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ON THE STABLE COHOMOLOGY ALGEBRA OF EXTENDED MAPPING CLASS GROUPS FOR SURFACES

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ON THE STABLE COHOMOLOGY ALGEBRA OF EXTENDED MAPPING CLASS GROUPS FOR SURFACES

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ABSTRACT. Let $\Sigma_{g,1}$ be an oriented compact surface of genus g with 1 boundary component, and $\Gamma_{g,1}$ the mapping class group of $\Sigma_{g,1}$. We determine the stable cohomology group of $\Gamma_{g,1}$ with coefficients in $H^1(\Sigma_{g,1};\mathbb{Z})^{\otimes n}$, $n \geq 1$, explicitly modulo the stable cohomology group with trivial coefficients. As a corollary the rational stable cohomology algebra of the semi-direct product $\Gamma_{g,1} \ltimes H_1(\Sigma_{g,1};\mathbb{Z})$ (which we call the *extended mapping class group*) is proved to be freely generated by the generalized Morita-Mumford classes $\widetilde{m_{i,j}}$'s ($i \geq 0$, $j \geq 1$, $i+j \geq 2$) [Ka] over the rational stable cohomology algebra of the group $\Gamma_{g,1}$.

INTRODUCTION

The most fundamental fact on (co)homology of mapping class groups for (compact C^∞) surfaces is the Harer stability theorem [H], which states the cohomology group of the mapping class group with trivial coefficients is independent of the genus g and the number of boundary components of the surface, provided that the degree is smaller than $g/3$ [H] or $g/2$ [I1]. Ivanov [I] has generalized this theorem to those with twisted coefficients in the case when the surface has boundaries, (which we call the 'bounded' case). It should be remarked, as for twisted coefficients, the existence of boundaries is essential. For example, the first homology group of the mapping class group with coefficients in the first homology group of the surface is isomorphic to $\mathbb{Z}$ (the 'bounded' case), $\mathbb{Z}/(2-2g)$ (the case when the surface has no boundary, which we call the 'closed'case) [Mo1]. These theorems enable us to consider the <u>stable</u> cohomology group of the mapping class groups for surfaces. When we consider trivial coefficients $\mathbb{Z}$ (resp. $\mathbb{Q}$), we denote it by $H^*(\Gamma_\infty;\mathbb{Z})$ (resp. $H^*(\Gamma_\infty;\mathbb{Q})$). At present the cohomology algebra $H^*(\Gamma_\infty;\mathbb{Q})$ (and so $H^*(\Gamma_\infty;\mathbb{Z})$) has not determined yet.

Recently Looijenga [L] proved that, in the 'closed'case, (i.e., the case when the surface has no boundary,) the <u>rational</u> stable cohomology group of the mapping class group with coefficients in any irreducible representation of the complex symplectic group was a free module over $H^*(\Gamma_\infty;\mathbb{Q})$, and described its free basis. His computation is involved with geometric consideration on the moduli orbifold of

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complex algebraic curves including a theorem on Hodge theory [D]. Here it is remarkable that his results are based only on the Harer stability theorem with trivial coefficients. This seems to suggest that the Harer stability theorem has not shown all its own power yet.

From a topological viewpoint the mapping class groups for surfaces with boundary (the 'bounded' ones) seem to play rather important roles than those for surfaces without boundary (the 'closed'ones). It is illustrated by the stabilization map in the Harer stability theorem [H], Johnson's important results on Torelli groups [J, J1, 2] [Mo3], Morita's descriptions of Casson invariants of 3-manifolds [Mo4, 5] and so on. So we think it would be significant to give a result for the 'bounded' case. Furthermore it is required for an infinitesimal approach to the stable cohomology of the moduli of complex analytic curves [ADKP] [Ka2,3].

In our previous paper [Ka] we independently constructed a bigraded series of cohomology classes of the mapping class group with coefficients in the exterior algebra on the first integral homology group of the surface in the 'bounded' case. This series is a generalizations of the Morita-Mumford classes [Mo] [Mu], and is easily modified to those with coefficients in the n-fold tensor product of the first integral homology group of the surface, $n \geq 1$.

