Coamoeba and equivariant homological mirror symmetry for the projective space

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1 Introduction

Let n be a natural number and Δ be a convex lattice polytope in $\mathbb{R}^n$, i.e., the convex hull of a finite subset of $\mathbb{Z}^n$. We always assume that the origin is in the interior of Δ. Homological mirror symmetry for toric Fano stacks, conjectured by Kontsevich [13], states that there is an equivalence

$$D^b \operatorname{coh} X \cong D^b \mathfrak{Fuk}\, W \tag{1}$$

of two triangulated categories of geometric origin associated to Δ.

The category on the left hand side is the derived category of coherent sheaves on a toric Fano stack X defined as follows: Let $\{v_i\}_{i=1}^r$ be the set of vertices of Δ and take a simplicial stacky fan $\mathbf{\Sigma}$ such that the set of generators of one-dimensional cones is given by $\{v_i\}_{i=1}^r$. The associated toric stack is the quotient stack

$$X = [(\mathbb{C}^r \setminus \mathrm{SR}(\Sigma))/K]$$

where the Stanley-Reisner locus $\mathrm{SR}(\Sigma)$ consists of points $(z_1, \dots, z_r)$ such that there is no cone in Σ which contains all v_i for which $z_i = 0$, and

$$K = \operatorname{Ker}(\phi \otimes \mathbb{C}^\times)$$

is the kernel of the tensor product with $\mathbb{C}^\times$ of the map $\phi : \mathbb{Z}^r \to \mathbb{Z}^n$ sending the i-th coordinate vector to v_i for $i = 1, \dots, r$. The torus $\operatorname{Spec} \mathbb{C}[\mathbb{Z}^n]$ acting on X will be denoted by $\mathbb{T}$. Although X depends not only on Δ but also on $\mathbf{\Sigma}$, the derived category $D^b \operatorname{coh} X$ is independent of this choice and depends only on Δ.

On the right hand side, one takes a sufficiently general Laurent polynomial

$$W = \sum_{\omega \in \Delta \cap \mathbb{Z}^n} a_\omega x^\omega$$

whose Newton polygon coincides with Δ. This defines an exact Lefschetz fibration

$$W : (\mathbb{C}^\times)^n \to \mathbb{C}$$

with respect to the standard cylindrical Kähler structure on $(\mathbb{C}^\times)^n$, and $\mathfrak{Fuk}\, W$ is the Fukaya category of Lefschetz fibration in the sense of Seidel [15].

The equivalence (1) is proved for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ by Seidel [14], weighted projective planes and Hirzebruch surfaces by Auroux, Katzarkov and Orlov [3], and toric del Pezzo surfaces by Ueda [16]. Here we discuss a proof of the torus-equivariant version of (1) for the projective space. The proof is based on the Picard-Lefschetz theory developed by Seidel [15] and the behavior of vanishing cycles by the argument map. The latter is a generalization of the relation between brane tilings and vanishing cycles, conjectured by Feng, He, Kennaway and Vafa [10] and proved in some cases [17, 18]. See also Auroux, Katzarkov and Orlov [4] for homological mirror symmetry for not necessarily toric del Pezzo surfaces, Abouzaid [1, 2] for an application of tropical geometry to homological mirror symmetry, and Kerr [11] for the behavior of homological mirror symmetry under weighted blowup of toric surfaces. The works of Bondal and Ruan [7] and Fang, Liu, Treumann and Zaslow [8, 9] use constructible sheaves on real tori and their universal covers, and it is an interesting problem to explore relationship between their works and ours.

2 Fukaya categories

For a $\mathbb{Z}$-graded vector space $N = \oplus_{j \in \mathbb{Z}} N^j$ and an integer i, the i-th shift of N to the left will be denoted by $N[i]$; $(N[i])^j = N^{i+j}$.

