Floer homology for the Gelfand-Cetlin system

Yuichi Nohara and Kazushi Ueda

1 Introduction

Let $F = GL(n,\mathbb{C})/P$ be a (full or partial) flag manifold. The *Gelfand-Cetlin system*, introduced by Guillemin and Sternberg [10], is a completely integrable system

$$\Phi : F \longrightarrow \mathbb{R}^{(\dim_{\mathbb{R}} F)/2}$$

on F, i.e., a set of functionally independent and Poisson commuting functions. The image $\Delta = \Phi(F)$ is a convex polytope, which we call the *Gelfand-Cetlin polytope*, and Φ gives a Lagrangian torus fibration structure over the interior $\operatorname{Int}\Delta$ of Δ. Because of non-smoothness of Φ, it has non-torus fibers on some faces of codim ≥ 3. In this paper we study Lagrangian intersection Floer theory for Lagrangian torus and non-torus fiber of the Gelfand-Cetlin system.

Lagrangian intersection Floer theory for torus orbits in a toric manifold has been developed by Fukaya, Oh, Ohta and Ono [7]. We recall some of the results which are relevant to this paper. Let (X,ω) be a compact toric manifold of $\dim_{\mathbb{C}} X = N$, and $\Phi : X \to \mathbb{R}^N$ be the toric moment map with moment polytope $\Delta = \Phi(X)$. For an interior point $u \in \operatorname{Int}\Delta$, let $L(u)$ denote the Lagrangian torus fiber $\Phi^{-1}(u)$.

- The potential function $\mathfrak{PO}$ of Lagrangian torus fibers is defined as a function on

$$\bigcup_{u \in \operatorname{Int}\Delta} H^1(L(u);\Lambda_0/2\pi\sqrt{-1}\mathbb{Z}) \cong \operatorname{Int}\Delta \times (\Lambda_0/2\pi\sqrt{-1}\mathbb{Z})^N,$$

Yuichi Nohara
Faculty of Education, Kagawa University, Takamatsu, Kagawa, 760-8522, Japan, e-mail: nohara@ed.kagawa-u.ac.jp

Kazushi Ueda
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043, Japan, e-mail: kazushi@math.sci.osaka-u.ac.jp

where Λ_0 is the Novikov ring. In the Fano case, $\mathfrak{PO}$ can be regarded as a Laurent polynomial, and it coincides with the superpotential of the Landau-Ginzburg mirror of X.
- Each critical point of $\mathfrak{PO}$ corresponds to a pair $(L(u), b)$ of a fiber $L(u)$ and $b \in H^1(L(u); \Lambda_0 / 2\pi\sqrt{-1}\mathbb{Z})$ with nontrivial Floer homology.
- The quantum cohomology of X is isomorphic to the Jacobi ring $\mathrm{Jac}(\mathfrak{PO})$ of the potential function.

See [7] or [8] for more detail. In particular, the number of critical points of $\mathfrak{PO}$ is equal to the rank of the cohomology group $H^*(X)$ of X, provided that $\mathfrak{PO}$ is a Morse function.

In the case of Gelfand-Cetlin system, Nishinou and we [11] compute the potential function of Lagrangian torus fibers by using a toric degeneration of the flag manifold, and show that it coincides with the superpotential of the Landau-Ginzburg mirror of the flag manifold ([9], [1]). In contrast to the toric case, the number of critical points of the potential function, and hence the number of Lagrangian torus fibers with nontrivial Floer homology, is smaller than the rank of $H^*(F)$ in general. Eguchi, Hori, and Xiong [3] and Rietsch [12] consider a partial compactification of the mirror of F to get as many critical points of the superpotential as $\operatorname{rank} H^*(F)$. It is natural to expect that the critical points at "infinity" correspond to Lagrangian fibers on the boundary of the Gelfand-Cetlin polytope. In this paper, we study Floer homology of such non-torus fibers in the 3-dimensional flag manifold $\mathrm{Fl}(3)$ and the Grassmannian $\mathrm{Gr}(2,4)$ of 2-planes in $\mathbb{C}^4$.