In the present paper we consider only the 'bounded' case, i.e., the case when the surface has boundaries. Our purpose is to prove the stable cohomology group of the mapping class group with coefficients in the n-fold tensor product of the first <u>integral</u> cohomology group of the surface, $n \geq 1$, is a free module over the algebra $H^*(\Gamma_\infty; \mathbb{Z})$ and combinations of the (modified) generalized Morita-Mumford classes give its free basis (Theorems 1.A and 1.B). These imply the Ivanov stability with coefficients in any finite dimensional <u>rational</u> symplectic coefficients. Following Looijenga [L] we deduce them from the Harer stability theorem with trivial coefficients, but geometric considerations including Hodge theory do not fit our situation where the surface is not closed. We use the Lyndon-Hochschild-Serre spectral sequence for a pair of groups introduced in [Ka] instead. This makes our computation purely algebraic. As a corollary the rational cohomology algebra of the semi-direct product of the mapping class group and the first integral homology group of the surface (which we call *the extended mapping class group*) is proved to be stabilized and to be freely generated by the generalized Morita-Mumford classes over the rational stable cohomology algebra of the mapping class group in the stable range (Theorem 1.C).

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Contents.

1. Results.

We should fix our notations. Let $g \geq 2$, $r, s \geq 0$ be integers. Let $\Sigma^s_{g,r}$ denote a 2-dimensional oriented C^∞ manifold (i.e., oriented surface) of genus g with r boundary components and (ordered) s punctures. The group of path-components $\pi_0(\mathrm{Diff}^+(\Sigma^s_{g,r}))$ is denoted by $\Gamma^s_{g,r}$ (or $\mathcal{M}^s_{g,r}$) and called the mapping class group of genus g with r boundary components and (ordered) s punctures. Here $\mathrm{Diff}^+(\Sigma^s_{g,r})$ denotes the topological group (endowed with C^∞ topology) consisting of all orientation preserving diffeomorphisms of $\Sigma^s_{g,r}$ which fix all the boundary points and the punctures pointwise. When $s = 0$, we drop the indices: $\Sigma_{g,r} = \Sigma^0_{g,r}$, $\Gamma_{g,r} = \Gamma^0_{g,r}$ and similarly $\Sigma_g = \Sigma^0_{g,0}$, $\Gamma_g = \Gamma^0_{g,0}$. Throughout this paper we often denote by $H^1(\Sigma^s_{g,r})$ the first integral singular cohomology group of the space $\Sigma^s_{g,r}$, on which the group $\Gamma^t_{g,q}$ acts in an obvious way provided that $q \geq r$ and $t \geq s$. When $s = 0$ and $r = 1$, we often abbreviate

$$H := H_1(\Sigma_{g,1};\mathbb{Z}) = H_1(\Sigma_g;\mathbb{Z}) = H^1(\Sigma_g;\mathbb{Z}) = H^1(\Sigma_{g,1}). \tag{1.1}$$

The third isomorphism is the Poincaré duality, which is invariant under the action of the mapping class group Γ_g.

In view of the Harer stability theorem [H] there exists an integer $N(g)$ depending only on the genus g such that the forgetful map $\Gamma^s_{g,r+1} \to \Gamma^s_{g,r}$ given by forgetting the $(r+1)$-th boundary component induces an isomorphism

$$H^*(\Gamma^s_{g,r+1};\mathbb{Z}) = H^*(\Gamma^s_{g,r};\mathbb{Z})$$

for any $* \leq N(g)$ and $s, r \geq 0$. Harer [H] proved $N(g) \geq g/3$, and later Ivanov [I1] proved $N(g) \geq g/2$. Substituting this isomorphism into the Gysin sequence induced by a natural central extension

$$0 \to \mathbb{Z} \to \Gamma^s_{g,r+1} \to \Gamma^{s+1}_{g,r} \to 1 \tag{1.2}$$

given by mapping the $(r+1)$-th boundary component to the $(s+1)$-th puncture, we obtain a natural decomposition

$$H^*(\Gamma^{s+1}_{g,r};\mathbb{Z}) = H^*(\Gamma^s_{g,r};\mathbb{Z}) \oplus eH^{*-2}(\Gamma^{s+1}_{g,r};\mathbb{Z}) = H^*(\Gamma^s_{g,r};\mathbb{Z})[e] \tag{1.3}$$

for $* \leq N(g)$, where we denote by $e \in H^2(\Gamma^{s+1}_{g,r};\mathbb{Z})$ the Euler class of the central extension (1.2) (cf. [Mo] [H1] [L]).