Definition 1. An A_∞-category $\mathcal{A}$ consists of

- the set $\mathfrak{Ob}(\mathcal{A})$ of objects,
- for $c_1,\ c_2 \in \mathfrak{Ob}(\mathcal{A})$, a $\mathbb{Z}$-graded vector space $\hom_{\mathcal{A}}(c_1, c_2)$ called the space of morphisms, and
- operations

  $$\mathfrak{m}_l : \hom_{\mathcal{A}}(c_{l-1}, c_l) \otimes \cdots \otimes \hom_{\mathcal{A}}(c_0, c_1) \longrightarrow \hom_{\mathcal{A}}(c_0, c_l)$$

  of degree $2 - l$ for $l = 1, 2, \dots$ and $c_i \in \mathfrak{Ob}(\mathcal{A})$, $i = 0, \dots, l$, satisfying the A_∞-*relations*

  $$\sum_{i=0}^{l-1} \sum_{j=i+1}^{l} (-1)^{\deg a_1 + \cdots + \deg a_i - i} \mathfrak{m}_{l+i-j+1}(a_l \otimes \cdots \otimes a_{j+1} \otimes \mathfrak{m}_{j-i}(a_j \otimes \cdots \otimes a_{i+1}) \otimes a_i \otimes \cdots \otimes a_1) = 0, \quad (2)$$

  for any positive integer l, any sequence $c_0, \dots, c_l$ of objects of $\mathcal{A}$, and any sequence of morphisms $a_m \in \hom_{\mathcal{A}}(c_{m-1}, c_m)$ for $m = 1, \dots, l$.

An A_∞-category satisfying $\mathfrak{m}_k = 0$ for $k \geq 3$ is the same thing as a *differential graded category*, i.e., a category whose spaces of morphisms are complexes such that the differential d satisfies the Leibniz rule with respect to the composition. The derived category of an A_∞-category is defined using *twisted complexes*, which are introduced by Bondal and Kapranov [6] for differential graded categories and generalized to A_∞-categories by Kontsevich [12].

The Fukaya category $\mathfrak{Fuk}\, M$ of a symplectic manifold (M, ω) is an A_∞-category whose objects are Lagrangian submanifolds of M (together with additional structures such as gradings, spin structures and flat $U(1)$ bundles on them) and whose spaces of morphisms are Lagrangian intersection Floer complexes: For two objects L_1 and L_2 intersecting transversely, $\hom(L_1, L_2)$ is a graded vector space spanned by intersection points $L_1 \cap L_2$. For a positive integer k, a sequence $(L_0, \dots, L_k)$ of objects, and morphisms $p_l \in L_{\ell-1} \cap L_\ell$ for $\ell = 1, \dots, k$, the A_∞-operation $\mathfrak{m}_k$ is given by counting the virtual number of holomorphic disks with Lagrangian boundary conditions;

$$\mathfrak{m}_k(p_k, \dots, p_1) = \sum_{p_0 \in L_0 \cap L_k} \# \overline{\mathcal{M}}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k) p_0.$$

Here, $\overline{\mathcal{M}}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k)$ is the stable compactification of the moduli space of holomorphic maps $\phi : D^2 \to M$ from the unit disk D^2 with $k+1$ marked points $(z_0, \dots, z_k)$ on the boundary respecting the cyclic order, with the following boundary condition: Let $\partial_l D^2 \in \partial D^2$ be the interval between z_l and z_{l+1}, where we set $z_{k+1} = z_0$. Then $\phi(\partial_l D^2) \subset L_\ell$ and $\phi(z_l) = p_l$ for $\ell = 0, \dots, k$.

A holomorphic function

$$\pi : E \to \mathbb{C}$$

on an exact Kähler manifold E is an *exact Lefschetz fibration* if all the critical points of π are non-degenerate. This means that for any critical point $p \in E$, one can choose a holomorphic local coordinate $(x_1, \dots, x_n)$ of E around p such that

$$\pi(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 + w, \tag{3}$$

where w is the critical value of π. For the moment, we assume that all the critical values are distinct and 0 is a regular value of π. We choose the origin as the base point and write

$$E_0 = \pi^{-1}(0).$$

A *vanishing path* is an embedded path $\gamma : [0, 1] \to \mathbb{C}$ such that

- $\gamma(0) = 0$,
- $\gamma(1)$ is a critical value of π, and
- $\gamma(t)$ is not a critical value of π for $t \in (0, 1)$.

A *distinguished set of vanishing paths* is an ordered set $(\gamma_i)_{i=1}^m$ of vanishing paths $\gamma_i : [0, 1] \to \mathbb{C}$ such that

- $\{\gamma_i(1)\}_{i=1}^m$ is the set of critical values of π,
- images of γ_i and γ_j for $i \neq j$ intersect only at the origin,
- $\gamma_i'(0) \neq 0$ for $i = 1, \dots, m$, and
- $\arg \gamma_1'(0) > \dots > \arg \gamma_m'(0)$ for a suitable choice of a branch of the argument map.