This paper is organized as follows. In Section 2 we recall the construction of the Gelfand-Cetlin system and see non-torus Lagrangian fibers in $\mathrm{Fl}(3)$ and $\mathrm{Gr}(2,4)$. In Section 3 we study the potential function for the Gelfand-Cetlin system. The computation of the Floer homologies of non-torus Lagrangian fibers in $\mathrm{Fl}(3)$ and $\mathrm{Gr}(2,4)$ is given in Section 4.

2 Gelfand-Cetlin System

Fix a sequence $0 = n_0 < n_1 < \cdots < n_r < n_{r+1} = n$ of integers, and set $k_i = n_i - n_{i-1}$ for $i = 1, \dots, r+1$. The flag manifold $F = F(n_1, \dots, n_r, n)$ is defined by

$$F = U(n)/(U(k_1) \times \cdots \times U(k_{r+1})).$$

Let $\mathrm{Fl}(n) := F(1, 2, \dots, n)$ and $\mathrm{Gr}(k,n) := F(k,n)$ denote the full flag manifold and the Grassmannian of k-planes in $\mathbb{C}^n$, respectively. The dimension of $F(n_1, \dots, n_r, n)$ is given by

$$N = N(n_1, \dots, n_r, n) := \dim_{\mathbb{C}} F(n_1, \dots, n_r, n) = \sum_{i=1}^{r} (n_i - n_{i-1})(n - n_i).$$

We identify the dual $\mathfrak{u}(n)^*$ of the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ with the space $\sqrt{-1}\mathfrak{u}(n)$ of Hermitian matrices by using an invariant inner product. Then F is identified with the adjoint orbit $\mathscr{O}_\lambda \subset \sqrt{-1}\mathfrak{u}(n)$ of a diagonal matrix $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with

$$\underbrace{\lambda_1 = \cdots = \lambda_{n_1}}_{k_1} > \underbrace{\lambda_{n_1+1} = \cdots = \lambda_{n_2}}_{k_2} > \cdots > \underbrace{\lambda_{n_r+1} = \cdots = \lambda_n}_{k_{r+1}}.$$

Note that $\mathscr{O}_\lambda$ consists of Hermitian matrices with fixed eigenvalues $\lambda_1, \dots, \lambda_n$. Let ω be the Kostant-Kirillov form on $\mathscr{O}_\lambda$.

For $x \in \mathscr{O}_\lambda$ and $k = 1, \dots, n-1$, let $x^{(k)}$ denote the upper-left $k \times k$ submatrix of x. Since $x^{(k)}$ is also a Hermitian matrix, it has real eigenvalues $\lambda_1^{(k)}(x) \geq \lambda_2^{(k)}(x) \geq \cdots \geq \lambda_k^{(k)}(x)$. By taking the eigenvalues for all $k = 1, \dots, n-1$, we obtain a set of $n(n-1)/2$ functions $(\lambda_i^{(k)})_{1 \leq i \leq k \leq n-1}$. Since the eigenvalues satisfy the following inequalities

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-1} & & \lambda_n \\
& \rotatebox{-45}{$\geq$} \;\; \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} \;\; \rotatebox{45}{$\leq$} & & & & \rotatebox{-45}{$\geq$} \;\; \rotatebox{45}{$\leq$} & \\
& \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & & & \lambda_{n-1}^{(n-1)} & \\
& & \rotatebox{-45}{$\geq$} \;\; \rotatebox{45}{$\leq$} & & & & \rotatebox{45}{$\leq$} & & \\
& & \lambda_1^{(n-2)} & & & & \lambda_{n-2}^{(n-2)} & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & \ddots & & \iddots & & & \\
& & & & \rotatebox{-45}{$\geq$} \;\; \rotatebox{45}{$\leq$} & & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}, \tag{1}$$

some of $\lambda_i^{(k)}$ are constant functions if F is not a full flag manifold. It is easy to see that the number of nonconstant $\lambda_i^{(k)}$ coincides with $N = \dim_{\mathbb{C}} F$. The Gelfand-Cetlin system is defined to be the tuple

$$\Phi = (\lambda_i^{(k)})_{i,k} : F(n_1, \dots, n_r, n) \longrightarrow \mathbb{R}^{N(n_1, \dots, n_r, n)}$$

of nonconstant $\lambda_i^{(k)}$.

