y}_{2,0}(\overline{\mathbb{F}}_p)$ is surjective (see [Bos14, §8.3, Proposition 8]). Thus, there exists some finite extension E' of E and a morphism $\mathrm{Spf}(\mathcal{O}_{E'}) \to \mathfrak{Y}_2$ whose special fiber is the underlying point of y_2. As α is étale, we know that $\alpha^{-1}(y_2)$ is a disjoint union of copies of $\mathrm{Spec}(\overline{\mathbb{F}}_p)$, and so by the topological invariance of the étale site, this implies that $\mathfrak{Y}_1 \times_{\mathfrak{Y}_2} \mathrm{Spf}(\mathcal{O}_{E'})$ is a disjoint union of copies of $\mathrm{Spf}(\mathcal{O}_{E'})$. As α_η is an isomorphism though, this number of copies must be one. The claim follows. ☐

As an immediately corollary of Proposition 5.2 and Theorem 5.27 we obtain a scheme-theoretic version of this result.

Definition 5.30. A smooth and separated $\mathcal{O}_E$-model $\mathscr{X}_{\mathsf{K}^p}$ of $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ is called a *prismatic integral canonical model* if $\widehat{\mathscr{X}_{\mathsf{K}^p}}$ is an integral canonical model of U_{K^p}. Equivalently, if

(1) $(\widehat{\mathscr{X}_{\mathsf{K}^p}})_\eta$ is the potentially crystalline locus of $\omega_{\mathsf{K}^p,\text{ét}}$,

(2) there exists a prismatic model ζ_{K^p} of $\omega_{\mathsf{K}^p,\text{ét}}$ (see Definition 5.12) such that for every point x of $\mathscr{X}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)$ there exists an isomorphism

$$\Theta_x^{\Delta} \colon R_{\mathcal{G}^c,\mu_h^c} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathscr{X}_{\mathsf{K}^p},x}$$

such that $(\Theta_x^{\Delta})^*(\omega_{b_x}^{\mathrm{univ}})$ is isomorphic to the pullback of ζ_{K^p} to $\widehat{\mathcal{O}}_{\mathscr{X}_{\mathsf{K}^p},x}$.

Corollary 5.31. *The unique prismatic integral canonical model of* $\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}$ *is* $\mathscr{S}_{\mathsf{K}^p}$.

Remark 5.32.

(1) Our proof of Theorem 5.27 follows the general strategy of [Pap23, Theorem 7.1.7], with three major deviations.

(a) We cannot use Tate's theorem about lifting generic isomorphisms of p-divisible groups on normal schemes. This is because despite the results of [ALB23], we do not know that $\mathfrak{X}''_{\mathsf{K}^p}$ is a priori quasi-syntomic, and thus we cannot work with p-divisible groups in our set-up involving prismatic objects. This does have the advantage of not requiring us to reduce to the p-divisible groups situation thus making the passage to the abelian type case.

(b) Our usage of formal schemes allows one to avoid showing that the morphisms π and π' (in Theorem 5.27) are proper, a pivotal part of the proof of [Pap23, Theorem 7.1.7]. This is the only place in loc. cit. where the entire system (opposed to working with a single level) is used, and is ultimately why we are able to work at the level of individual levels. In some sense, this properness-like property is baked into our formulation using formal schemes and rigid geometry.

(c) The commutative algebra portion of our proof is rather simpler given our set-up (e.g., doesn't require thinking about things closed embedded in some ambient deformation ring).

(2) The analogous result in the work of Pappas–Rapoport characterizing (the system of) integral canonical models, namely [PR22, Theorem 4.2.4], also imitates [Pap23, Theorem 7.1.7]. But, Pappas–Rapoport have an extra intervening complication in that their usage of moduli spaces of shtukas forces them to consider the technical issue of producing a framing (see [PR22, §3.1.1]). This issue does not occur in our formulation or proof essentially due to its absence in the universal deformation space of Ito.

5.4.2. *Relationship to work of Pappas and Rapoport.* We now discuss the relationship between our work and that in [PR22]. Below we shall refer to the conjunction of [PR22, Conjecture 4.2.2] and [Dan22, Conjecture 4.4] as the *Pappas–Rapoport conjecture.*

We begin by formulating a version of a prismatic integral canonical model as in Definition 5.30, but for the entirety system $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$. Namely, by a *smooth* $\mathbf{G}(\mathbb{A}_f^p)$*-model* of $\{\mathrm{Sh}_{\mathsf{K}^p\mathsf{K}_0}\}$, we mean a system $\{\mathscr{X}_{\mathsf{K}^p}\}$ of separated smooth $\mathcal{O}_E$-schemes together with finite étale morphisms $f_{\mathsf{K}^p,\mathsf{K}^{p\prime}}(g^p)$ modeling $t_{\mathsf{K}^p,\mathsf{K}^{p\prime}}(g^p)$.

Definition 5.33. A *prismatic integral canonical model* of $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$ is a smooth $\mathbf{G}(\mathbb{A}_f^p)$-model $\{\mathscr{X}_{\mathsf{K}^p}\}$ such that $\mathscr{X}_{\mathsf{K}^p}$ is a prismatic integral canonical model of $\mathscr{S}_{\mathsf{K}^p}$ for all K^p.

We have the following which is an essentially trivial corollary of Theorem 5.27.

Theorem 5.34. *The system* $\{\mathscr{S}_{\mathsf{K}^p}\}$ *is the unique prismatic integral canonical model of* $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$.

Proof. By Theorem 5.27 it suffices to observe the following. Let $\{\mathscr{X}_{\mathsf{K}^p}\}$ be a prismatic integral canonical model of $\{\mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}\}$. As each $\mathscr{X}_{\mathsf{K}^p}$ is integral and separated, there is at most one finite étale morphism $f_{\mathsf{K}^p,\mathsf{K}^{p\prime}}(g^p)$ modeling $t_{\mathsf{K}^p,\mathsf{K}^{p\prime}}(g^p)$, by a density argument. $\square$

That said, independent of Theorem 5.27 we can show a prismatic integral canonical model satisfies the conditions of the Pappas–Rapoport conjecture. This gives an alternative proof of Theorem 5.34, as well as verifying the cases of the Pappas–Rapoport conjecture at hyperspecial level not addressed in [PR22] and [Dan22] (i.e., abelian type but not of toral or Hodge type).

Proposition 5.35. *Suppose* $\{\mathscr{X}_{\mathsf{K}^p}\}$ *is a prismatic integral canonical model of* $\{\mathrm{Sh}_{\mathsf{K}^p\mathsf{K}_0}\}$*. Then,* $\{\mathscr{X}_{\mathsf{K}^p}\}$ *satisfies the conditions of the Pappas–Rapoport conjecture.*

Proof. That condition (a) of [Dan22, Conjecture 4.4] holds for $\{\mathscr{X}_{\mathsf{K}^p}\}$ follows by combining Proposition 5.2 and the Néron–Ogg–Shafarevich criterion (see Lemma 5.36 below). To show that condition (b) of loc. cit. holds, observe that $(\omega_{\mathsf{K}^p,\text{ét}},\zeta_{\mathsf{K}^p},\jmath_{\mathsf{K}^p})$, where $\jmath_{\mathsf{K}^p}\colon T_{\text{ét}}\circ\zeta_{\mathsf{K}^p}\xrightarrow{\sim}\omega_{\mathsf{K}^p,\mathrm{an}}$ is an isomorphism, defines a $\mathcal{G}^c(\mathbb{Z}_p)$-$\boldsymbol{\mu}_h^c$-local system with prismatic model on $\mathscr{X}_{\mathsf{K}^p}$ (in the sense of Definition 5.12), by Corollary 5.10 and Corollary 5.20. Set $(\mathscr{P}_{\mathsf{K}^p},\varphi_{\mathscr{P}_{\mathsf{K}^p}})$ to be the value $T_{\mathrm{sht}}(\omega_{\mathsf{K}^p,\text{ét}},\zeta_{\mathsf{K}^p},\jmath_{\mathsf{K}^p})$ under the shutka realization functor. By definition, we have that

$(\mathscr{P}_{\mathsf{K}^p})_E = U_{\mathrm{sht}}(\omega_{\mathsf{K}^p,\text{ét}}) \cong \mathscr{P}_{\mathsf{K}^p,E}$, and thus condition (b) is satisfied. Finally, to verify condition (c) of loc. cit., it suffices by setup to show that there exists an isomorphism

$$\Theta_x\colon \widehat{\mathcal{M}}^{\mathrm{int}}_{(\mathcal{G}^c,b_x,\mu_h^c)/x_0} \xrightarrow{\sim} \mathrm{Spd}(R_{\mathcal{G}^c,\mu_h^c}),$$

with the property that $T_{\mathrm{sht}}(\Theta_x^*(\omega_{b_x}^{\mathrm{univ}}))$ is isomorphic to the universal shutka. Here $\mathcal{M}^{\mathrm{int}}_{(\mathcal{G}^c,b_x,\mu_h^c)}$ is the integral moduli space of shtukas as in [SW20, Definition 25.1], and $\widehat{\mathcal{M}}^{\mathrm{int}}_{(\mathcal{G}^c,b_x,\mu_h^c)/x_0}$ is the completion at the neutral point x_0 in the sense of [Gle22]. But, the existence of such an isomorphism follows from [Ito23a, Theorem 5.3.5] and its proof. ☐

Lemma 5.36 (Néron–Ogg–Shafarevich criterion). *Let R be a discrete valuation ring over $\mathcal{O}_E$ of mixed characteristic $(0,p)$. For an element $\{x_{\mathsf{K}^p}\}$ of $\varprojlim_{\mathsf{K}^p} \mathrm{Sh}_{\mathsf{K}_0\mathsf{K}^p}(R[1/p])$, each induced morphism*

$$x^{\mathrm{an}}_{\mathsf{K}^p}\colon \mathrm{Spa}(R[1/p]) \to \mathrm{Sh}^{\mathrm{an}}_{\mathsf{K}_0\mathsf{K}^p}$$

factorizes through U_{K^p}.

