NOTES ON QUANDLE INVARIANTS OF KNOTS AND EXTENSIONS

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I would like to express my sincere gratitude to
Professor Yukio Matsumoto
on the occasion of his 70th birthday.

1. Introduction

This note is a brief summary of [6, 7]. Sets with certain self-distributive operations called *quandles* have been studied since the 1940s [18] in various areas with different names. The *fundamental quandle* of a knot was defined in a manner similar to the fundamental group [12, 13] of a knot, and became an important tool in knot theory. The fundamental quandle classifies knots up to reversed mirror.

The number of homomorphisms from the fundamental quandle to a fixed finite quandle has an interpretation as colorings of knot diagrams by quandle elements, and has been widely used as a knot invariant. Algebraic homology theories for quandles were defined in [3, 11], and investigated in [14, 15, 16], for example. A variety of knot invariants have been defined using quandle colorings and cocycles, and applied to various properties of knots and knotted surfaces (see, for example, [5] and references therein). Extensions of quandles by cocycles have been studied [1, 2, 10].

This note is organized as follows. Preliminary material follows this section. A summary on computational results of quandle colorings is given in Section 3. In Section 4, a method of computing quandle cocycle invariants from colorings of composite knots is studied. Relations to abelian extensions of quandles are examined in Section 5.

2. Preliminaries

We briefly review some definitions and examples of quandles. More details can be found, for example, in [1, 5, 11].

A *quandle* X is a set with a binary operation $(a, b) \mapsto a * b$ satisfying the following conditions.

(1) For any $a \in X$, $a * a = a$.

(2) For any $b, c \in X$, there is a unique $a \in X$ such that $a * b = c$.

(3) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

For example, a *generalized Alexander quandle* is defined by a pair (G, f) where G is a group, $f \in \mathrm{Aut}(G)$, and the quandle operation is defined by $x * y = f(xy^{-1})y$. If G is abelian, this is called an *Alexander* (or *affine*) quandle.

Let X be a quandle. The *right translation* $\mathcal{R}_a : X \to X$, by $a \in X$, is defined by $\mathcal{R}_a(x) = x * a$ for $x \in X$. Then $\mathcal{R}_a$ is a quandle automorphism of X by Axiom (2) and (3). The subgroup of $\mathrm{Aut}(X)$, the quandle automorphism group of X, generated by $\mathcal{R}_a$, $a \in X$, is called the *inner automorphism group* of X, and is denoted by $\mathrm{Inn}(X)$. A quandle is *connected* if $\mathrm{Inn}(X)$ acts transitively on X. A quandle is *faithful* if the mapping $\varphi : X \to \mathrm{Inn}(X)$ defined by $\varphi(a) = \mathcal{R}_a$ is an injection from X to $\mathrm{Inn}(X)$. A quandle X is called a *kei* [18], or *involutory*, if $(x * y) * y = x$ for all $x, y \in X$.

A *coloring* of an oriented knot diagram by a quandle X is a map $\mathcal{C}$ from the set of arcs $\mathcal{A}$ of the diagram to X such that the image of the map satisfies the relation depicted in Figure 1 at each crossing. More details can be found in [5, 10], for example. The number of colorings of a diagram of a knot K by a finite quandle X does not depend on the choice of a diagram, and denoted by $\mathrm{Col}_X(K)$.

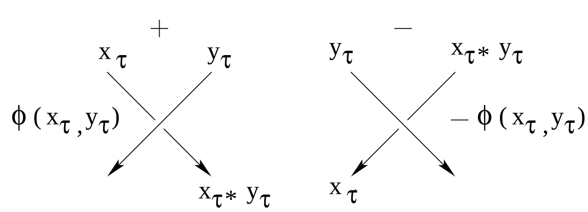

FIGURE 1. Colored crossings and cocycle weights

A function $\phi : X \times X \to A$ for an abelian group A is called a *quandle 2-cocycle* if it satisfies

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0$$

for any $x, y, z \in X$ and $\phi(x, x) = 0$ for any $x \in X$ (see [3] for a homology theory). For a quandle 2-cocycle ϕ, $E = X \times A$ becomes a quandle by

$$(x, a) * (y, b) = (x * y, a + \phi(x, y))$$

for $x, y \in X$, $a, b \in A$, denoted by $E(X, A, \phi)$ or simply $E(X, A)$, and it is called an *abelian extension* of X by A. See [2] for more information on abelian extensions of quandles.

Let X be a quandle, and ϕ be a 2-cocycle with coefficient group A, a finite abelian group. For a coloring of a knot diagram by a quandle X as depicted in Figure 1 at a positive (left) and negative (right) crossing, respectively, the pair (x_τ, y_τ) of colors assigned to a pair of nearby arcs is called the *source* colors. The third arc receives the color $x_\tau * y_\tau$.

The 2-cocycle (or cocycle, for short) invariant is an element of the group ring $\mathbb{Z}[A]$ defined by $\Phi_\phi(K) = \sum_{\mathcal{C}} \prod_\tau \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where the product ranges over all crossings τ (the image is in multiplicative notation of A), the sum ranges over all colorings of a given knot diagram, (x_τ, y_τ) are source colors at the crossing τ, and $\epsilon(\tau)$ is the sign of τ as specified in Figure 1.