Proposition 2.1 (Guillemin-Sternberg [10]). *The map Φ is a completely integrable system on $(F(n_1, \dots, n_r, n), \omega)$, and the functions $\lambda_i^{(k)}$ are action variables. The image $\Delta = \Phi(F)$ is a convex polytope defined by* (1)*, and the fiber $L(u) = \Phi^{-1}(u)$ over each interior point $u \in \operatorname{Int}\Delta$ is a Lagrangian torus.*

Example 2.1. The Gelfand-Cetlin polytope for the 3-dimensional flag manifold $\mathrm{Fl}(3)$ is defined by

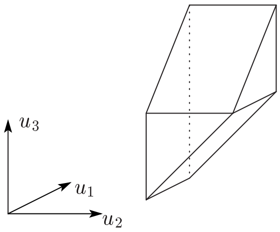

Fig. 1 The Gelfand-Cetlin polytope for $\mathrm{Fl}(3)$

$$\begin{array}{ccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 \\
& \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & \\
& & u_1 & & & & u_2 & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & & u_3 & & & &
\end{array}.$$

The Gelfand-Cetlin system has a non-torus fiber over the vertex $u_0 = (\lambda_2, \lambda_2, \lambda_2)$, where four edges are intersecting (See Fig. 1). The fiber $L_0 = \Phi^{-1}(u_0)$ is given by

$$L_0 = \left\{ \begin{pmatrix} \lambda_2 & 0 & z_1 \\ 0 & \lambda_2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & \lambda_1 - \lambda_2 + \lambda_3 \end{pmatrix} \in \mathscr{O}_\lambda \;\middle|\; |z_1|^2 + |z_2|^2 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) \right\},$$

which is diffeomorphic to a 3-sphere S^3.

Example 2.2. Next we consider the case of $\mathrm{Gr}(2,4)$. After a translation, we may assume that $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = \lambda$ for $\lambda > 0$. Then Δ is given by

$$\begin{array}{ccccccccccccc}
\lambda & & & & \lambda & & & & -\lambda & & & & -\lambda \\
& \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & \\
& & \lambda & & & & u_1 & & & & -\lambda & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & & u_2 & & & & u_3 & & & & \\
& & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & & & \\
& & & & & & u_4 & & & & & &
\end{array}$$

Figure 2 shows the projection $\Delta \to [-\lambda, \lambda]$, $u = (u_1, u_2, u_3, u_4) \mapsto u_1$. In this case non-torus fibers appear along the edge $u_1 = u_2 = u_3 = u_4$. For $-\lambda < t < \lambda$, the fiber $L_t = \Phi^{-1}(u_t)$ over $u_t = (t,t,t,t)$ is given by

$$L_t = \left\{ \begin{pmatrix} tI_2 & \sqrt{\lambda^2 - t^2}P \\ \sqrt{\lambda^2 - t^2}P^* & (-t)I_2 \end{pmatrix} \in \sqrt{-1}\mathfrak{u}(4) \;\middle|\; P \in U(2) \right\} \cong U(2).$$

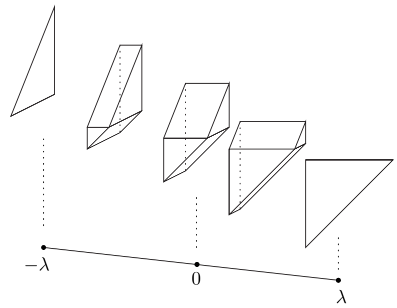

Fig. 2 The Gelfand-Cetlin polytope for $\mathrm{Gr}(2,4)$

3 Potential Functions for Gelfand-Cetlin Systems

Let $\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \geq 0,\ \lim_{i\to\infty} \lambda_i = \infty \right\}$ be the Novikov ring. The maximal ideal and the quotient field of the local ring Λ_0 will be denoted by Λ_+ and Λ respectively. For a spin and oriented Lagrangian submanifold L in a symplectic manifold (X,ω), one can equip an A_∞-structure