Here we are viewing $R[1/p]$ as a (non-complete) Huber ring given the unique topological ring structure having R (endowed with the p-adic topology) as an open subring.

Proof of Lemma 5.36. As $\{\mathscr{S}_{\mathsf{K}^p}\}$ satisfies the extension property, we know that there exists a unique element $\{y_{\mathsf{K}^p}\}$ in $\varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}^p}(R)$ such that $(y_{\mathsf{K}^p})_\eta = x_{\mathsf{K}^p}$. Let $\widehat{y}_{\mathsf{K}^p}\colon \mathrm{Spf}(\widehat{R}) \to \mathscr{S}_{\mathsf{K}^p}$ denote the completion of y_{K^p}. Then, we observe that $x^{\mathrm{an}}_{\mathsf{K}^p} = (\widehat{y}_{\mathsf{K}^p})_\eta$. But, $(\widehat{y}_{\mathsf{K}^p})_\eta$ takes values in U_{K^p} by Proposition 4.16 ☐

Remark 5.37. While we used the system $\{\mathscr{S}_{\mathsf{K}^p}\}$ and its extension property in the proof of Lemma 5.36, we could eliminate this and prove the result directly, by using the method in the proof of Proposition 4.16 to reduce to the Siegel-type case, and thus to the classic Néron–Ogg–Shafarevich theorem.

Corollary 5.38. *The Pappas–Rapoport conjecture holds in all cases of hyperspecial level.*

Appendix A. Tannakian formalism of torsors

In this appendix we collect some results concerning torsors and the Tannakian formalism.

A.1. **Basic definitions and results.** A topos $\mathscr{T}$ is the category of sheaves on a site (as in [SP, Tag 03NH]), with the topology where $\{T_i \to T\}$ is a cover if it is a universal effective epimorphism (equiv. $\bigsqcup T_i \to T$ is a surjection of sheaves). Denote the final object of $\mathscr{T}$ by $*$.

Fix $\mathcal{G}$ to be a group object of $\mathscr{T}$. An object P of $\mathscr{T}$ equipped with a right action of $\mathcal{G}$ is a *pseudo-torsor* for $\mathcal{G}$ if the following morphism is an isomorphism

$$P \times \mathcal{G} \to P \times P, \qquad (p,g) \mapsto (p, p\cdot g),$$

or, equivalently, $\mathcal{G}(S)$ acts simply transitively on $P(S)$ if the latter is non-empty. A pseudo-torsor P is a *torsor* if $P \to *$ is an epimorphism or, equivalently, P is locally non-empty.[33] A morphism of pseudo-torsors for $\mathcal{G}$ is a $\mathcal{G}$-equivariant morphism in $\mathscr{T}$, which is automatically an isomorphism if the source is a torsor. A torsor P is trivial, if and only if $P(*) \neq \varnothing$.

Denote the category of pseudo-torsors for $\mathcal{G}$ on $\mathscr{T}$ by $\mathbf{PseuTors}_{\mathcal{G}}(\mathscr{T})$, by $\mathbf{Tors}_{\mathcal{G}}(\mathscr{T})$ the full subcategory of torsors, and by $H^1(\mathscr{T},\mathcal{G})$ the set of isomorphism classes in $\mathbf{Tors}_{\mathcal{G}}(\mathscr{T})$. For an object T of $\mathscr{T}$ with localized topos $\mathscr{T}/T$ (see [SP, Tag 04GY]), for $\mathcal{G}_T := \mathcal{G}|_{\mathscr{T}/T}$ we have

$$\mathbf{Tors}_{\mathcal{G}}(\mathscr{T}) \to \mathbf{Tors}_{\mathcal{G}_T}(\mathscr{T}/T), \qquad P \mapsto P_T := P \times T,$$

(we shorten the target to $\mathbf{Tors}_{\mathcal{G}}(\mathscr{T}/T)$). The association of $\mathbf{Tors}_{\mathcal{G}}(\mathscr{T}/T)$ to T is a stack on $\mathscr{T}$.

[33]An object $\mathcal{Q}$ of $\mathscr{T}$ is *locally non-empty* if there exists a cover $\{U_i \to *\}$ with $\mathcal{Q}(U_i)$ non-empty for all i.

For a $\mathcal{G}$-torsor P and an object Q of $\mathscr{T}$ with a left action of $\mathcal{G}$, we denote by $P \wedge^{\mathcal{G}} Q$ the *contracted product* obtained as the quotient of $P \times Q$ by the $\mathcal{G}$-action $g \cdot (p, q) := (pg^{-1}, gq)$. For a morphism $f\colon \mathcal{G} \to \mathcal{H}$ of group objects, $\mathcal{H}$ inherits a left $\mathcal{G}$-action and we have a functor

$$f_*\colon \mathbf{Tors}_{\mathcal{G}}(\mathscr{T}) \to \mathbf{Tors}_{\mathcal{H}}(\mathscr{T}), \qquad P \mapsto f_*(P) := P \wedge^{\mathcal{G}} \mathcal{H},$$

where $\mathcal{H}$ acts on $f_*(P)$ in the obvious way (see [Gir71, Chapitre III, Proposition 1.4.6]).

Let $\mathcal{C}$ be a site and set $\mathscr{C} := \mathbf{Shv}(\mathcal{C})$ to be its category of sheaves. For an object X of $\mathcal{C}$ denote by h_X the associated representable presheaf and by $h_X^\sharp$, or just X, its sheafification. We shall freely abuse the identification $\mathscr{T} \xrightarrow{\sim} \mathbf{Shv}(\mathscr{T})$ (cf. [SGA4-1, Exposé IV, Corollaire 1.2.1]).

Lemma A.1. *Let $\{X_i\}$ be a set of objects of $\mathcal{C}$ and $\mathcal{A}$ an object of $\mathscr{C}$. Then, a collection of elements f_i of $\mathcal{A}(X_i)$ corresponds to a cover $\{h_{X_i}^\sharp \xrightarrow{f_i} \mathcal{A}\}$ if and only if for every object X of $\mathcal{C}$ and element $f \in \mathcal{A}(X)$ there is a cover $\{U_j \xrightarrow{g_j} X\}$ so that for all j there is a morphism $k_j\colon U_j \to X_i$ with $f_i \circ k_j = f \circ g_j$.*

Lemma A.2. *Let X be an object of $\mathcal{C}$, and $\{\mathcal{A}_j \to h_X^\sharp\}$ a cover in $\mathscr{C}$. Then, there exists a cover $\{X_i \to X\}$ in $\mathcal{C}$ such that $\{h_{X_i}^\sharp \to h_X^\sharp\}$ refines $\{\mathcal{A}_j \to h_X^\sharp\}$.*

Set $\mathbf{PseuTors}_{\mathcal{G}}(\mathcal{C})$ to be $\mathbf{PseuTors}_{\mathcal{G}}(\mathscr{C})$, and define $\mathbf{Tors}_{\mathcal{G}}(\mathcal{C})$ and $H^1(\mathcal{C}, \mathcal{G})$ similarly. By Lemma A.1, an object $\mathcal{A}$ of $\mathscr{C}$ with right $\mathcal{G}$-action belongs to $\mathbf{PseuTors}_{\mathcal{G}}(\mathcal{C})$ if and only if $\mathcal{G}(X)$ acts simply transtively on $\mathcal{A}(X)$ whenever X is an object of $\mathcal{C}$ with $\mathcal{A}(X) \neq \varnothing$. By the following lemma an object $\mathcal{A}$ of $\mathbf{PseuTors}_{\mathcal{G}}(\mathcal{C})$ belongs to $\mathbf{Tors}_{\mathcal{G}}(\mathcal{C})$ if and only if for every object X of $\mathcal{C}$, there exists a cover $\{X_i \to X\}$ in $\mathcal{C}$ with $\mathcal{A}(X_i)$ non-empty.

Lemma A.3. *An object $\mathcal{A}$ of $\mathscr{C}$ is locally non-empty if and only if for all objects X of $\mathcal{C}$ there exists a cover $\{X_i \to X\}$ in $\mathcal{C}$ with $\mathcal{A}(X_i)$ non-empty for all i.*

By [Gir71, Chapitre III, 1.7.3.3], for any object X of $\mathcal{C}$ there is a natural identification between $\mathbf{Tors}_{\mathcal{G}}(\mathscr{C}/h_X^\sharp)$ and $\mathbf{Tors}_{\mathcal{G}}(\mathcal{C}/X)$, with $\mathcal{C}/X$ as in [SP, Tag 00XZ]. Thus, these objects are unambiguous in their definition, and so we use the latter notation in practice.

A.2. **Vector bundles and torsors.** Let $\mathcal{O}$ be a ring object of a topos $\mathscr{T}$. A *vector bundle* on $(\mathscr{T}, \mathcal{O})$ is an $\mathcal{O}$-module $\mathcal{E}$ for which there exists a cover $\{U_i \to *\}$ with $\mathcal{E}|_{U_i}$ isomorphic to $\mathcal{O}_{U_i}^{n_i}$ for some n_i.[34] Define $\mathbf{Vect}(\mathscr{T}, \mathcal{O})$ to be the category of vector bundles on $(\mathscr{T}, \mathcal{O})$, and $\mathbf{Vect}_n(\mathscr{T}, \mathcal{O})$ the full subcategory where $n_i = n$ for all i. Let $\mathbf{Vect}_n^{\mathrm{iso}}(\mathscr{T}, \mathcal{O})$ be the groupoid with the same objects as $\mathbf{Vect}_n(\mathscr{T}, \mathcal{O})$ but with only the isomorphisms as morphisms. If $\mathscr{C} = \mathbf{Sh}(\mathcal{C})$ we use the notation $\mathbf{Vect}(\mathcal{C}, \mathcal{O})$ and $\mathbf{Vect}_n(\mathcal{C}, \mathcal{O})$, instead.