3. Computer calculations

Computations using GAP [20] significantly expanded the list for connected quandles. These quandles may be found in the GAP package Rig [19]. Rig includes all connected quandles of order less than 48 (extended this year from 36 previously). There are 790 (431 for order less than 36) such quandles. We refer to these quandles as *Rig* quandles (short for Rack In GAP), and use the notation $Q(n, i)$ for the i-th quandle of order n in the list of Rig quandles (see the Wiki page of Rig [19]). The following computer calculations have been made recently.

• The package Rig [19] includes homology groups, 2-cocycles, abelian extensions and cocycle invariants for some Rig quandles and some knots in the KnotInfo table [9].

• The properties of quandles such as faithful, kei, Alexander, abelian extensions, were determined for Rig quandles of order up to 35, and posted at http://math.usf.edu/∼saito/QuandleColor/.

• The number of quandle colorings by Rig quandles for knots in the table [9] up to 12 crossings have been computed. The output is posted at the same site as above.

• In [6], the coloring numbers were used to give some new information on the tunnel number and the unknotting number.

• For two quandles Q_1 and Q_2, and a set $\mathcal{K}$ of knots, we write $Q_1 \approx_{\mathcal{K}} Q_2$ if $\mathrm{Col}_{Q_1}(K) = \mathrm{Col}_{Q_2}(K)$ for all $K \in \mathcal{K}$. Let $\mathcal{K}$ be the set of all 2977 knots in the table in KnotInfo [9] up to 12 crossings. For this set $\mathcal{K}$, the equivalence classes of Rig quandles of order up to 35 under $\approx_{\mathcal{K}}$ were determined [6]. For example, $\{Q(6, 1), Q(6, 2)\}$ is such a class. Out of 431 Rig quandles, 151 classes consist of more than one element, and among these, 145 classes consist of two quandles.

4. Quandle colorings of composite knots

A quandle coloring of an oriented 1-tangle diagram is defined in a manner similar to those for knots. We do not require that the end points receive the same color for a quandle coloring of 1-tangle diagrams.

Definition 4.1 ([7]). Let K be a 1-tangle diagram and X be a quandle. We say that (K, X) is *end monochromatic*, or K is *end monochromatic* with X, if any coloring of K by X assigns the same color on the two end points.

Let K be a knot diagram with a base point b. Then we say that (K, X) is *end monochromatic*, or K is *end monochromatic* with X, if a corresponding 1-tangle diagram is end monochromatic. This does not depend on the choice of a base point, and if a diagram of a knot K is end monochromatic with X for a base point b, then we say that a knot K is *end monochromatic* with X.

A faithful quandle X is end monochromatic for any knot. Computer calculations show that there are non-faithful quandles that are end monochromatic for all knots up to 12 crossings. Most are abelian extensions (abelian extensions are not faithful).

The following result, originally stated for faithful quandles, naturally extends to end monochromatic quandles.

Proposition 4.2 ([16]). *Let ϕ be a 2-cocycle of a finite connected quandle X with coefficient group A. Suppose that K_1 and K_2 are end monochromatic with X. Then $|X|\Phi_\phi(K_1 \# K_2) = \Phi_\phi(K_1)\Phi_\phi(K_2)$.*

Let X be a quandle, A be a finite abelian group, and ϕ be a 2-cocycle with coefficient group A. Let $\Phi_\phi(K) = \sum_{g \in A} a_g g \in \mathbb{Z}[A]$ be the cocycle invariant of a knot K. We write $C_g(\Phi_\phi(K)) = a_g$. In particular, $C_e(\Phi_\phi(K)) \in \mathbb{Z}$ denotes the coefficient of the identity element $e \in A$.

In [2] it was shown that a coloring of a knot K by a quandle X contributes a non-identity element of the coefficient group A if and only if it extends to a coloring by the corresponding abelian extension E. Using this and Proposition 4.2, we obtain the following.

Proposition 4.3. *Let X, A, ϕ be as above. Suppose that X is end monochromatic with K. Suppose further that for an element $v \in A$ that is not the identity element e, there exists a knot R_v such that $\Phi_\phi(R_v) = r_e e + r_v v \in \mathbb{Z}[A]$.*

Then

$$C_{v^{-1}}(\Phi_\phi(K)) = \frac{1}{r_v|A|}(\ |X|\mathrm{Col}_E(R_v\#K) - r_e\,\mathrm{Col}_E(K)\ ).$$

Example 4.4. Let $X = Q(6,2)$ and ϕ be a generating 2-cocycle over $A = \mathbb{Z}_4$ such that the abelian extension of X with respect to ϕ is $E = Q(24,2)$. Since X is faithful, any knot is end monochromatic with X.