$$\mathfrak{m}_k = \sum_{\beta\in\pi_2(X,L)} T^{\omega(\beta)} \mathfrak{m}_{k,\beta} : H^*(L;\Lambda_0)^{\otimes k} \longrightarrow H^*(L;\Lambda_0)$$

on the cohomology group of L with coefficients in Λ_0 by "counting" pseudo-holomorphic disks ([5, Theorem A]). An element b in $H^1(L;\Lambda_+)$ (or $H^1(L;\Lambda_0)$) is called a *weak bounding cochain* if it satisfies the *Maurer-Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\dots,b) \equiv 0 \mod \mathrm{PD}([L]). \tag{2}$$

The set of weak bounding cochains will be denoted by $\widehat{\mathscr{M}}_{\mathrm{weak}}(L)$. For any $b \in \widehat{\mathscr{M}}_{\mathrm{weak}}(L)$, one can twist the Floer differential as

$$\mathfrak{m}_1^b(x) = \sum_{k,l} \mathfrak{m}_{k+l+1}(b^{\otimes k} \otimes x \otimes b^{\otimes l}).$$

The Maurer-Cartan equation (2) implies $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$, and the *Floer homology* of the pair (L,b) is defined by

$$HF((L,b),(L,b);\Lambda_0) = \mathrm{Ker}\,\mathfrak{m}_1^b / \mathrm{Im}\,\mathfrak{m}_1^b.$$

The *potential function* $\mathfrak{PO} : \widehat{\mathscr{M}}_{\mathrm{weak}}(L) \to \Lambda_0$ is defined by

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\dots,b) = \mathfrak{PO}(b) \cdot \mathrm{PD}([L]).$$

Now we consider the Gelfand-Cetlin system $\Phi : F = F(n_1,\dots,n_r,n) \to \Delta$. Take primitive vectors $v_i \in \mathbb{Z}^N$ and $\tau_i \in \mathbb{R}$ so that the Gelfand-Cetlin polytope is given by

$$\Delta = \{u \in \mathbb{R}^N \mid \ell_i(u) = \langle v_i, u\rangle - \tau_i \geq 0,\ i = 1,\dots,m\},$$

where m is the number of codimension one faces of Δ. For each interior point $u \in \operatorname{Int}\Delta$, we will identify $H^1(L(u);\Lambda_0)$ with Λ_0^N using the angle coordinate dual to the functions $\lambda_i^{(k)}$. The following theorem is a Gelfand-Cetlin analogue of [2, Section 15] and [6, Proposition 3.2 and Theorem 3.4].

Theorem 3.1 ([11, Theorem 10.1]). *For any interior point $u \in \operatorname{Int}\Delta$, we have an inclusion $H^1(L(u);\Lambda_0) \subset \widehat{\mathscr{M}}_{\mathrm{weak}}(L(u))$, and the potential function on $H^1(L(u);\Lambda_0) \cong \Lambda_0^N$ is given by*

$$\mathfrak{PO}(x) = \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(u)}.$$

After the coordinate change

$$\begin{aligned} y_k &= e^{x_k} T^{u_k}, \qquad k = 1,\dots,N(n_1,\dots,n_r,n),\\ Q_j &= T^{\lambda_{n_j}}, \qquad j = 1,\dots,r+1, \end{aligned}$$

the potential function can be regarded as a Laurent polynomial in $y_1,\dots,y_N$ with coefficients in $\mathbb{Q}[Q_1^{\pm1},\dots,Q_{r+1}^{\pm1}]$.