Define $\mathrm{GL}_{n,\mathcal{O}}$ to be the group object of $\mathscr{T}$ given by $\mathrm{GL}_{n,\mathcal{O}}(T) := \mathrm{Aut}_{\mathcal{O}_T}(\mathcal{O}_T^n)$. Consider

$$\underline{\mathrm{Isom}}(\mathcal{O}^n, \mathcal{E})\colon \mathscr{T} \to \mathbf{Set}, \qquad T \mapsto \mathrm{Isom}(\mathcal{O}_T^n, \mathcal{E}_T),$$

for $\mathcal{E}$ an object of $\mathbf{Vect}_n(\mathscr{T}, \mathcal{O})$, which carries the natural structure of a $\mathrm{GL}_{n,\mathcal{O}}$-torsor. Conversely, for a $\mathrm{GL}_{n,\mathcal{O}}$-torsor P the contracted product $P \wedge^{\mathrm{GL}_{n,\mathcal{O}}} \mathcal{O}^n$, which inherits the structure of an $\mathcal{O}$-module from $\mathcal{O}^n$, is a vector bundle.

Proposition A.4 (cf. [Gir71, Chapitre III, Théorème 2.5.1])**.** *The functor*

$$\mathbf{Vect}_n^{\mathrm{iso}}(\mathscr{T}, \mathcal{O}) \to \mathbf{Tors}_{\mathrm{GL}_{n,\mathcal{O}}}(\mathscr{T}), \qquad \mathcal{E} \mapsto \underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}^n)$$

is an equivalence with quasi-inverse given by

$$\mathbf{Tors}_{\mathrm{GL}_{n,\mathcal{O}}}(\mathscr{T}) \to \mathbf{Vect}_n^{\mathrm{iso}}(\mathscr{T}, \mathcal{O}), \qquad P \mapsto P \wedge^{\mathrm{GL}_{n,\mathcal{O}}} \mathcal{O}^n.$$

[34]For all ringed sites we consider this agrees with the notion of vector bundle defined in [BS23, Notation 2.1].

A.3. **Torsors and morphism of topoi.** Let $\mathcal{C}$ (resp. $\mathcal{D}$) be a site and set $\mathscr{C}$ (resp. $\mathscr{D}$) to be the associated topos. Fix a morphism of topoi $(u_*, u^{-1})\colon \mathscr{C} \to \mathscr{D}$ (see [SGA4-1, Exposé IV, Definition 3.1] or [SP, Tag 00XA]) and a group object $\mathcal{G}$ of $\mathscr{C}$. Observe that as u_* is left exact it induces a morphism

$$u_*\colon \mathbf{PseuTors}_{\mathcal{G}}(\mathscr{C}) \to \mathbf{PseuTors}_{u_*(\mathcal{G})}(\mathscr{D}).$$

On the other hand, if $\mathcal{H}$ is a group object of $\mathscr{D}$ then we similarly obtain a functor

$$u^{-1}\colon \mathbf{PseuTors}_{\mathcal{H}}(\mathscr{D}) \to \mathbf{PseuTors}_{u^{-1}(\mathcal{H})}(\mathscr{C}).$$

When $\mathcal{H} = u_*(\mathcal{G})$, the counit map $\epsilon : u^{-1}(u_*(\mathcal{G})) \to \mathcal{G}$ gives us a functor

$$\epsilon_*\colon \mathbf{PseuTors}_{u^{-1}(u_*(\mathcal{G}))}(\mathscr{C}) \to \mathbf{PseuTors}_{\mathcal{G}}(\mathscr{C}).$$

By composing these two functors we obtain a functor

$$u^* := \epsilon_* \circ u^{-1}\colon \mathbf{PseuTors}_{u_*(\mathcal{G})}(\mathscr{D}) \to \mathbf{PseuTors}_{\mathcal{G}}(\mathscr{C}).$$

We then obtain an adjoint pair $(u_*, u^*)\colon \mathbf{PseuTors}_{\mathcal{G}}(\mathscr{C}) \to \mathbf{PseuTors}_{u_*(\mathcal{G})}(\mathscr{D})$.

The following result follows quickly by applying the adjointness of u^* and u_*.

Proposition A.5. *Suppose that $u^{-1}(\mathcal{B})$ is locally non-empty for all objects $\mathcal{B}$ of $\mathbf{Tors}_{u_*(\mathcal{G})}(\mathscr{D})$. Then, $(u_*, u^*)\colon \mathbf{Tors}_{\mathcal{G}}(\mathscr{C})' \to \mathbf{Tors}_{u_*(\mathcal{G})}(\mathscr{D})$ is a pair of quasi-inverses, where $\mathbf{Tors}_{\mathcal{G}}(\mathscr{C})'$ is the full subcategory of $\mathbf{Tors}_{\mathcal{G}}(\mathscr{C})$ consisting of those $\mathcal{A}$ such that $u_*(\mathcal{A})$ is locally non-empty.*

If $u\colon \mathcal{D} \to \mathcal{C}$ is a continuous functor (see [SGA4-1, Exposé III, Definition 1.1] or [SP, Tag 00WV]) then, by [SP, Tag 00WU] we get an adjoint pair $(u_*, u^{-1})\colon \mathscr{C} \to \mathscr{D}$ where $u_*(\mathcal{A})$ is $\mathcal{A} \circ u$, and $u^{-1}(\mathcal{B})$ is the sheafification of

$$(u^{-1})^{\mathrm{pre}}(\mathcal{B})(C) := \operatorname*{colim}_{(D,\psi)\in \mathcal{I}_C^{\mathrm{opp}}} \mathcal{B}(D)$$

with $\mathcal{I}_C^{\mathrm{opp}}$ the category of pairs (D, ψ) where D is an object of $\mathcal{D}$ and $\psi\colon C \to u(D)$. If u induces a morphism of sites (i.e., that u^{-1} is left exact), then (u_*, u^{-1}) is a morphism of topoi.

Corollary A.6 (cf. [Gir71, Chapitre V, Proposition 3.1.1])**.** *If $u\colon \mathcal{D} \to \mathcal{C}$ induces a morphism of sites, then we obtain a pair of quasi-inverse functors $(u_*, u^*)\colon \mathbf{Tors}_{\mathcal{G}}(\mathscr{C})' \to \mathbf{Tors}_{u_*(\mathcal{G})}(\mathscr{D})$.*

If $u\colon \mathcal{C} \to \mathcal{D}$ is a cocontinuous functor (see [SGA4-1, Exposé III, §2] or [SP, Tag 00XI]), then by [SP, Tag 00XN] u induces a morphism of topoi $(u_*, u^{-1})\colon \mathscr{C} \to \mathscr{D}$. Here, $u^{-1}(\mathcal{B})$ is the sheafification of $\mathcal{B} \circ u$, and

$$u_*(\mathcal{A})(D) = \lim_{(C,\psi)\in {}_D\mathcal{I}^{\mathrm{opp}}} \mathcal{A}(C)$$

where ${}_D\mathcal{I}^{\mathrm{opp}}$ is the the category of pairs (C, ψ) where C is an object of $\mathcal{C}$ and $\psi\colon u(C) \to D$. Combining Proposition A.5 and Lemma A.8 below we obtain the following corollary.

Corollary A.7. *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Then, for any group object $\mathcal{G}$ of $\mathscr{C}$ we obtain a pair of quasi-inverse functors $(u_*, u^*)\colon \mathbf{Tors}_{\mathcal{G}}(\mathscr{C})' \to \mathbf{Tors}_{u_*(\mathcal{G})}(\mathscr{D})$.*

Lemma A.8. *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Then, for any locally non-empty object $\mathcal{B}$ of $\mathscr{D}$, the object $u^{-1}(\mathcal{B})$ of $\mathscr{C}$ is locally non-empty.*

Proof. Take a cover $\{\mathcal{B}_i \to *\}$ in $\mathscr{D}$ with $\mathrm{Hom}(\mathcal{B}_i, \mathcal{B}) \neq \varnothing$ for all i, and an arbitrary cover $\{h^\sharp_{Z_\gamma} \to *\}$ in $\mathscr{C}$. By Lemma A.2, for each γ there exists a cover $\{D_{\beta\gamma} \to u(Z_\gamma)\}$ in $\mathcal{D}$ such that $\{h^\sharp_{D_{\beta\gamma}} \to h^\sharp_{u(Z_\gamma)}\}$ refines $\{\mathcal{B}_i \times h^\sharp_{u(Z_\gamma)} \to h^\sharp_{u(Z_\gamma)}\}$. By cocontinuity, for each γ there exists a cover $\{X_{\alpha\gamma} \to Z_\gamma\}$ in $\mathcal{C}$ such that $\{u(X_{\alpha\gamma}) \to u(Z_\gamma)\}$ refines $\{D_{\beta\gamma} \to u(Z_\gamma)\}$, and therefore $\mathcal{B}(u(X_{\alpha\gamma}))$ is non-empty, and there is a map $\mathcal{B}(u(X_{\alpha\gamma})) \to u^{-1}(\mathcal{B})(X_{\alpha\gamma})$, so $u^{-1}(\mathcal{B})(X_{\alpha\gamma})$ is non-empty. As $\{h^\sharp_{X_{\alpha\gamma}} \to *\}$ is a cover in $\mathscr{C}$, the claim follows. $\square$

Let $(u_*, u^{-1})\colon \mathscr{C} \to \mathscr{D}$ be a morphism of topoi defined by a (co)continuous functor u. For a ring object $\mathcal{O}$ of $\mathscr{C}$ one has an identification $u_*(\mathrm{GL}_{n,\mathcal{O}}) \xrightarrow{\sim} \mathrm{GL}_{n,u_*(\mathcal{O})}$. Define $\mathbf{Vect}(\mathscr{C},\mathcal{O})'$ to be the full subcategory of $\mathbf{Vect}(\mathscr{C},\mathcal{O})$ of vector bundles $\mathcal{E}$ with $u_*(\mathcal{E})$ a vector bundle over $u_*(\mathcal{O})$. Moreover, define the functor

$$u^*\colon \mathbf{Vect}(\mathscr{D}, u_*(\mathcal{O})) \to \mathbf{Vect}(\mathscr{C},\mathcal{O})', \quad u^*(\mathcal{W}) := u^{-1}(\mathcal{W}) \otimes_{u^{-1}(u_*(\mathcal{O}))} \mathcal{O}$$

which is compatible under Proposition A.4 with $u^*\colon \mathbf{Tors}_{u_*(\mathrm{GL}_{n,\mathcal{O}})}(\mathscr{D}) \to \mathbf{Tors}_{\mathrm{GL}_{n,\mathcal{O}}}(\mathscr{C})'$. Using similar ideas to above, one obtains the following.