For a point $x \in E_0$ and a curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = 0$, one can define the horizontal lift $\widetilde{\gamma}_x : [0,1] \to E$ starting from x by the condition that the tangent vector of the curve $\widetilde{\gamma}$ is orthogonal to the tangent space of the fiber with respect to the Kähler form.

Let γ be a vanishing path and y be the critical point of π above $\gamma(1)$. Then the *vanishing cycle* along γ is the cycle of E_0 which collapses to the critical point y by the symplectic parallel transport along γ;

$$V_\gamma = \left\{ x \in E_0 \,\middle|\, \lim_{t \to 1} \widetilde{\gamma}_x(t) = y \right\}.$$

The vanishing cycle is a Lagrangian $(n-1)$-sphere E_0. The trajectory

$$\Delta_\gamma = \bigcup_{x \in V_\gamma} \operatorname{Im} \widetilde{\gamma}_x$$

of the vanishing cycle is called the *Lefschetz thimble*. It is a Lagrangian ball in E whose boundary is the corresponding vanishing cycle;

$$\partial \Delta_\gamma = V_\gamma.$$

If the Kähler structure is Euclidean with respect to the local coordinate $(x_1, \dots, x_n)$ such that π is of the form (3), w is a negative real number, and the vanishing path is the straight line from the origin to w, then the vanishing cycle is

$$\begin{aligned} V_\gamma &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = |w|\} \\ &\subset E_0 = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1^2 + \cdots + x_n^2 + w = 0\} \end{aligned}$$

and the Lefschetz thimble is given by

$$\Delta_\gamma = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \le |w|\}.$$

For a distinguished set $(\gamma_i)_{i=1}^m$ of vanishing paths, the ordered set

$$\boldsymbol{V} = (V_{\gamma_1}, \dots, V_{\gamma_n})$$

is called the *distinguished basis of vanishing cycles*.

To define the Fukaya category of Lefschetz fibration, let

$$\beta : \widetilde{E} = \{(x,y) \in E \times \mathbb{C} \mid \pi(x) = y^2\} \to E$$

be the double cover of E branched along the fiber $E_0 = \pi^{-1}(0)$ over the origin. Then the covering transformation $\iota : (x,y) \mapsto (x,-y)$ defines a $\mathbb{Z}/2\mathbb{Z}$-action on $\widetilde{E}$, which induces an action on the Fukaya category $\mathfrak{Fuk}\, \widetilde{E}$ of $\widetilde{E}$. Roughly speaking, the Fukaya category $\mathfrak{Fuk}\, \pi$ of the Lefschetz fibration π is defined as the ι-invariant part of $\mathfrak{Fuk}\, \widetilde{E}$; objects of $\mathfrak{Fuk}\, \pi$ are ι-invariant Lagrangian submanifolds of $\widetilde{E}$, and the space of morphisms in $\mathfrak{Fuk}\, \pi$ are ι-invariant part of morphisms in $\mathfrak{Fuk}\, \widetilde{E}$.

There are two important classes of ι-invariant Lagrangian submanifolds in $\widetilde{E}$. One of them, called of type (U) in [15], is the inverse image

$$\widetilde{L} = \beta^{-1}(L) = \widetilde{L}_+ \coprod \widetilde{L}_-$$

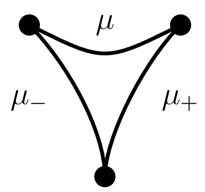

Figure 1: A matching path

of a Lagrangian submanifold L whose image by π is contained in a simply-connected domain inside $\mathbb{C}^\times$ (i.e., $\mathbb{C}$ minus the base point). It is the disjoint union of two connected components $\widetilde{L}_+$ and $\widetilde{L}_-$. The other, called of type (B), is the inverse image

$$\widetilde{\Delta}_\gamma = \beta^{-1}(\Delta_\gamma)$$

of the Lefschetz thimble Δ_γ for a vanishing path γ. It is a Lagrangian n-sphere in $\widetilde{E}$.