We abbreviate the identity element. For example, $6+24u$ means $6e+24u$ for the identity element e. The cocycle invariants of $X = Q(6,2)$ using this cocycle are given in the Wiki page of Rig [19], for knots up to 10 crossings. In particular, in order to use Proposition 4.3, we use the following invariant values:

$$\Phi_\phi(3_1) = 6+24u, \qquad \Phi_\phi(8_5) = 30+24u^2, \qquad \Phi_\phi(9_1) = 6+24u^3.$$

Proposition 4.3 implies that

$$\begin{array}{rcl} C_u(\Phi_\phi(K)) &=& (\ 1/(24\cdot 4)\ )\ (\ 6\cdot \mathrm{Col}_E(9_1\#K) - 6\cdot\mathrm{Col}_E(K)\ ),\\ C_{u^2}(\Phi_\phi(K)) &=& (\ 1/(24\cdot 4)\ )\ (\ 6\cdot \mathrm{Col}_E(8_5\#K) - 30\cdot\mathrm{Col}_E(K)\ ),\\ C_{u^3}(\Phi_\phi(K)) &=& (\ 1/(24\cdot 4)\ )\ (\ 6\cdot \mathrm{Col}_E(3_1\#K) - 6\cdot\mathrm{Col}_E(K)\ ). \end{array}$$

We also have $C_e(\Phi_\phi(K)) = (1/|A|)\mathrm{Col}_E(K) = (1/4)\mathrm{Col}_E(K)$. Therefore we obtain

$$\begin{aligned}\Phi_\phi(K) = \frac{1}{16}\,[\ & 4\,\mathrm{Col}_E(K) + (\mathrm{Col}_E(9_1\#K) - \mathrm{Col}_E(K))u \\ &+ (\mathrm{Col}_E(8_5\#K) - 5\,\mathrm{Col}_E(K))u^2 + (\mathrm{Col}_E(3_1\#K) - \mathrm{Col}_E(K))u^3\ \ ].\end{aligned}$$

See [7] for computational outputs, and formulas for other quandles.

5. Properties of abelian extensions

We summarize our findings on extensions of Rig quandles in this section. Using a group cocycle, we obtain the following.

Proposition 5.1. *Let X be a finite quandle, and $0 \to C \xrightarrow{\iota} A \xrightarrow{p_B} B \to 0$ be an exact sequence of finite abelian groups. Let $\phi : X \times X \to A$ be a quandle 2-cocycle. Then $E(X, A, \phi)$ is an abelian extension of $E(X, B, p_B\phi)$ with coefficient group C.*

We examined some connected abelian extensions of Rig quandles of order up to 12. In the following, we use the notation $E \xrightarrow{n} X$ if $E = E(X, \mathbb{Z}_n.\phi)$ for some 2-cocycle ϕ such that E is connected. We write $E_2 \xRightarrow{m} E_1 \xRightarrow{d} X$ if there is a short exact sequence $0 \to \mathbb{Z}_m \to \mathbb{Z}_n \to \mathbb{Z}_d \to 0$ such that $\mathbb{Z}_n \subset H^2_Q(X, \mathbb{Z}_n)$ and E_1, E_2 are corresponding extensions as in Proposition 5.1.

In this case $E_2 \xrightarrow{n} X$ where $n = md$. The notation $\emptyset \xrightarrow{1} X$ indicates that $H^2_Q(X, A) = 0$ for any coefficient group A, and hence there is no non-trivial abelian extension. It is noted to the left when all quandles in question are keis.

$$\emptyset \xrightarrow{1} Q(8,1) \xrightarrow{2} Q(4,1)$$
$$\text{(Kei)} \quad \emptyset \xrightarrow{1} Q(24,1) \xrightarrow{2} Q(12,1) \xrightarrow{2} Q(6,1)$$
$$\emptyset \xrightarrow{1} Q(24,2) \xRightarrow{2} Q(12,2) \xRightarrow{2} Q(6,2)$$
$$\text{(Kei)} \quad \emptyset \xrightarrow{1} Q(27,1) \xrightarrow{3} Q(9,2) = Q(3,1) \times Q(3,1)$$
$$\emptyset \xrightarrow{1} Q(27,6) \xrightarrow{3} Q(9,3) = \mathbb{Z}_3[t]/(t^2+1)$$
$$\emptyset \xrightarrow{1} Q(27,14) \xrightarrow{3} Q(9,6) = \mathbb{Z}_3[t]/(t^2+2t+1)$$
$$\emptyset \xrightarrow{1} Q(24,8) = Q(3,1) \times Q(8,1) \xrightarrow{2} Q(12,4) = Q(3,1) \times Q(4,1)$$

These computations raise the following questions.

• What is a condition on cocycles for abelian, or non-abelian extensions to be connected?

In [1], a condition for an extension to be connected was given in terms of elements of the inner automorphism group.

• Is there an infinite sequence of abelian extensions of connected quandles $\cdots \to Q_n \to \cdots \to Q_1$?

We note that sequences of abelian extensions of connected quandles terminate as much as we were able to compute.

• Is any abelian extension of a finite kei a kei?

Acknowledgements. The author was partially supported by NIH R01GM109459.

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