Proposition A.9. *Suppose that $(u_*, u^{-1})\colon \mathscr{C} \to \mathscr{D}$ is a morphism of topoi induced by either continuous or cocontinuous morphism. Then, $u_*\colon \mathbf{Vect}(\mathcal{C},\mathcal{O})' \to \mathbf{Vect}(\mathcal{D}, u_*(\mathcal{O}))$ is a rank preserving $\otimes$-equivalence with quasi-inverse u^* (see §A.5 for this terminology).*

A.4. **Torsors and vector bundles on formal schemes.** Let $\mathfrak{X}$ be a formal scheme, and denote by $\mathfrak{X}_{\mathrm{fl}}$ the category consisting of morphisms of formal schemes $\mathfrak{Y} \to \mathfrak{X}$, morphisms being $\mathfrak{X}$-morphisms, and endowed with the Grothendieck topology where $\{\mathfrak{Y}_i \to \mathfrak{Y}\}$ is a cover if $\coprod_i \mathfrak{Y}_i \to \mathfrak{Y}$ is adically faithfully flat (see [FK18, Chapter I, Definition 4.8.12.(2)]) and quasi-compact. This site is subcanonical by [FK18, Chapter I, Proposition 6.1.5]. Denote by $\mathfrak{X}_{\mathrm{fl}}^{\mathrm{adic}}$, $\mathfrak{X}_{\text{ét}}$, and $\mathfrak{X}_{\mathrm{Zar}}$ the full subcategories of $\mathfrak{X}_{\mathrm{fl}}$ consisting of objects whose structure morphism is adic, étale, and an open embedding, respectively, with the induced topology. Denote by $\mathfrak{X}_{\mathrm{ZAR}}$, the full subcategory of $\mathfrak{X}_{\mathrm{fl}}$ whose covers are given by Zariski covers. When $\mathfrak{X}$ is a scheme, we use the notation $\mathfrak{X}_{\mathrm{fpqc}}$ for $\mathfrak{X}_{\mathrm{fl}}^{\mathrm{adic}}$. Each of these has a variant consisting only of affine (formal) schemes, but as these variants give rise to the same topos, we often confuse the two. Each of these sites is ringed via the usual structure sheaf (see [FK18, Chapter I, Proposition 6.1.2]).

Let $\mathcal{G}$ be a smooth affine group (formal) $\mathfrak{X}$-scheme. As affine adic morphisms satisfy effective descent in $\mathfrak{X}_{\mathrm{fl}}$ (see [FK18, Chapter I, Corollary 6.1.13]), and smoothness can be checked locally in $\mathfrak{X}_{\mathrm{fl}}$ (see [FK18, Chapter I, Proposition 6.1.8]) one may observe the following.

Lemma A.10. *A $\mathcal{G}$-torsor on $\mathfrak{X}_{\mathrm{fl}}$ is representable by a smooth and affine surjection $\mathfrak{P} \to \mathfrak{X}$.*

As smooth surjections have étale local sections, we obtain the following from Corollary A.6, and we denote the common category of $\mathcal{G}$-torsors on these three sites by $\mathbf{Tors}_{\mathcal{G}}(\mathfrak{X})$.

Corollary A.11. *The inclusions $\mathfrak{X}_{\text{ét}} \to \mathfrak{X}_{\mathrm{fl}}^{\mathrm{adic}} \to \mathfrak{X}_{\mathrm{fl}}$ give equivalences on categories of $\mathcal{G}$-torsors.*

Suppose that R is a ring which is J-adically complete with respect to a finitely generated ideal $J \subseteq R$. Consider the left exact functor

$$\widehat{(-)}\colon \mathbf{PSh}(\mathrm{Spec}(R)_{\mathrm{fpqc}}) \to \mathbf{PSh}(\mathrm{Spf}(R)_{\mathrm{fl}}^{\mathrm{adic}}), \qquad \mathcal{F} \mapsto \left(\widehat{\mathcal{F}}(\mathfrak{Y}) := \varprojlim_n \mathcal{F}(\mathfrak{Y}_n)\right),$$

where $\mathfrak{Y}_n := (|\mathfrak{Y}|, \mathcal{O}_{\mathfrak{Y}}/J^n\mathcal{O}_{\mathfrak{Y}})$. If $\mathcal{F}$ is a, then $\widehat{\mathcal{F}}$ is a sheaf by as the inverse limit functor is left exact. If $\mathcal{F} = h_P$ for a morphism $P \to \mathrm{Spec}(R)$, then $\widehat{\mathcal{F}}$ is represented by $\widehat{P} \to \mathrm{Spf}(R)$.

Let $\mathcal{G}$ be a smooth affine group R-scheme. Note that $\widehat{\mathcal{G}}(\mathrm{Spf}(S)) = \mathcal{G}(\mathrm{Spec}(S))$ when S is J-adically complete, and we denote this common group by $\mathcal{G}(S)$, and confuse $\mathcal{G}$ and $\widehat{\mathcal{G}}$. As $\widehat{(-)}$ commutes with products, it naturally sends pseudo-torsors for $\mathcal{G}$ to pseudo-torsors for $\widehat{\mathcal{G}}$. Due to the following, we can unambiguously denote $\mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(R))$ and $\mathbf{Tors}_{\mathcal{G}}(\mathrm{Spec}(R))$ by the common symbol $\mathbf{Tors}_{\mathcal{G}}(R)$.

Proposition A.12. *The functor $\widehat{(-)}$ functor induces an equivalence*

$$\widehat{(-)}\colon \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spec}(R)_{\mathrm{fpqc}}) \to \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(R)_{\mathrm{fl}}^{\mathrm{adic}}).$$

Proof. Let us first establishing bijectivity on the sets of isomorphism classes. To do this, first observe that by Corollary A.11, we are free replace $\mathrm{Spec}(R)_{\mathrm{fpqc}}$ (resp. $\mathrm{Spf}(R)_{\mathrm{fl}}^{\mathrm{adic}}$) with $\mathrm{Spec}(R)_{\mathrm{et}}$

(resp. $\mathrm{Spf}(R)_{\text{ét}}$). Observe then that we have a commutative diagram

$$\begin{tikzcd} H^1(\mathrm{Spec}(R)_{\text{ét}}, \mathcal{G}) \arrow[r, "\widehat{(-)}"] \arrow[d] & H^1(\mathrm{Spf}(R)_{\text{ét}}, \mathcal{G}) \arrow[dl, "\sim"] \\ \varprojlim_n H^1(\mathrm{Spec}(R/J^n)_{\text{ét}}, \mathcal{G}) & \end{tikzcd} , \tag{A.4.1}$$

As the completion of a $\mathcal{G}$-pseudo-torsor is a $\mathcal{G}$-pseudo-torsor, that the arrow labeled by $\widehat{(-)}$ is well-defined (i.e., sends a torsor to a torsor) follows from the observation that the completion of an étale cover of $\mathrm{Spec}(R)$ is an étale cover of $\mathrm{Spf}(R)$. The arrow labeled as an isomorphism is obtained from the equivalence of categories

$$\mathbf{Tors}_{\mathcal{G}}(\mathrm{Spf}(R)_{\text{ét}}) \xrightarrow{\sim} 2\text{-}\lim_n \mathbf{Tors}_{\mathcal{G}}(\mathrm{Spec}(R/J^n)),$$

given by sending $\mathcal{A}$ to $(\mathcal{A}_{\mathrm{Spec}(R/J^n)})$ with quasi-inverse taking $(\mathcal{A}_n)$ to the $\mathcal{G}$-torsor sending $\mathrm{Spf}(S)$ to $\varprojlim \mathcal{A}_n(\mathrm{Spec}(S/J^nS))$. That this quasi-inverse is well-defined (i.e., actually produces torsors) follows from the topological invariance of étale sites $\mathrm{Spf}(R)_{\text{ét}} \xrightarrow{\sim} \mathrm{Spec}(R/J)_{\text{ét}}$, and the smoothness of each $\mathcal{A}_n$, which shows that any étale cover of $\mathrm{Spec}(R/J)$ trivializing $\mathcal{A}_1$ lifts uniquely to an étale cover of $\mathrm{Spf}(R)$ trivializing $\varprojlim \mathcal{A}_n$. So, the claim follows as the vertical arrow in (A.4.1) is bijective by [BČ22, Theorem 2.1.6.(b)].

To show fully faithfulness we must show that for any two $\mathcal{G}$-torsors $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathrm{Spec}(R)_{\text{fpqc}}$ that the induced map $\mathrm{Hom}_{\mathcal{G}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathcal{G}}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2)$ is a bijection. As $\underline{\mathrm{Aut}}(\mathcal{F}_2)$ is locally isomorphic to $\mathcal{G}$, we deduce from effective descent for affine morphisms in $\mathrm{Spec}(R)_{\text{fpqc}}$ that $\underline{\mathrm{Aut}}(\mathcal{F}_2)$ is represented by some smooth affine group R-scheme H. Thus, $\underline{\mathrm{Aut}}(\widehat{\mathcal{F}}_2)$ is represented by $\widehat{H}$. Moreover, we may assume that $\mathcal{F}_1$ is isomorphic to $\mathcal{F}_2$, and thus by the bijectivity of isomorphism classes, that $\widehat{\mathcal{F}}_1$ is isomorphic to $\widehat{\mathcal{F}}_2$. So, $\mathrm{Hom}_{\mathcal{G}}(\mathcal{F}_1, \mathcal{F}_2)$ (resp. $\mathrm{Hom}_{\mathcal{G}}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2)$) is a torsor for $\mathrm{Aut}(\mathcal{F}_2) = H(R)$ (resp. $\mathrm{Aut}(\widehat{\mathcal{F}}_2) = \widehat{H}(R)$). The claim follows as $\mathrm{Hom}_{\mathcal{G}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathcal{G}}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2)$ is equivariant for the bijection $H(R) \to \widehat{H}(R)$. ☐

Let $\mathbf{FPMod}(R)$ denote the category of finite projective R-modules. The following is a vector bundle analogue of Proposition A.12.