For type (U) Lagrangian submanifolds $\widetilde{L}_0$ and $\widetilde{L}_1$ of $\widetilde{E}$, their intersections are two disjoint copies of intersections between L_0 and L_1 in E. By taking ι-invariant, one can show that there is a natural isomorphism

$$\hom_{\mathfrak{Fuk}\,\pi}(\widetilde{L}_0, \widetilde{L}_1) \cong \hom_{\mathfrak{Fuk}\,E}(L_0, L_1)$$

of vector spaces, which lifts to a full and faithful A_∞-functor

$$\mathfrak{Fuk}\,E \to \mathfrak{Fuk}\,\pi.$$

For type (B) Lagrangian submanifolds, the situation is a little more complicated, but the conclusion is that the full A_∞-subcategory of $\mathfrak{Fuk}\,\pi$ consisting of $\widetilde{\boldsymbol{\Delta}} = (\widetilde{\Delta}_{\gamma_1}, \dots, \widetilde{\Delta}_{\gamma_m})$ for a distinguished set $(\gamma_i)_{i=1}^m$ of vanishing paths is quasi-isomorphic to the *directed subcategory* $\mathfrak{Fuk}^{\to}(\boldsymbol{V})$ of $\mathfrak{Fuk}\,E_0$, whose set of objects is the distinguished basis $\boldsymbol{V} = (V_{\gamma_1}, \dots, V_{\gamma_m})$ of vanishing cycles, whose spaces of morphisms are given by

$$\hom_{\mathfrak{Fuk}^{\to}(\boldsymbol{V})}(V_{\gamma_i}, V_{\gamma_j}) = \begin{cases} \mathbb{C} \cdot \mathrm{id}_{V_{\gamma_i}} & i = j, \\ \hom_{\mathfrak{Fuk}\,E_0}(V_{\gamma_i}, V_{\gamma_j}) & i < j, \\ 0 & \text{otherwise}, \end{cases}$$

and non-trivial A_∞-operations coincide with those in $\mathfrak{Fuk}\,E_0$.

Let $\mu : [-1, 1]$ be an embedded path in $\mathbb{C}$ such that $\mu^{-1}(\mathrm{Critv}(\pi)) = \{-1, 1\}$. One can deform μ and split it into two pieces $\mu_\pm(t) = \mu(\pm t)$ to obtain a pair of vanishing paths as shown in Figure 1. If the vanishing cycles V_{μ_-} and V_{μ_+} are isotopic as exact framed Lagrangian $(n-1)$-spheres in E_0, then μ is called a *matching cycle*. In this case, one can perturb $\Delta_{\mu_+} \cup \Delta_{\mu_-}$ to obtain a Lagrangian n-sphere Σ_μ called the *matching cycle*.

Picard-Lefschetz theory describes the behavior of Fukaya category under symplectic Dehn-twist along Lagrangian spheres. One of the consequences is that the type (U) Lagrangian submanifold $\widetilde{\Sigma}_\mu = \beta^{-1}(\widetilde{\Sigma}_\mu)$ of $\widetilde{E}$ coming from a matching path μ is isomorphic to the mapping cone over the (unique up to scalar) non-trivial morphism from $\widetilde{\Delta}_{\mu_-}$ to $\widetilde{\Delta}_{\mu_-}$ in the derived Fukaya category $D^b\,\mathfrak{Fuk}\,\pi$ of Lefschetz fibration;

$$\widetilde{\Sigma}_\mu \cong \mathrm{Cone}(\widetilde{\Delta}_{\mu_-} \to \widetilde{\Delta}_{\mu_+}).$$

This is important since it allows to reduce computation for matching cycles in $\mathfrak{Fuk}\, E$ to that of vanishing cycles in $\mathfrak{Fuk}\, E_0$. By iterating this process, one ends up in the case of symplectic 2-manifolds, where Lagrangian submanifolds are simple closed curves and counting of holomorphic disks is a combinatorial problem of "painting polygons".

A natural source of matching paths is a *Lefschetz bifibration*. It is a diagram

$$\mathcal{E} \xrightarrow{\varpi} \mathbb{C}^2 \xrightarrow{\psi} \mathbb{C}.$$

with certain genericity conditions, which implies that for any critical point of $\Psi = \psi \circ \varpi$, there are local holomorphic coordinates of $\mathcal{E}$ and $\mathbb{C}^2$ such that

$$\varpi(x_1, \dots, x_{2n}) = (x_1^2 + x_2^2 + \dots + x_{2n}^2, x_1), \qquad \psi(y_1, y_2) = y_1.$$

Then the map

$$\mathcal{E}_w \xrightarrow{\varpi_w} \mathcal{S}_w$$

from $\mathcal{E}_w = \Psi^{-1}(w)$ to $\mathcal{S}_w = \psi^{-1}(w)$ for a general $w \in \mathbb{C}$ is a Lefschetz fibration, and by chasing the trajectory of critical values of ϖ_w as w varies along a vanishing path γ, one obtains a matching path μ in $\mathcal{S}_0$ such that the matching cycle Σ_μ is Hamiltonian isotopic to the vanishing cycle V_γ.