Our first theorem in the present paper is

Theorem 1.A. *If $s \geq 0$, $r \geq 1$ and $n \geq 0$, we have*

$$H^*(\Gamma^s_{g,r}; H^1(\Sigma^s_{g,r};\mathbb{Z})^{\otimes n}) = H^*(\Gamma_{g,1}; H^{\otimes n}) \otimes_{H^*(\Gamma_{g,1};\mathbb{Z})} H^*(\Gamma^s_{g,r};\mathbb{Z})$$

for $* \leq N(g) - n$.

As a consequence one deduce the Ivanov stability theorem [I] for the $\Gamma^s_{g,r}$-module $H^1(\Sigma^s_{g,r};\mathbb{Z})^{\otimes n}$ and those for any finite dimensional rational Sp-modules.

To describe the cohomology group $H^*(\Gamma_{g,1}; H^{\otimes n})$ we need to introduce some notions related to the mapping class groups. From the observation that the surface $\Sigma^1_{g,1}$ is obtained by glueing the surfaces $\Sigma_{g,1}$ and $\Sigma^1_{0,2}$ along the boundaries the group $\Gamma_{g,1} \times \mathbb{Z}$ is embedded into the group $\Gamma^1_{g,1}$ (cf. e.g., [Is]). Here the infinite cyclic group $\mathbb{Z}$ acts on the surface $\Sigma^1_{0,2}$ by rotating the puncture and fixing all the boundary points pointwise. The Lyndon-Hochschild-Serre spectral sequence of the pair of groups $(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z})$ introduced in [Ka]§1 induces the fiber integral

$$\pi_! : H^q(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M) \to H^{q-2}(\Gamma_{g,1}; M) \tag{1.4}$$

for any $\Gamma_{g,1}$-module M. Here we denote by $H^q(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M)$ the q-th cohomology group of the kernel of the restriction map

$$C^*(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; M) := \mathrm{Ker}(C^*(\Gamma^1_{g,1}; M) \to C^*(\Gamma_{g,1} \times \mathbb{Z}; M))$$

of the <u>normalized</u> standard cochain complexes $C^*(\cdot;\cdot)$.

The cohomology class ω defined below plays an important role throughout this paper. Regard the surface $\Sigma^1_{g,1}$ as a subsurface obtained by deleting one interior point from the surface $\Sigma_{g,1}$. The cohomology exact sequence of the pair of spaces $(\Sigma_{g,1}, \Sigma^1_{g,1})$ gives a $\Gamma^1_{g,1}$-exact sequence

$$0 \to H^1(\Sigma_{g,1}) = H \to H^1(\Sigma^1_{g,1}) \to H^2(\Sigma_{g,1}, \Sigma^1_{g,1}) = \mathbb{Z} \to 0. \tag{1.5}$$

We denote by ω the image of $1 \in \mathbb{Z} = H^0(\Gamma^1_{g,1}; \mathbb{Z})$ under the connecting homomorphism δ^* induced by (1.5):

$$\omega := \delta^*(1) \in H^1(\Gamma^1_{g,1}; H). \tag{1.6}$$

The restriction of ω to the subgroup $\Gamma_{g,1} \times \mathbb{Z} (\subset \Gamma^1_{g,1})$ is null-cohomologous. In fact, choose a simple curve l inside the subsurface $\Sigma^1_{0,2} (\subset \Sigma^1_{g,1})$ connecting the puncture to a point on the boundary of $\Sigma^1_{g,1}$. The 1-cocycle $\omega_l \in Z^1(\Gamma^1_{g,1}; H)$ given by

$$\omega_l(\gamma) = \gamma l - l \in H, \quad \gamma \in \Gamma^1_{g,1}, \tag{1.7}$$

represents the cohomology class $\omega \in H^1(\Gamma^1_{g,1}; H)$. Clearly we have $\omega_l(\gamma) = 0$ for any $\gamma \in \Gamma_{g,1} \times \mathbb{Z}$.