Proposition A.13. *The global section functor* $\mathbf{Vect}(\mathrm{Spf}(R)_{\text{fl}}, \mathcal{O}_{\mathrm{Spf}(R)}) \to \mathbf{FPMod}(R)$ *is a bi-exact R-linear $\otimes$-equivalence (see §A.5 for this terminology) which preserves rank.*

Proof. We claim the source is equal to $\mathbf{Vect}(\mathrm{Spf}(R)_{\text{Zar}}, \mathcal{O}_{\mathrm{Spf}(R)})$. By Proposition A.9 it suffices to show that for an object $\mathcal{E}$ of $\mathbf{Vect}_n(\mathrm{Spf}(R), \mathcal{O}_{\mathrm{Spf}(R)})$ that $P = \underline{\mathrm{Isom}}(\mathcal{O}^n_{\mathrm{Spf}(R)}, \mathcal{E})$ has a section Zariski locally on $\mathrm{Spf}(R)$. Up to replacing R by a completed localization, we may assume by [SP, Tag 05VG] that $P(R/JR)$ is non-empty. But, as P is represented by a smooth formal R-scheme by Lemma A.10 we deduce from Hensel's lemma that $P(R)$ is non-empty. Then, by [FK18, Chapter I, Theorem 3.2.8], and the fact that any finite projective R-module M is automatically J-adically complete,[35] it suffices to show that for an adically quasi-coherent sheaf $\mathcal{E}$ on $\mathrm{Spf}(R)$, that $M = \mathcal{E}(\mathrm{Spf}(R))$ is finite projective if and only if $\mathcal{E}$ is a vector bundle, and the only if direction is clear. So, suppose that $\mathcal{E}$ is a vector bundle. Then, by [FK18, Chapter I, Theorem 3.2.8] M is a finitely generated J-adically complete R-module. Moreover, as $\mathcal{E}|_{\mathrm{Spec}(R/J^m)}$ is a vector bundle for all m, we know from [SP, Tag 05JM] that M/J^mM is finite projective for all m. Then, M is finite projective by [SP, Tag 0D4B]. ☐

Because of Proposition A.13 and its proof, the category of vector bundles on a formal scheme $\mathfrak{X}$ is independent of the above-defined sites. We denote the common category by $\mathbf{Vect}(\mathfrak{X})$ (omitting

[35]Find an R-module N such that $M \oplus N \cong R^m$ for some m. As R^m is J-adically complete the morphism $M \oplus N \to \widehat{M \oplus N} = \widehat{M} \oplus \widehat{N}$ is an isomorphism, from where it follows that $M \to \widehat{M}$ is an isomorphism.

the structure sheaf from the notation). If $\mathfrak{X} = \mathrm{Spf}(R)$, we shorten this notation further to $\mathbf{Vect}(R)$, and abusively identity it with $\mathbf{FPMod}(R)$.

A.5. **Tannakian formalism.** For a ring R,[36] we say $\mathcal{C}$ is an *(exact) R-linear $\otimes$-category* if

(• it is an exact category (see [Kel90, Appendix A]),)
- $\mathcal{C}$ is an additive R-linear category (see [SP, Tag 0104] and [SP, Tag 09MI]),
- the underlying category is Karoubian (see [SP, Tag 09SF])),
- there is an R-bilinear symmetric monoidal structure $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ (see [SP, Tag 0FFJ]).

For the unit object $\mathbf{1}$ of $\mathcal{C}$ and an object X of $\mathcal{C}$, an *element* of X means a morphism $\mathbf{1}\to X$.

For (exact) R-linear $\otimes$-categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F\colon \mathcal{C}\to\mathcal{D}$ is an *(exact) R-linear $\otimes$-functor* if it (preserves exact sequences and it) is R-linear (see [SP, Tag 09MK]) and symmetric monoidal (see [SP, Tag 0FFL] and [SP, Tag 0FFY]). From [SR72, I. Proposition 4.4.2], a quasi-inverse of an R-linear $\otimes$-functor F is automatically an R-linear $\otimes$-functor, in which case we call F an *R-linear $\otimes$-equivalence*. If F is exact then it is not guaranteed that the same holds for its quasi-inverse.[37] If an exact R-linear $\otimes$-functor has an exact R-linear $\otimes$-functor quasi-inverse, we say that F is a *bi-exact R-linear $\otimes$-equivalence*.

Let $\mathcal{C}$ be an R-linear $\otimes$-category and X a dualizable object of $\mathcal{C}$ (see [SP, Tag 0FFP]). As $\mathcal{C}$ is Karoubian, and by [SP, Tag 0FFU] and [SP, Tag 0FFT], we may construct in $\mathcal{C}$ an object obtained from X by taking any finite combination of direct sums, duals, symmetric products, and alternating products. By a *set of tensors* $\mathbb{T}$ on X we mean a finite set of elements in an object built in this way. We write this symbolically as $\mathbb{T}\subseteq X^{\otimes}$.[38] For an R-linear $\otimes$-functor $F\colon \mathcal{C}\to\mathcal{D}$ and a set of tensors $\mathbb{T}\subseteq X^{\otimes}$ we obtain the set $F(\mathbb{T})\subseteq F(X)^{\otimes}$ of tensors on $F(X)$.

We define a *tensor package* over R to be a pair $(\Lambda_0,\mathbb{T}_0)$ where Λ_0 is a finite projective R-module and $\mathbb{T}_0\subseteq\Lambda_0^{\otimes}$. Given a tensor package $(\Lambda_0,\mathbb{T}_0)$ we have the group R-scheme

$$\mathrm{Fix}(\mathbb{T}_0)\colon \mathbf{Alg}_R\to\mathbf{Set},\qquad S\mapsto\{g\in\mathrm{GL}(\Lambda_0\otimes_R S) : g(t)=t,\ \text{for all } t\in\mathbb{T}_0\},$$

a closed subgroup scheme of $\mathrm{GL}(\Lambda_0)$.

Theorem A.14 ([Bro13, Theorem 1.1])**.** *Suppose that R is a Dedekind domain. Then, for every flat finite type affine group R-scheme $\mathcal{G}$, and faithful representation $\mathcal{G}\to\mathrm{GL}(\Lambda_0)$, there exists a tensor package $(\Lambda_0,\mathbb{T}_0)$ with $\mathcal{G}=\mathrm{Fix}(\mathbb{T}_0)$.*

Remark A.15. Theorem A.14 was previously proven by Kisin in [Kis10, Proposition 1.3.2] in the case when R is a discrete valuation ring with reductive generic fiber. As pointed out by Deligne in [Del11] it is possible to, in the case considered by Kisin, only consider $\mathbb{T}_0$ contained in $\bigoplus_{m,n}\Lambda_0^{\otimes m}\otimes_R(\Lambda_0^\vee)^{\otimes n}$. This observation also applies to the situation of Theorem A.14.

Suppose now that $(\Lambda_0,\mathbb{T}_0)$ is a tensor package over R with $\mathcal{G}=\mathrm{Fix}(\mathbb{T}_0)$. Denote by $\mathbf{Rep}_R(\mathcal{G})$ the natural exact R-linear $\otimes$-category of representations $\mathcal{G}\to\mathrm{GL}(\Lambda)$, where Λ is a finite projective R-module. For any exact R-linear $\otimes$-category $\mathcal{C}$ a *$\mathcal{G}$-object* in $\mathcal{C}$ is an exact R-linear $\otimes$-functor $\omega\colon \mathbf{Rep}_R(\mathcal{G})\to\mathcal{C}$. An *isomorphism* $\omega\to\omega'$ is an invertible natural transformation, and we denote the groupoid of $\mathcal{G}$-objects in $\mathcal{C}$ by $\mathcal{G}$-$\mathcal{C}$.

Let $\mathscr{X}$ be a topos and $\mathcal{O}$ an R-algebra object of $\mathscr{X}$. Then, $\mathbf{Vect}(\mathscr{X},\mathcal{O})$ is an exact R-linear $\otimes$-category, with exactness inherited from the category of $\mathcal{O}$-modules. The following is a sheaf

$$\mathcal{G}_{\mathcal{O}}\colon \mathscr{X}\to\mathbf{Grp},\qquad T\mapsto\mathcal{G}(\mathcal{O}(T)),$$

as $\mathcal{G}$ preserves all limits of rings. Observe that $(\mathrm{GL}_{n,R})_{\mathcal{O}}=\mathrm{GL}_{n,\mathcal{O}}$. We write $\mathbf{PseuTors}_{\mathcal{G}}(\mathscr{X})$ (resp. $\mathbf{Tors}_{\mathcal{G}}(\mathscr{X})$) for $\mathbf{PseuTors}_{\mathcal{G}_{\mathcal{O}}}(\mathscr{X})$ (resp. $\mathbf{Tors}_{\mathcal{G}_{\mathcal{O}}}(\mathscr{X})$), when $\mathcal{O}$ is clear from context. For

[36]In this article R will almost always be $\mathbb{Z}_p$ or $\mathbb{Q}_p$, and which it is should always be clear from context.

[37]For example, the restriction functor $\mathbf{Rflx}(X)\to\mathbf{Rflx}(U)$ for a large open subset $U\subsetneq X$ when both are endowed with the exact structure inherited from the usual one on the category of coherent modules (see §A.6).