3 Equivariant homological mirror symmetry for $\mathbb{P}^3$

The mirror of the projective space $\mathbb{P}^3$ is given by the Laurent polynomial

$$W(x, y, z) = x + y + z + \frac{1}{xyz}$$

with critical points $x = y = z = \pm 1, \pm\sqrt{-1}$ and critical values $\pm 4, \pm 4\sqrt{-1}$. Choose a distinguished set of vanishing paths $(\gamma_i)_{i=1}^4$ as the straight lines from the origin to the critical values. To use Picard-Lefschetz theory, consider the diagram

$$(\mathbb{C}^\times)^3 \xrightarrow{\varpi} \mathbb{C} \times \mathbb{C}^\times \xrightarrow{\psi} \mathbb{C}$$

where

$$\varpi(x, y, z) = \left(x + y + z + \frac{1}{xyz},\ z \right), \qquad \psi(t, z) = t.$$

The critical values of

$$\varpi_t : W^{-1}(t) \to \psi^{-1}(t) \cong \mathbb{C}^\times$$

are $(-3)^{3/4}$ at $t = 0$, which moves as shown in Figure 2 along the vanishing paths $(\gamma_i)_{i=1}^4$. These four paths are matching paths for ϖ_0 and one can reduce computations in $\mathfrak{Fuk}\, W$ to those in $\mathfrak{Fuk}\, \varpi_0$. Take $z = 1$ as a base point and choose a distinguished set $(\delta_i)_{i=1}^4$ of vanishing paths for ϖ_0 as straight lines from the base point. The fiber $\varpi_0^{-1}(z)$ is a branched double cover of $\mathbb{C}^\times$ by the y-projection

$$\begin{array}{cccc} \pi_z : & \varpi_0^{-1}(z) & \to & \mathbb{C}^\times \\ & \cup & & \cup \\ & (x, y, z) & \mapsto & y. \end{array}$$

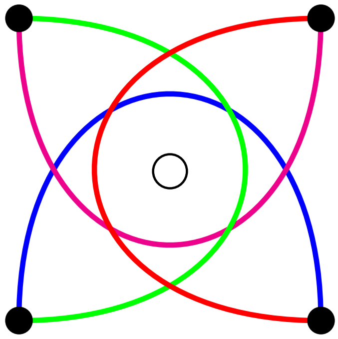

Figure 2: Matching cycles on the z-plane

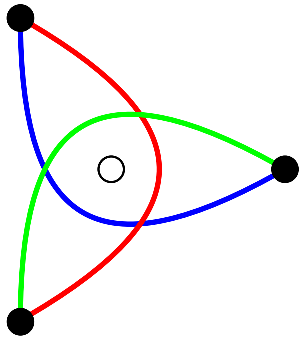

Figure 3: Matching cycles on the y-plane

Figure 3 shows the behavior of these branch points along vanishing paths $(\delta_i)_{i=1}^4$, which can be considered as matching paths coming from the Lefschetz bifibration

$$\begin{array}{ccccc} W^{-1}(0) & \to & \mathbb{C}^\times \times \mathbb{C}^\times & \to & \mathbb{C}^\times \\ \Cup & & \Cup & & \Cup \\ (x, y, z) & \mapsto & (y, z) & \mapsto & z. \end{array}$$

Now one can use Picard-Lefschetz theory to compute the Fukaya category of W. Strictly speaking, one has to go to the universal cover of the z-plane to apply Seidel's theory, since the z-plane is not simply-connected. This passage to the universal cover can be taken into account by studying the behavior of the branch points of π_z as one goes around the origin in the z-plane.