Thus, in view of the cohomology exact sequence

$$0 \to H^1(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H) \to H^1(\Gamma^1_{g,1}; H) \to H^1(\Gamma_{g,1} \times \mathbb{Z}; H),$$

we may regard ω as a (uniquely determined) element of $H^1(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H)$:

$$\omega \in H^1(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H).$$

For a finite subset S of $\mathbb{N}$ we form the power of ω

$$\omega^S \in H^{\sharp S}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H^{\otimes S}),$$

which we multiply in numerical order. Let $i \geq 0$ be an integer. Under the condition

$$i + \sharp S \geq 2. \tag{1.8}$$

we define *the generalized Morita-Mumford class* $m_{i,S}$ by

$$m_{i,S} := \pi_!(e^i \omega^S) \in H^{2i+\sharp S-2}(\Gamma_{g,1}; H^{\otimes S}), \tag{1.9}$$

where $\pi_!$ is the fiber integral given in (1.4) and $e \in H^2(\Gamma_{g,1}; \mathbb{Z})$ is the Euler class of the central extension (1.2) ($r = 1$, $s = 0$).

Definition 1.10. *A set $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \dots, (S_\nu, i_\nu)\}$ is a weighted partition* * *of the index set $\{1, 2, \dots, n\}$ if*

(1) *The set $\{S_1, S_2, \dots, S_\nu\}$ is a partition of the set $\{1, 2, \dots, n\}$:*

$$\{1, 2, \dots, n\} = \coprod_{a=1}^{\nu} S_a, \quad S_a \neq \emptyset \quad (1 \leq \forall a \leq \nu).$$

(2) *$i_1, i_2, \dots, i_\nu \geq 0$ are non-negative integers.*

(3) *Each (S_a, i_a) satisfies the condition (1.8): $i_a + \sharp S_a \geq 2$.*

We denote by $\mathcal{P}_n$ the set consisting of all weighted partition of the index set $\{1, 2, \dots, n\}$. For each weighted partition $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \dots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$ we define *the generalized Morita-Mumford class*

$$m_{\widehat{P}} := m_{i_1, S_1} m_{i_2, S_2} \cdots m_{i_\nu, S_\nu} \in H^{2(\sum i_a) + n - 2\nu}(\Gamma_{g,1}; H^{\otimes n}). \tag{1.11}$$

Theorem 1.B. *For $* \leq N(g) - n$*

$$H^*(\Gamma_{g,1}; H^{\otimes n}) = \bigoplus_{\widehat{P} \in \mathcal{P}_n} H^*(\Gamma_{g,1}; \mathbb{Z}) m_{\widehat{P}} \cong H^*(\Gamma_{g,1}; \mathbb{Z})^{\oplus \mathcal{P}_n}.$$

By the *extended mapping class group* we mean the semi-direct product

$$\widetilde{\Gamma^s_{g,r}} := H \rtimes \Gamma^s_{g,r} = H_1(\Sigma_{g,1}; \mathbb{Z}) \rtimes \Gamma^s_{g,r}.$$

The generalized Morita-Mumford classes $\widetilde{m_{i,j}} \in H^*(\widetilde{\Gamma_{g,1}}; \mathbb{Z})$ are constructed as follows [Ka]. In a similar way to $\Gamma_{g,1} \times \mathbb{Z} \subset \Gamma^1_{g,1}$ the group $\widetilde{\Gamma_{g,1}} \times \mathbb{Z}$ is embedded into the group $\widetilde{\Gamma^1_{g,1}}$. Using the simple curve l in (1.7), we define a 2-cocycle $\widetilde{\omega_l} \in Z^2(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z})$ by

$$\widetilde{\omega_l}(u_1\gamma_1, u_2\gamma_2) := \gamma_1(\gamma_2 l - l) \cdot u_1, \quad u_1, u_2 \in H, \gamma_1, \gamma_2 \in \Gamma^1_{g,1}, \tag{1.12}$$

where $\cdot$ denotes the intersection product on $H = H_1(\Sigma_g; \mathbb{Z})$. Its image $\widetilde{\omega}$ in $H^2(\widetilde{\Gamma^1_{g,1}}; \mathbb{Z})$ is equal to the Euler class of the central extension

$$0 \to \mathbb{Z} \to H_1(\Sigma^1_{g,1}; \mathbb{Z}) \rtimes \Gamma^1_{g,1} \to H_1(\Sigma_{g,1}; \mathbb{Z}) \rtimes \Gamma^1_{g,1} = \widetilde{\Gamma^1_{g,1}} \to 1.$$