[38]We often implicitly interpret $X^{\otimes}$ as the direct sum of these finite constructions in a larger R-linear $\otimes$-category that is closed under arbitrary direct sums when such a larger category is naturally given.

any object T of $\mathscr{X}$ there is a restriction functor

$$\mathcal{G}\text{-}\mathbf{Vect}(\mathscr{X},\mathcal{O}) \to \mathcal{G}\text{-}\mathbf{Vect}(\mathscr{X}/T,\mathcal{O}), \qquad \omega \mapsto \omega_T := (-)|_T \circ \omega,$$

where $(-)|_T$ denotes restriction. Denote by ω_{triv} the $\mathcal{G}$-object given by $\omega_{\mathrm{triv}}(\Lambda) = \Lambda \otimes_R \mathcal{O}$.

For an object $\mathcal{P}$ of $\mathbf{Tors}_{\mathcal{G}}(\mathscr{X})$ we obtain the object $\omega_{\mathcal{P}}$ of $\mathcal{G}\text{-}\mathbf{Vect}(\mathscr{X},\mathcal{O})$ given by

$$\omega_{\mathcal{P}} \colon \mathbf{Rep}_R(\mathcal{G}) \to \mathbf{Vect}(\mathscr{X},\mathcal{O}), \qquad \Lambda \mapsto \mathcal{P} \wedge^{\mathcal{G}} (\Lambda \otimes_R \mathcal{O}),$$

which is locally on $\mathscr{X}$ isomorphic to ω_{triv}. Observe that for a representation $\rho\colon \mathcal{G} \to \mathrm{GL}(\Lambda)$, the vector bundle $\omega_{\mathcal{P}}(\Lambda)$ agrees, functorially in $\mathcal{P}$ and Λ, with the vector bundle associated to $\rho_*(\mathcal{P})$ by Proposition A.4, and so we sometimes confuse the two.

For a pair $(\mathcal{E},\mathbb{T})$, where $\mathcal{E}$ is an object of $\mathbf{Vect}(\mathscr{X},\mathcal{O})$ and $\mathbb{T} \subseteq \mathcal{E}^{\otimes}$, the functor

$$\begin{aligned}\underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_R \mathcal{O}, \mathbb{T}_0 \otimes 1), (\mathcal{E},\mathbb{T})\right) &: \mathscr{X} \to \mathbf{Set},\\ T &\mapsto \{f \colon \Lambda_0 \otimes_R \mathcal{O}_T \xrightarrow{\sim} \mathcal{E}_T : f(\mathbb{T}_0 \otimes 1) = \mathbb{T}\},\end{aligned}$$

has the structure of an object of $\mathbf{PseuTors}_{\mathcal{G}}(\mathscr{X})$. We call $(\mathcal{E},\mathbb{T})$ a *twist* of $(\Lambda_0,\mathbb{T}_0)$ if this pseudo-torsor is a torsor. By an *isomorphism* of twists $(\mathcal{E},\mathbb{T}) \to (\mathcal{E}',\mathbb{T}')$ we mean an isomorphism $\mathcal{E} \to \mathcal{E}'$ carrying $\mathbb{T}$ to $\mathbb{T}'$. Denote by $\mathbf{Twist}_{\mathcal{O}}(\Lambda_0,\mathbb{T}_0)$ the groupoid of twists of $(\Lambda_0,\mathbb{T}_0)$.

Proposition A.16. *The functor*

$$\mathbf{Twist}_{\mathcal{O}}(\Lambda_0,\mathbb{T}_0) \to \mathbf{Tors}_{\mathcal{G}_{\mathcal{O}}}(\mathscr{X}), \qquad (\mathcal{E},\mathbb{T}) \mapsto \underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_R \mathcal{O}, \mathbb{T}_0 \otimes 1), (\mathcal{E},\mathbb{T})\right),$$

is an equivalence of groupoids with quasi-inverse given by sending $\mathcal{P}$ to $(\omega_{\mathcal{P}}(\Lambda_0), \omega_{\mathcal{P}}(\mathbb{T}_0))$.

Proof. The association of T in $\mathscr{X}$ to the groupoid of pairs $(\mathcal{E},\mathbb{T})$ of an object $\mathcal{E}$ of $\mathbf{Vect}(\mathscr{X},\mathcal{O}/T)$ and $\mathbb{T} \subseteq \mathcal{E}^{\otimes}$, forms a stack over $\mathscr{X}$ which we denote C. This proposition is then a special case of [Gir71, Chapitre III, Théorème 2.5.1] as the natural map $\mathcal{G}_{\mathcal{O}} \to \underline{\mathrm{Aut}}(\Lambda_0 \otimes_R \mathcal{O}_X, \mathbb{T}_0 \otimes 1)$ is an isomorphism, and (with notation in loc. cit.) $C(\Lambda_0 \otimes_R \mathcal{O}, \mathbb{T}_0 \otimes 1) = \mathbf{Twist}_{\mathcal{O}}(\Lambda_0,\mathbb{T}_0)$. ☐

For an object ω of $\mathcal{G}\text{-}\mathbf{Vect}(\mathscr{X},\mathcal{O})$ we have a pseudo-torsor

$$\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}},\omega) \colon \mathscr{X} \to \mathbf{Set}, \qquad T \mapsto \mathrm{Isom}(\omega_{\mathrm{triv},T},\omega_T),$$

for the group sheaf $\underline{\mathrm{Aut}}(\omega_{\mathrm{triv}})$. Call ω *locally trivial* if this pseudo-torsor is a torsor, and by $\mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{lt}}(\mathscr{X},\mathcal{O})$ the full subgroupoid of $\mathcal{G}\text{-}\mathbf{Vect}(\mathscr{X},\mathcal{O})$ of locally trivial objects. Say that $\mathcal{G}$ is *reconstructable in* $(\mathscr{X},\mathcal{O})$ if the natural map $\mathcal{G}_{\mathcal{O}} \to \underline{\mathrm{Aut}}(\omega_{\mathrm{triv}})$ is an isomorphism. In this case, there a natural equivalence $\mathbf{Tors}_{\mathcal{G}}(\mathscr{X}) \to \mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{lt}}(\mathscr{X},\mathcal{O})$ given by sending $\mathcal{P}$ to $\omega_{\mathcal{P}}$, with quasi-inverse sending ω to $\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}},\omega)$.

If $\mathcal{G}$ is reconstructable in $(\mathscr{X},\mathcal{O})$, then for an object ω of $\mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{lt}}(\mathscr{X},\mathcal{O})$, the map

$$\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}},\omega) \to \underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_R \mathcal{O}, \mathbb{T}_0 \otimes 1), (\omega(\Lambda_0),\omega(\mathbb{T}_0))\right)$$

given by evaluation is a morphism of pseudo-torsors where the source is a torsor, and so an isomorphism. Thus, $(\omega(\Lambda_0),\omega(\mathbb{T}_0))$ is an object of $\mathbf{Twist}_{\mathcal{O}}(\Lambda_0,\mathbb{T}_0)$. We deduce the following.

Proposition A.17. *Suppose that $\mathcal{G}$ is reconstructable in $(\mathscr{X},\mathcal{O})$. Then,*

$$\begin{tikzcd}
\mathbf{Twist}_{\mathcal{O}}(\Lambda_0,\mathbb{T}_0) \arrow[rr, "(\mathcal{E},\mathbb{T})\mapsto\underline{\mathrm{Isom}}((\Lambda_0\otimes_R\mathcal{O},\mathbb{T}_0\otimes1),(\mathcal{E},\mathbb{T}))"] & & \mathbf{Tors}_{\mathcal{G}}(\mathscr{X}) \arrow[d, "\mathcal{P}\mapsto\omega_{\mathcal{P}}"] \\
& & \mathcal{G}\text{-}\mathbf{Vect}^{\mathrm{lt}}(\mathscr{X},\mathcal{O}) \arrow[ull, "\omega\mapsto(\omega(\Lambda_0),\omega(\mathbb{T}_0))"]
\end{tikzcd}$$

is a commuting triangle of equivalences.

The following shows that the assumptions in Proposition A.17 are often satisfied.

Theorem A.18 ([Bro13, Theorem 1.2], [SW20, Theorem 19.5.1])**.** *Assume R is a Dedekind domain and that $\mathcal{G}$ is R-flat. Then for an R-scheme X, $\mathcal{G}$ is reconstructable in $(X_{\mathrm{fpqc}},\mathcal{O}_X)$ and every object of $\mathcal{G}\text{-}\mathbf{Vect}(X)$ is locally trivial.*

Proof. The only thing not contained in loc. cit. is the reconstructability claim, but this follows from the fully faithfulness of the the functors in loc. cit. applied to the trivial objects. □

Remark A.19. There is an error in [Bro13, Lemma 4.4 (iii)], which is necessary for [Bro13, Theorem 1.2]. Namely, the inclusion $\mathcal{O}_X \to \mathcal{O}_G$[39] is not split as $\mathcal{O}_G$-comodules, and thus one cannot use additivity to conclude that $F(\mathcal{O}_X)$ is a summand of $F(\mathcal{O}_G)$. But, observe that

$$0 \to \mathcal{O}_X \to \mathcal{O}_G \to \mathcal{O}_G/\mathcal{O}_X \to 0,$$

is an exact sequence in $\mathbf{Rep}'(G)$, where $\mathcal{O}_G/\mathcal{O}_X$ is flat as $\mathcal{O}_X \to \mathcal{O}_G$ is split as $\mathcal{O}_X$-modules. By the exactness of F, and the flatness of $F(\mathcal{O}_G/\mathcal{O}_X)$, we obtain the (universally) exact sequence

$$0 \to F(\mathcal{O}_X) \to F(\mathcal{O}_G) \to F(\mathcal{O}_G/\mathcal{O}_X) \to 0.$$

As $F(\mathcal{O}_X) = \mathcal{O}_Y$ universally injects into $F(\mathcal{O}_G)$, we deduce that $F(\mathcal{O}_G)$ is faithfully flat.