Example 3.1. In the case of 3-dimensional flag manifold $\mathrm{Fl}(3)$, the potential function is given by

$$\begin{aligned} \mathfrak{PO} &= e^{-x_1}T^{-u_1+\lambda_1} + e^{x_1}T^{u_1-\lambda_2} + e^{-x_2}T^{-u_2+\lambda_2}\\ &\qquad + e^{x_2}T^{u_2-\lambda_3} + e^{x_1-x_3}T^{u_1-u_3} + e^{-x_2+x_3}T^{-u_2+u_3}\\ &= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2} \end{aligned}$$

Critical points of $\mathfrak{PO}$ are given by

$$\begin{aligned} &y_1 = y_3^2/y_2,\\ &y_2 = \pm\sqrt{Q_3(y_3+Q_2)},\\ &y_3 = \sqrt[3]{Q_1Q_2Q_3},\ e^{2\pi\sqrt{-1}/3}\sqrt[3]{Q_1Q_2Q_3},\ e^{4\pi\sqrt{-1}/3}\sqrt[3]{Q_1Q_2Q_3}. \end{aligned}$$

It is easy to see that all critical points are nondegenerate and have the same valuation which lies in the interior of the Gelfand-Cetlin polytope. Hence we have as many critical point as $\dim H^*(\mathrm{Fl}(3)) = 6$ in this case.

Example 3.2. Next we discuss the case of $\mathrm{Gr}(2,4)$, where $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$. The potential function is given by

$$\begin{aligned}\mathfrak{PO} &= e^{-x_2}T^{-u_2+\lambda_1} + e^{-x_1+x_2}T^{-u_1+u_2} + e^{x_1-x_3}T^{u_1-u_3} \\ &\quad + e^{x_3}T^{u_3-\lambda_3} + e^{x_2-x_4}T^{u_2-u_4} + e^{-x_3+x_4}T^{-u_3+u_4} \\ &= \frac{Q_1}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_3}{Q_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3},\end{aligned}$$

whose critical points are given by

$$y_1 = y_4 = \pm\sqrt{Q_1Q_3}, \quad y_2 = Q_1Q_3/y_3, \quad y_3 = \pm\sqrt{2Q_3y_1}.$$

These four critical points are non-degenerate and have a common valuation in the interior of the Gelfand-Cetlin polytope. Since $\dim H^*(\mathrm{Gr}(2,4)) = 6$, one has less critical point than $\dim H^*(\mathrm{Gr}(2,4))$.

4 Floer Homologies of Non-torus Fibers

In this section we discuss Floer homologies of non-torus Lagrangian fibers in $\mathrm{Fl}(3)$ and $\mathrm{Gr}(2,4)$. Proofs of the results in this section will be given in a forthcoming paper.

4.1 Floer Homology of Lagrangian S^3 in $\mathrm{Fl}(3)$

Recall that $\pi_2(\mathrm{Fl}(3)) \cong \mathbb{Z}^2$ is generated by 1-dimensional Schubert varieties X_1, X_2. Since the fiber L_0 is diffeomorphic to S^3, the exact homotopy sequence yields $\pi_2(\mathrm{Fl}(3), L_0) \cong \pi_2(\mathrm{Fl}(3)) \cong \mathbb{Z}^2$. Let β_1, β_2 be generators of $\pi_2(\mathrm{Fl}(3), L_0)$ corresponding to X_1 and X_2, respectively. The Maslov index and the symplectic area of β_i are given by

$$\mu_{L_0}(\beta_1) = \mu_{L_0}(\beta_2) = 4, \quad \omega(\beta_1) = \lambda_1 - \lambda_2, \quad \omega(\beta_2) = \lambda_2 - \lambda_3.$$

Theorem 4.1. *The Floer homology of L_0 over the Novikov ring Λ_0 is*

$$HF(L_0, L_0; \Lambda_0) \cong \Lambda_0 / T^{\min\{\lambda_1-\lambda_2, \lambda_2-\lambda_3\}}\Lambda_0.$$

Hence the Floer homology over the Novikov field Λ is trivial: $HF(L_0, L_0; \Lambda) = 0$.

Sketch of proof. Since the minimal Maslov number is four, the only nontrivial parts of the Floer differential are

$$\mathfrak{m}_{1,\beta_i} : H^3(L_0;\Lambda_0) \cong H_0(L_0;\Lambda_0) \longrightarrow H^0(L_0;\Lambda_0) \cong H_3(L_0;\Lambda_0)$$

for $i = 1, 2$.