The forgetful map $\widetilde{\pi} : \widetilde{\Gamma^1_{g,1}} \to \widetilde{\Gamma_{g,1}}$ induces the fiber integral

$$\widetilde{\pi}_! : H^q(\widetilde{\Gamma^1_{g,1}}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z}) \to H^{q-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}). \tag{1.13}$$

Thus we can define *the generalized Morita-Mumford class*

$$\widetilde{m_{i,j}} := \widetilde{\pi}_!(e^i \widetilde{\omega}^j) \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z})$$

for $i \geq 0$, $j \geq 0$ with $i + j \geq 2$. Clearly $\widetilde{m_{i+1,0}}$ is equal to (the image of) the i-th Morita-Mumford (tautological) class $e_i (= \kappa_i) \in H^{2i}(\Gamma_g; \mathbb{Z})$ [Mo][Mu]:

$$\widetilde{m_{i+1,0}} = e_i \in H^{2i}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}).$$

Theorem 1.C.

$$H^*(\widetilde{\Gamma_{g,1}}; \mathbb{Q}) = H^*(\Gamma_{g,1} \ltimes H_1(\Sigma_{g,1}; \mathbb{Z}); \mathbb{Q}) = H^*(\Gamma_{g,1}; \mathbb{Q}) \otimes \mathbb{Q}[\widetilde{m_{i,j}}]$$

for $* \leq N(g)$, where integers i and j run over the domain

$$\{(i, j) \in \mathbb{Z} \times \mathbb{Z}; \quad i \geq 0, j \geq 1 \, \text{and} \, i + j \geq 2\}.$$

---

*We use the term 'partition of a set' following Stanley [S] p.33.

2. Stable Cohomology with Coefficients in $H^1(\Sigma_{g,1};\mathbb{Z})^{\otimes n}$.

This section is devoted to the proof of Theorems 1.A and 1.B.

Suppose $r, s \geq 1$. We define the forgetful map

$$\pi : \Gamma_{g,r}^{s} \to \Gamma_{g,r}^{s-1}$$

by forgetting the s-th puncture, and define

$$\varpi : \Gamma_{g,r}^{s} \to \Gamma_{g,1}^{1} \quad \text{and} \quad \varpi : \Gamma_{g,r}^{s-1} \to \Gamma_{g,1}$$

by forgetting the punctures from the first to the $(s-1)$-th and the boundary components except the first. We have a natural commutative diagram

$$\begin{array}{ccc} \Gamma_{g,r}^{s} & \xrightarrow{\pi} & \Gamma_{g,r}^{s-1} \\ {\scriptstyle\varpi}\downarrow & & {\scriptstyle\varpi}\downarrow \\ \Gamma_{g,1}^{1} & \xrightarrow{\pi} & \Gamma_{g,1}. \end{array} \tag{2.1}$$

We regard the surface $\Sigma_{g,r}^{s}$ as a subsurface obtained by deleting one interior point from the surface $\Sigma_{g,r}^{s-1}$ and numbering the resulting puncture the s-th. The inclusion homomorphism $H^1(\Sigma_{g,r}^{s-1}) \to H^1(\Sigma_{g,r}^{s})$ is equivariant under the forgetful map $\pi : \Gamma_{g,r}^{s} \to \Gamma_{g,r}^{s-1}$, and so induces a $\Gamma_{g,r}^{s}$-exact sequence

$$0 \to H^1(\Sigma_{g,r}^{s-1}) \to H^1(\Sigma_{g,r}^{s}) \to H^2(\Sigma_{g,r}^{s-1}, \Sigma_{g,r}^{s}) = \mathbb{Z} \to 0. \tag{2.2}$$

We denote by $\omega = \omega_{(s-1)}$ the image of $1 \in \mathbb{Z} = H^0(\Gamma_{g,r}^{s};\mathbb{Z})$ under the connecting homomorphism δ^* induced by (2.2):

$$\omega = \omega_{(s-1)} := \delta^*(1) \in H^1(\Gamma_{g,r}^{s}; H^1(\Sigma_{g,r}^{s-1})). \tag{2.3}$$

From the commutative diagram (2.1) the homomorphism induced by the forgetful map ϖ

$$H^1(\Gamma_{g,1}^{1}; H) \to H^1(\Gamma_{g,r}^{s}; H) \to H^1(\Gamma_{g,r}^{s}; H^1(\Sigma_{g,r}^{s-1}))$$

maps ω defined in (1.6) to $\omega = \omega_{(s-1)}$ defined in (2.3).