Remark A.20. If $\mathcal{G}$ is R-smooth we may replace X_{fl} in Theorem A.18 with $X_{\text{ét}}$. When $X = \mathrm{Spec}(A)$, for A complete with respect to a finitely generated ideal, we may replace all instances of X in Theorem A.18 with $\mathrm{Spf}(A)$ by Proposition A.12 and Proposition A.13.

A.6. **Reflexive pseudo-torsors.** Let R be a ring and X an integral locally Noetherian normal R-scheme. An open embedding $j\colon U \hookrightarrow X$ is *large* if $j(U)$ contains all points of codimension 1. If $X' \to X$ is an étale map, then then X' is normal and $U \times_X X' \hookrightarrow X'$ is large. Denote by $\mathbf{A}(X_{\text{ét}})$ the full subcategory of $\mathbf{Shv}(X_{\text{ét}})$ of sheaves represented by an affine finite type X-scheme.

Proposition A.21 (cf. [CTS79, Lemme 2.1])**.** *Let $j\colon U \hookrightarrow X$ be a large open embedding. Then, $(j_*, j^*)\colon \mathbf{A}(U_{\text{ét}}) \to \mathbf{A}(X_{\text{ét}})$ is a pair of quasi-inverse functors.*

Proof. To see that j_* is well-defined, suppose that $\mathcal{F} = \underline{\mathrm{Spec}}(\mathcal{A})$, where $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_U$-algebra of finite type. As $\mathcal{O}_X \to j_*j^*\mathcal{O}_X = j_*\mathcal{O}_U$ is an isomorphism (see [CTS79, Lemme 2.1]), applying [SP, Tag 01LQ] shows that $j_*\mathcal{F}$ is represented by $\underline{\mathrm{Spec}}(j_*\mathcal{A})$. We claim that $j_*\mathcal{A}$ is a finitely generated algebra over $j_*\mathcal{O}_U = \mathcal{O}_X$. We may assume that X is affine and so U is quasi-affine, and thus $\mathcal{A} = \mathcal{O}_U[T_1, \ldots, T_n]/\mathcal{I}$ for some quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_U[T_1, \ldots, T_n]$ (c.f. [EGA2, Proposition 5.I.2] and [EGA1, Proposition 9.6.5]). Thus, $j_*\mathcal{A}$ is isomorphic to $\mathcal{O}_X[T_1, \ldots, T_n]/j_*\mathcal{I}$ and so finite type. As j_* and j^* are adjoint it suffices to show that the unit and counit are isomorphisms. The former is [CTS79, Lemme 2.1] and the latter is clear. □

Recall that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *reflexive* if the natural map

$$\mathcal{F} \to \mathcal{F}^{**} := \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism. Equivalently, $\mathcal{F}$ is reflexive if there exists a large open embedding $j\colon U \hookrightarrow X$ such that $j^*\mathcal{F}$ is a vector bundle and for which the unit map $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism (see [SP, Tag 0AY6]). Denote by $\mathbf{Rflx}(X)$ the category of reflexive $\mathcal{O}_X$-modules, which has the structure of an R-linear $\otimes$-category where the tensor product is as in [SP, Tag 0EBH]. We endow $\mathbf{Rflx}(X)$ with an exact structure by declaring an sequence exact if it is exact at every codimension 1 point. Thus, $\mathbf{Rflx}(X)$ contains $\mathbf{Vect}(X)$ as a full R-linear tensor subcategory but *not* as an exact subcategory. With this exact structure, if $j\colon U \hookrightarrow X$ is a large open embedding, then $j_*\colon \mathbf{Rflx}(U) \to \mathbf{Rflx}(X)$ is a bi-exact R-linear $\otimes$-equivalence (see [SP, Tag 0EBJ]). Let $\mathbf{Rflx}_n^{\mathrm{iso}}(X)$ be the category of reflexive $\mathcal{O}_X$-modules $\mathcal{F}$ for which there is a large open embedding $j\colon U \to X$ with $j^*\mathcal{F}$ a rank n vector bundle, with morphisms being isomorphisms of $\mathcal{O}_X$-modules.

For a smooth group X-scheme $\mathcal{G}$, a pseudo-torsor $\mathcal{Q}$ for $\mathcal{G}$ is called *reflexive* if there is a large open embedding $j\colon U \hookrightarrow X$ with $j^*\mathcal{Q}$ a $\mathcal{G}$-torsor and the unit map $\mathcal{Q} \to j_*j^*\mathcal{Q}$ an isomorphism. Denote by $\mathbf{Rflx}_{\mathcal{G}}(X)$ the full subcategory of $\mathbf{PseuTors}_{\mathcal{G}}(X_{\text{ét}})$ of reflexive pseudo-torsors.

Proposition A.22. *Suppose $\mathcal{G}$ is a smooth group X-scheme. Then, the following is true.*

(1) An object $\mathcal{Q}$ of $\mathbf{PseuTors}_{\mathcal{G}}(X_{\text{ét}})$ belongs to $\mathbf{Rflx}_{\mathcal{G}}(X)$ if and only if $\mathcal{Q}$ belongs to $\mathbf{A}(X_{\text{ét}})$, and $\mathcal{Q}_x$ belongs to $\mathbf{Tors}_{\mathcal{G}}(\mathcal{O}_{X,x})$ for all codimension 1 points x of X.

[39] In this remark we use notation as in loc. cit. In particular, our X (resp. $\mathrm{Spec}(R)$) is Y (resp. X) in loc. cit.

(2) For a large open embedding $j\colon U \hookrightarrow X$, the pair $(j_*, j^*)\colon \mathbf{Rflx}_{\mathcal{G}}(U) \to \mathbf{Rflx}_{\mathcal{G}}(X)$ are quasi-inverse.

(3) The natural functors $\mathcal{Q} \mapsto \mathcal{Q} \wedge^{\mathrm{GL}_{n,\mathcal{O}_X}} \mathcal{O}_X^n$ is an equivalence of categories

$$\mathbf{Rflx}_{\mathrm{GL}_{n,\mathcal{O}_X}}(X) \to \mathbf{Rflx}_n^{\mathrm{iso}}(X),$$

with quasi-inverse given by $\mathcal{E} \mapsto \underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}_X^n)$.

Proof. By Proposition A.21 and Proposition A.4, it only remains to show the if part of (1). Let $Y \to X$ be a finite type affine X-scheme representing $\mathcal{Q}$. By Proposition A.21, it suffices to show that $Y_U \to U$ is faithfully flat for some large open U. As $Y_{\mathcal{O}_{X,x}}$ is faithfully flat over $\mathcal{O}_{X,x}$ for all codimension 1 points x, we may conclude by [SP, Tag 04AI] and [SP, Tag 07RR]. $\square$

For any étale map $X' \to X$ there is a restriction functor

$$\mathcal{G}\text{-}\mathbf{Rflx}(X) \to \mathcal{G}\text{-}\mathbf{Rflx}(X'), \qquad \omega \mapsto \omega_{X'} := (-)|_{X'} \circ \omega,$$

where $(-)|_{X'}$ denotes restriction. There is a fully faithful embedding of $\mathcal{G}$-$\mathbf{Vect}(X)$ into $\mathcal{G}$-$\mathbf{Rflx}(X)$. For an object $\mathcal{Q}$ of $\mathbf{Rflx}_{\mathcal{G}}(\mathscr{X})$ we obtain the object $\omega_{\mathcal{Q}}$ of $\mathcal{G}$-$\mathbf{Rflx}(X)$ given by

$$\omega_{\mathcal{Q}}\colon \mathbf{Rep}_R(\mathcal{G}) \to \mathbf{Rflx}(X), \qquad \Lambda \mapsto \mathcal{Q} \wedge^{\mathcal{G}} (\Lambda \otimes_R \mathcal{O}_X).$$

For $\rho\colon \mathcal{G} \to \mathrm{GL}(\Lambda)$ one checks that $\omega_{\mathcal{Q}}(\Lambda)$ agrees, functorially in $\mathcal{Q}$ and Λ, with the reflexive module associated to $\rho_*(\mathcal{Q})$ by Proposition A.22. If $\mathcal{G}$ is reconstructable in $(X_{\text{ét}}, \mathcal{O}_X)$, an object ω of $\mathcal{G}$-$\mathbf{Rflx}(X)$ is called *locally trivial* if the pseudo-torsor $\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}}, \omega)$ is reflexive. Denote by $\mathcal{G}$-$\mathbf{Rflx}^{\mathrm{lt}}(X, \mathcal{O}_X)$ the full subcategory of locally trivial objects.

Suppose $(\Lambda_0, \mathbb{T}_0)$ is a tensor package over R and $\mathcal{G}$ is the base change to X of $\mathrm{Fix}(\mathbb{T}_0)$, which we abusively also denote $\mathcal{G}$. The category $\mathbf{Twist}_X^{\mathrm{rflx}}(\Lambda_0, \mathbb{T}_0)$ of *reflexive twists* consists of pairs $(\mathcal{E}, \mathbb{T})$ where $\mathcal{E}$ is an object of $\mathbf{Rflx}(X)$ and $\mathbb{T} \subseteq \mathcal{E}^{\otimes}$ such that $\underline{\mathrm{Isom}}\left((\Lambda_0 \otimes_R \mathcal{O}_X, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T})\right)$ is reflexive. If $\mathcal{Q}$ is a reflexive pseudo-torsor for $\mathcal{G}$, then $(\omega_{\mathcal{Q}}(\Lambda_0), \omega_{\mathcal{Q}}(\mathbb{T}_0))$ is reflexive.

Combining Proposition A.17 and Proposition A.22, one deduces the following.