Lemma 4.1. *For each $p_0 \neq p_1 \in L_0$ and β_i, there exists a holomorphic disk $v : (D^2, \partial D^2) \to (\mathrm{Fl}(3), L_0)$ such that $v(1) = p_0$, $v(-1) = p_1$, and $[v] = \beta_i$. Such v is unique up to the action of $\{g \in \mathrm{Aut}(D^2) \,|\, g(1) = 1, g(-1) = -1\}$.*

Let J be the standard complex structure on $\mathrm{Fl}(3)$. Since $(\mathrm{Fl}(3), L_0)$ is $SU(2)$-homogeneous in the sense of Evans and Lekili [4, Definition 1.1.1], the result [4, Proposition 3.2.1] implies that any J-holomorphic disk in $(\mathrm{Fl}(3), L_0)$ is Fredholm regular. Hence Lemma 4.1 implies the following.

Lemma 4.2. *The moduli space $\mathscr{M}_2(J, \beta_i)$ of J-holomorphic disks in the class β_i with two boundary marked points is a smooth manifold of dimension 6, and the evaluation map $\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1) : \mathscr{M}_2(J, \beta_i) \to L_0 \times L_0$ is generically one-to-one.*

Then for the generator $[p] \in H_0(L_0; \mathbb{Z})$ we have

$$\mathfrak{m}_{1,\beta_1}([p]) = \mathfrak{m}_{1,\beta_2}([p]) = \mathrm{ev}_{0*}[\mathscr{M}_2(J, \beta_1)_{\mathrm{ev}_1} \times \{p\}] = [L_0],$$

and thus

$$\mathfrak{m}_1([p]) = \sum_{i=1}^{2} T^{\omega(\beta_i)} \mathfrak{m}_{1,\beta_i}([p]) = (T^{\lambda_1 - \lambda_2} + T^{\lambda_2 - \lambda_3})[L_0],$$

which proves the theorem.

4.2 Floer Homologies of $U(2)$-fibers in $\mathrm{Gr}(2,4)$

Assume that $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = \lambda > 0$, and set $L_t = \Phi^{-1}(t,t,t,t)$ as in Example 2.2. Recall that $\pi_2(\mathrm{Gr}(2,4)) \cong \mathbb{Z}$ is generated by a 1-dimensional Schubert variety X_1. Since $\pi_1(\mathrm{Gr}(2,4)) = \pi_2(L_t) = 0$ and $\pi_1(L_t) \cong \mathbb{Z}$, the exact sequence

$$0 \longrightarrow \pi_2(\mathrm{Gr}(2,4)) \longrightarrow \pi_2(\mathrm{Gr}(2,4), L_t) \longrightarrow \pi_1(L_t) \longrightarrow 0$$

implies that $\pi_2(\mathrm{Gr}(2,4), L_t) \cong \mathbb{Z}^2$. Let β_1, β_2 be generators of $\pi_2(\mathrm{Gr}(2,4), L_t)$ such that $\beta_1 + \beta_2 = [X_1] \in \pi_2(\mathrm{Gr}(2,4))$. The Maslov index and the symplectic area are given by

$$\mu_{L_t}(\beta_1) = \mu_{L_t}(\beta_2) = 4, \quad \omega(\beta_1) = \lambda + t, \quad \omega(\beta_2) = \lambda - t.$$

Since L_t is diffeomorphic to $U(2) \cong S^1 \times S^3$, we have $H^*(L_t) \cong H^*(S^1) \otimes H^*(S^3)$. Let $e_1 \in H^1(L_t; \mathbb{Z}) \cong H^1(S^1; \mathbb{Z})$ and $e_3 \in H^3(L_t; \mathbb{Z}) \cong H^3(S^3; \mathbb{Z})$ be generators. Since the minimal Maslov number is four, the only nontrivial parts of the Floer differential $\mathfrak{m}_1^b$ are

$$\mathfrak{m}_{1,\beta_i}^b : H^4(L_t; \Lambda_0) \longrightarrow H^1(L_t; \Lambda_0), \quad H^3(L_t; \Lambda_0) \longrightarrow H^0(L_t; \Lambda_0) \cong \Lambda_0$$

for $i = 1, 2$. By a similar argument to the proof of Theorem 4.1, we have the following.

