Towards Formalization of Four-Dimensional Topology

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Recently, computers are used to settle or confirm lengthy and difficult mathematical theorems. In fact, last August this year, the Flyspeck project headed by T. Hales completed a formal proof of the Kepler conjecture, confirming this 400 year old problem [1]. Two years ago, in 2012, a lengthy proof of the significant theorem in finite group theory, the Feit-Thompson Theorem, was formalized by a team lead by G. Gonthier [2]. Back in 2004, Gonthier announced the computer-checked proof of the Four-Color-theorem [3] ....

Then why not computer-confirm the core theorem in topology of four-dimensional manifolds: The 'Casson Handles are Topological Handles' theorem proved by M. Freedman in 1981 [4] ?

So, we began to use COQ, one of the proof systems used in those formalizations, and tried to formalize some basic theorems in geometric topology toward this direction.

Our first goal, then, becomes the Bing Shrinking Criterion, since this is the basic theorem needed to construct final homeomorphisms.

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Theorem Bing_Shrinking_Criterion(unfinished):
 ∀ f:U→V, continuous f → surjective f →
 (Bing_shrinkable f ↔ approximable_by_homeos f).

Definition Bing_shrinkable (f:U→V): Prop:=
  ∀ eps:R, eps>0 →
    ∃ hu:point_set (MetricTopology du du_metric)
          → point_set (MetricTopology du du_metric),
    homeomorphism hu ∧
    ∀ u:U, dv (f u) (f (hu u)) < eps ∧
    ∀ u1 u2:U, (f u1) = (f u2) → (du u1 u2) < eps.
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Although we haven't quite finished formalizing this first goal yet, most of the necessary mathematical propositions are formalized and it now seems easy to finish it. Among those propositions, we formalized the classical Baire Category Theorem:

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Theorem BaireCategoryTheorem : complete d d_metric → baire_space.
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Next goal should be the generalized Shoeflies Theorem. In this respect, The Jordan Curve Theorem was formalized only in 2005. Basic algebraic topology such as singular homology theory doesn't seem to be formalized yet. There's a long way to go.

References

1. https://code.google.com/p/flyspeck/wiki/AnnouncingCompletion

2. "Feit-Thompson theorem has been totally checked in Coq". www.msr-inria.fr/news/feit-thomson-proved-in-coq/

3. Gonthier, Georges, "A computer-checked proof of the Four Colour Theorem"

4. Freedman, Michael H.,"The topology of four-dimensional manifolds", Journal of Differential Geometry 17 (3): 357-453(1982).