Proposition A.23. *Suppose that $\mathcal{G}$ is reconstructable in $(X_{\text{ét}}, \mathcal{O}_X)$. Then,*

$$\begin{tikzcd}
\mathbf{Twist}_X^{\mathrm{rflx}}(\Lambda_0, \mathbb{T}_0) \arrow[rr, "(\mathcal{E},\mathbb{T})\mapsto\underline{\mathrm{Isom}}((\Lambda_0\otimes_R\mathcal{O},\mathbb{T}_0\otimes 1),(\mathcal{E},\mathbb{T}))"] & & \mathbf{Rflx}_{\mathcal{G}}(X) \arrow[d, "\mathcal{P}\mapsto\omega_{\mathcal{P}}"] \\
& & \mathcal{G}\text{-}\mathbf{Rflx}^{\mathrm{lt}}(X) \arrow[llu, "\omega\mapsto(\omega(\Lambda_0),\omega(\mathbb{T}_0))"]
\end{tikzcd}$$

is a commuting triangle of equivalences.

If R is a Dedekind domain, then we have an analogue of Theorem A.18.

Proposition A.24. *Assume that R is a Dedekind domain. Then, $\mathcal{G}$ is reconstructable in $(X_{\text{ét}}, \mathcal{O}_X)$ and every object of $\mathcal{G}$-$\mathbf{Rflx}(X)$ is locally trivial.*

Proof. Every representation Λ of $\mathcal{G}$ occurs as a subquotient of some $T^{m,n} := \Lambda_0^{\otimes m} \otimes_R (\Lambda_0^\vee)^{\otimes n}$ (cf. [dS09, Proposition 12]). Thus, if ω is an object of $\mathcal{G}$-$\mathbf{Rflx}(X)$ then the natural map $\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}}, \omega) \to \underline{\mathrm{Isom}}(\Lambda_0 \otimes_R \mathcal{O}_X, \omega(\Lambda_0))$ is an isomorphism onto the closed subscheme of those f such that for all m and n, and all subrepresentations $\Lambda \subseteq T^{m,n}$, the induced isomorphism $f^{m,n}\colon T^{m,n} \otimes_R \mathcal{O}_X \to \omega(T^{m,n})$ satisfies $f^{m,n}(\Lambda \otimes_R \mathcal{O}_X) \subseteq \Lambda$ and $(f^{m,n})^{-1}(\omega(\Lambda)) \subseteq \Lambda \otimes_R \mathcal{O}_X$. As $\underline{\mathrm{Isom}}(\Lambda_0 \otimes_R \mathcal{O}_X, \omega(\Lambda_0))$ is an affine finite type X-scheme, the same is true for $\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}}, \omega)$. By Proposition A.22, it then suffices to show that for all codimension 1 points x that $\underline{\mathrm{Isom}}(\omega_{\mathrm{triv}}, \omega)_x$ is a torsor. But, this corresponds to the exact R-linear $\otimes$-functor $\Lambda \mapsto \omega(\Lambda)_x$. As $\mathcal{O}_{X,x}$ is dimension 1 all reflexive modules are vector bundles, and thus this an object of $\mathcal{G}$-$\mathbf{Vect}(\mathcal{O}_{X,x})$. The claim then follows from Theorem A.18. $\square$

We end with some results inspired by [CTS79]. Suppose that $\rho\colon \mathcal{G} \hookrightarrow \mathcal{H}$ is a closed embedding of reductive group X-schemes. Denote by $p\colon \mathcal{H} \to \mathcal{H}/\mathcal{G}$ the quotient sheaf in the fppf topology. Combining [Alp14, Corollary 9.7.7] and [SP, Tag 02KK] shows that $\mathcal{H}/\mathcal{G}$ belongs to $\mathbf{A}(X_{\text{ét}})$.

Proposition A.25. *Let $\mathcal{Q}$ be an object of $\mathbf{Rflx}_{\mathcal{G}}(X)$. Then, for any large open embedding $j\colon U \hookrightarrow X$, the natural map $\rho_*\mathcal{Q} \to j_*\rho_*j^*\mathcal{Q}$ is an isomorphism.*

Proof. For $X' \to X$ étale and $U' := U \times_X X'$, we show that $\rho_*\mathcal{Q}(X') \to \rho_*j^*\mathcal{Q}(U')$ is bijective. By [Gir71, Chapitre III, Proposition 3.1.2], the source (resp. target) is identified with the set of $\mathcal{G}$-subtorsors $\mathcal{A}$ (resp. $\mathcal{B}$) of $\mathcal{H}_{X'} \times \mathcal{Q}_{X'}$ (resp. $\mathcal{H}_{U'} \times \mathcal{Q}_{U'}$). By [Gir71, Chapitre III, Proposition 1.3.6], $\mathrm{Hom}_{\mathcal{H}_{X'}}(\rho_*(\mathcal{A}), \mathcal{H}_{X'})$ is in bijection with $\mathrm{Hom}_{\mathcal{G}_{X'}}(\mathcal{A}, \mathcal{H}_{X'})$ which is non-empty by composing $\mathcal{A} \to \mathcal{H}_{X'} \times \mathcal{Q}_{X'}$ with the projection to $\mathcal{H}_{X'}$. So, $\rho_*\mathcal{A}$, and by a similar argument $\rho_*(\mathcal{B})$, are trivial. Then, [Gir71, Chapitre III, Proposition 3.2.2] implies that $\mathcal{A}$ (resp. $\mathcal{B}$) is of the form $p^{-1}(s)$ (resp. $p^{-1}(t)$) where s (resp. t) is an element of $(\mathcal{H}/\mathcal{G})(X')$ (resp. $(\mathcal{H}/\mathcal{G})(U')$). Consider

$$\begin{tikzcd} \rho_*\mathcal{Q}(X') \arrow[r] \arrow[d] & \rho_*j^*\mathcal{Q}(U') \arrow[d] \\ \{p^{-1}(s) \hookrightarrow \mathcal{H}_{X'} \times \mathcal{Q}_{X'}\} \arrow[r, "a"] & \{p^{-1}(t) \hookrightarrow \mathcal{H}_{U'} \times \mathcal{Q}_{U'}\} \end{tikzcd}$$

where the vertical arrows are bijections, and a is the obvious map. By Proposition A.21, $(\mathcal{H}/\mathcal{G})(X') \to (\mathcal{H}/\mathcal{G})(U')$ is bijective, and so the s and t occurring in this diagram are in bijective correspondence, and applying Proposition A.21 and Proposition A.22 shows a is bijective. $\square$

Proposition A.26. *Suppose that $(\Lambda_0, \mathbb{T}_0)$ is a tensor package and set $\mathcal{G} = \mathrm{Fix}(\mathbb{T}_0)$. Denote by $\rho_0\colon \mathcal{G} \to \mathrm{GL}(\Lambda_0)$ the tautological map and let $(\mathcal{E}, \mathbb{T})$ be an object of $\mathbf{Twist}_X^{\mathrm{rflx}}(\Lambda_0, \mathbb{T}_0)$. Then,*

$$(\rho_0)_*\mathcal{Q} = \underline{\mathrm{Isom}}\,(\Lambda_0 \otimes_R \mathcal{O}_X, \mathcal{E}), \quad \textit{where } \mathcal{Q} := \underline{\mathrm{Isom}}\,((\Lambda_0 \otimes_R \mathcal{O}_X, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}))\,.$$

In particular, $\mathcal{Q}$ is a torsor if and only if $\mathcal{E}$ is locally free.

Proof. Let $j\colon U \hookrightarrow X$ be a large open embedding such that $j^*\mathcal{Q}$ is a torsor. Applying [Gir71, Chapitre III, Proposition 1.3.6], we obtain a map $(\rho_0)_*j^*\mathcal{Q} \to \underline{\mathrm{Isom}}\,(\Lambda_0 \otimes_R \mathcal{O}_U, \mathcal{E}|_U)$ of torsors. As the natural map $\underline{\mathrm{Isom}}\,(\Lambda_0 \otimes_R \mathcal{O}_X, \mathcal{E}) \to j_*\underline{\mathrm{Isom}}\,(\Lambda_0 \otimes_R \mathcal{O}_U, \mathcal{E}|_U)$ is an isomorphism from Proposition A.21 and Proposition A.22, we conclude by Proposition A.25. For the second claim, it suffices to show the if statement, which follows as there is a tautological surjection of sheaves $\mathrm{GL}_{n,X} \times \mathcal{Q} \to (\rho_0)_*\mathcal{Q}$ and thus if the target is locally non-empty, so then must be the source. $\square$

Remark A.27. We introduced reflexive pseudo-torsors because we feel they are natural extensions of ideas present in the current article, that may be useful in a Tannakian formalism of analytic prismatic F-torsors. That said, while we do use reflexive pseudo-torsors in the body of this article, this is mainly through the final claim in Proposition A.26. The reader uninterested in the formalism of reflexive pseudo-torsors should note that such a result is already obtained by the method of proof in [CTS79, Théorème 6.13], which we briefly sketch.

Let $j\colon U \hookrightarrow X$ be a large open such that $j^*\mathcal{Q}$ is a torsor. Moving to an étale extension of X, we may assume that $\mathcal{E}$ is trivial. Thus $\rho_*j^*\mathcal{Q}$ is trivial, and so comes from an element of $(\mathrm{GL}(\Lambda_0)/\mathcal{G})(U)$. As $(\mathrm{GL}(\Lambda_0)/\mathcal{G})_X$ is affine over X, this can be extended to an element of $(\mathrm{GL}(\Lambda_0)/\mathcal{G})(X)$ by Proposition A.21, which gives rise to a $\mathcal{G}$-torsor $\mathcal{Q}'$. Evidently there exists an isomorphism of $\mathcal{G}$-torsors $f\colon j^*\mathcal{Q} \to j^*\mathcal{Q}'$ which as $\mathcal{Q}$ and $\mathcal{Q}'$ are affine (the former as it is a closed subscheme of the affine scheme $\underline{\mathrm{Isom}}(\Lambda_0 \otimes_R \mathcal{O}_X, \mathcal{E})$, and the latter by Lemma A.10), extends to an isomorphism of sheaves $\mathcal{Q} \to \mathcal{Q}'$ by A.21. That this is $\mathcal{G}$-equivariant, and thus an isomorphism of pseudo-torsors follows as $f|_U$ is $\mathcal{G}$-equivariant, and $j(U)$ is Zariski dense in X.

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