Theorem 4.2. *For $b = xe_1 \in H^1(L_0; \Lambda_0/2\pi\sqrt{-1}\mathbb{Z}) \cong \Lambda_0/2\pi\sqrt{-1}\mathbb{Z}$, the Floer differential $\mathfrak{m}_1^b$ is given by*

$$\mathfrak{m}_1^b(e_3) = e^x T^{\lambda+t} + e^{-x} T^{\lambda-t},$$
$$\mathfrak{m}_1^b(e_1 \otimes e_3) = (e^x T^{\lambda+t} + e^{-x} T^{\lambda-t}) e_1.$$

Hence the Floer homologies of (L_t, b) are

$$HF((L_t,b),(L_t,b);\Lambda_0) \cong \begin{cases} H^*(L_0;\Lambda_0) & \text{if } t=0 \text{ and } x = \pm\pi\sqrt{-1}/2, \\ (\Lambda_0/T^{\min\{\lambda-t,\lambda+t\}}\Lambda_0)^2 & \text{otherwise}, \end{cases}$$
$$HF((L_t,b),(L_t,b);\Lambda) \cong \begin{cases} H^*(L_0;\Lambda) & \text{if } t=0 \text{ and } x = \pm\pi\sqrt{-1}/2, \\ 0 & \text{otherwise}. \end{cases}$$

Acknowledgements Y. N. is supported by Grant-in-Aid for Young Scientists (No.23740055). K. U. is supported by Grant-in-Aid for Young Scientists (No.24740043).

References

1. Batyrev, V., Ciocan-Fontanine, I., Kim, B., and van Straten, D.: Mirror symmetry and toric degenerations of partial flag manifolds. Acta Math. **184**, no. 1, 1–39 (2000)
2. Cho, C.-H., and Oh, Y.-G.: Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. Asian J. Math. **10**, 773–814 (2006)
3. Eguchi, T., Hori, K., and Xiong, C.-S.: Gravitational quantum cohomology. Internat. J. Modern Phys. A **12**, no. 9, 1743–1782 (1997)
4. Evans, J. D., and Lekili, Y.: Floer cohomology of the Chiang Lagrangian. arXiv:1401.4073
5. Fukaya, K., Oh, Y.-G., Ohta, H., and Ono, K.: Lagrangian Intersection Floer theory — Anomaly and obstructions—, Part I and Part II. AMS/IP Studies in Advanced Mathematics, 46 (2009)
6. Fukaya, K., Oh, Y.-G., Ohta, H., and Ono, K.: Lagrangian Floer theory on compact toric manifolds I. Duke Math. J. **151**, no. 1, 23–174 (2010)
7. Fukaya, K., Oh, Y.-G., Ohta, H., and Ono, K.: Lagrangian Floer theory and mirror symmetry on compact toric manifolds. arXiv:1009.1648
8. Fukaya, K., Oh, Y.-G., Ohta, H., and Ono, K.: Lagrangian Floer theory on compact toric manifolds: survey. In: Surveys in differential geometry. Vol. XVII, 229–298, Surv. Differ. Geom., 17, Int. Press, Boston, MA (2012)
9. Givental, A.: Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture. In: Topics in singularity theory, 103–115, Amer. Math. Soc. Transl. Ser. 2, 180, Amer. Math. Soc., Providence, RI (1997)
10. Guillemin, V., and Sternberg, S.: The Gelfand-Cetlin system and quantization of the complex flag manifolds. J. Funct. Annal. **52**, 106–128 (1983)
11. Nishinou, T., Nohara, Y., and Ueda, K.: Toric degenerations of Gelfand-Cetlin systems and potential functions. Adv. Math. **224**, 648–706 (2010)
12. Rietsch, K.: A mirror symmetric construction of $qH_T^*(G/P)_{(q)}$. Adv. Math. **217**, no. 6, 2401–2442 (2008)