On the mirror side, the passage to the universal cover of the z-plane corresponds to working equivariantly with respect to a certain subgroup $\mathbb{C}^\times \subset \mathbb{T}$ of the torus $\mathbb{T}$ acting on $\mathbb{P}^3$. From this point of view, it is more natural to work equivariantly with respect to the whole torus $\mathbb{T}$ and consider the derived category $D^b \operatorname{coh}^{\mathbb{T}} \mathbb{P}^3$ of $\mathbb{T}$-equivariant coherent sheaves. This in turn corresponds to passing to the universal cover

$$\phi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$$

on the symplectic side and replacing the Lefschetz fibration W with its pull-back;

$$\widetilde{W} = W \circ \phi : \mathbb{C}^n \to \mathbb{C}.$$

The fact that $\widetilde{W}$ has infinitely many critical points does not cause any problem in defining its Fukaya category and formulating a torus-equivariant version of (1):

Conjecture 2. *For any convex lattice polytope, there is an equivalence*

$$D^b \operatorname{coh}^{\mathbb{T}} X \cong D^b \mathfrak{Fuk}\, \widetilde{W}$$

of triangulated categories.

The main result is the following:

Theorem 3. *Conjecture 2 holds when X is the projective space.*

The key ingredient in the proof of Theorem 3 is a division of the real 3-torus $T = \mathbb{R}^3/\mathbb{Z}^3$ into the union of four truncated octahedra, which encodes the information of the Fukaya category $\mathfrak{Fuk}\, W$ in a nice way. A *truncated octahedron* is obtained from an octahedron by truncating its six vertices. The truncation of each vertex increases the numbers of faces, edges and vertices by one, four and three respectively. As a result, a truncated octahedron has fourteen faces, thirty-six edges and twenty-four vertices. The *bitruncated cubic honeycomb* is a cell-transitive space-filling tessellation consisting of truncated octahedron. It can be realized as the Voronoi tessellation of the body-centered cubic lattice. By taking a quotient of the bitruncated cubic honeycomb with respect to a suitable lattice $M \subset \mathbb{R}^3$, one obtains a tessellation of a torus $T = \mathbb{R}^3/M$ by four truncated octahedra. To this tessellation, one can associate an A_∞-category $\mathcal{A}$ as follows:

- The set of objects is the set of cells.
- The space of morphisms between two cells are spanned by their common faces.
- For each edge e, one has an A_∞-operation
$$\mathfrak{m}_k(f_1, \dots, f_k) = \pm f_{k+1}$$
where $(f_1, \dots, f_{k+1})$ is the set of faces around e.

To be more precise, we fix an order on the set of cells which comes from the order in the distinguished basis of vanishing cycles, and color faces and edges according to some Floer-theoretic data (grading and sign). The order of the faces $(f_1, \dots, f_{k+1})$ around an edge is chosen in accordance with the order on the set of cells.

Using Picard-Lefschetz theory and the information of matching paths shown in Figures 2 and 3, one can show that $\mathfrak{Fuk}\, W$ is equivalent to $\mathcal{A}$;

$$\mathfrak{Fuk}\, W \cong \mathcal{A}.$$

On the other hand, it is easy to see that the minimal model of the full subcategory of (the enhancement of) $D^b \operatorname{coh} \mathbb{P}^3$ consisting of the full exceptional collection

$$(\mathcal{O}_{\mathbb{P}^3}, \Omega_{\mathbb{P}^3}(1)[1], \Omega^2_{\mathbb{P}^3}(2)[2], \Omega^3_{\mathbb{P}^3}(3)[3])$$

constructed by Beilinson [5] is equivalent to $\mathcal{A}$. This shows that

$$D^b \operatorname{coh} \mathbb{P}^3 \cong D^b \mathcal{A},$$

and homological mirror symmetry for $\mathbb{P}^3$ follows.

The passage to the equivariant situation can be achieved by working on the universal cover $\mathbb{R}^3$ of the 3-torus T. This gives an A_∞-category $\widetilde{\mathcal{A}}$ whose set of object is the (infinite) set of cells of bitruncated cubic honeycomb, with the space of morphisms and A_∞-operations analogous to those of $\mathcal{A}$. The generator of $D^b \operatorname{coh}^{\mathbb{T}} \mathbb{P}^3$ can be obtained by tensoring the exceptional collection above with the set $\operatorname{Irrep}(\mathbb{T})$ of irreducible representations of $\mathbb{T}$, which can be identified with the lattice M. One can show

$$\mathfrak{Fuk}\, \widetilde{W} \cong \widetilde{\mathcal{A}}$$

and

$$D^b \operatorname{coh}^{\mathbb{T}} X \cong D^b \widetilde{\mathcal{A}}$$

just as in the non-equivariant case, and Theorem 3 follows.

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