The kernel of the forgetful map $\pi : \Gamma_{g,r}^{s+1} \to \Gamma_{g,r}^{s}$ is naturally isomorphic to $\pi_1(\Sigma_{g,r}^{s})$ $(s \geq 0)$, and so we have a Gysin exact sequence

$$\begin{aligned} \cdots \to H^q(\Gamma_{g,r}^{s}; M) \xrightarrow{\pi^*} H^q(\Gamma_{g,r}^{s+1}; M) \xrightarrow{\pi_\sharp} \\ H^{q-1}(\Gamma_{g,r}^{s}; H^1(\Sigma_{g,r}^{s}) \otimes M) \to H^{q+1}(\Gamma_{g,r}^{s}; M) \to \cdots \end{aligned} \tag{2.4}$$

for any $\Gamma_{g,r}^{s}$-module M. Here we denote the Gysin map (the fiber integral) by $\pi_\sharp$ to know it from the fiber integral $\pi_!$ introduced in (1.4).

Theorem 2.5. *Let $s \geq 0$, $r \geq 1$ and let I, J be mutually disjoint finite index sets. Assume $s \geq 1$ if $I \neq \emptyset$. Then the forgetful map ϖ induces an isomorphism*

$$\begin{aligned} &H^*(\Gamma^s_{g,r}; H^1(\Sigma^{s-1}_{g,r})^{\otimes I} \otimes H^1(\Sigma^s_{g,r})^{\otimes J}) \\ =&(\bigoplus_{S \subset I} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J \cup I - S)})) \otimes_{H^*(\Gamma_{g,1};\mathbb{Z})} H^*(\Gamma^s_{g,r}; \mathbb{Z}) \end{aligned}$$

for $* \leq N(g) - \sharp(I \cup J)$, where $\omega^S \in H^{\sharp S}(\Gamma^1_{g,1}; H^{\otimes S})$ is the power of the class ω defined in (1.6).

Especially, if $I = \emptyset$ and $J = \{1, 2, \ldots, n\}$, we obtain Theorem 1.A stated in §1.

Proof. We abbreviate $H_{(s)} := H^1(\Sigma^s_{g,r}) = H^1(\Sigma^s_{g,r}; \mathbb{Z})$. So the $\Gamma^s_{g,r}$-exact sequence (2.2) is rewritten to

$$0 \to H_{(s-1)} \to H_{(s)} \to \mathbb{Z} \to 0. \tag{2.6}$$

We prove the theorem by a double induction on $\sharp(I \cup J)$ and $\sharp I$. When $I \cup J = \emptyset$, the theorem is trivial. Suppose $\sharp(I \cup J) \geq 1$.

(A). The case $I = \emptyset$: Choose an index $j_0 \in J$ and set $J_- := J - \{j_0\}$. From (1.3) and the inductive assumption applied to $H^*(\Gamma^{s+1}_{g,r}; H_{(s)}{}^{\otimes J_-})$ and $H^*(\Gamma^s_{g,r}; H_{(s)}{}^{\otimes J_-})$ the forgetful homomorphism

$$\pi^* : H^*(\Gamma^s_{g,r}; H_{(s)}{}^{\otimes J_-}) \to H^*(\Gamma^{s+1}_{g,r}; H_{(s)}{}^{\otimes J_-})$$

has a left inverse over $H^*(\Gamma^s_{g,r})$, and so the Gysin sequence (2.4) decomposes itself into the $H^*(\Gamma^s_{g,r})$-split exact sequences

$$\begin{aligned} &0 \to H^*(\Gamma^s_{g,r}; H_{(s)}{}^{\otimes J_-}) \to \\ &\qquad H^*(\Gamma^{s+1}_{g,r}; H_{(s)}{}^{\otimes J_-}) \xrightarrow{\pi_!} H^{*-1}(\Gamma^s_{g,r}; H_{(s)} \otimes H_{(s)}{}^{\otimes J_-}) \to 0, \end{aligned} \tag{2.7}$$

for $* \leq N(g) - \sharp J_-$. When $s = 0$ and $r = 1$, we have

$$0 \to H^*(\Gamma_{g,1}; H^{\otimes J_-}) \to H^*(\Gamma^1_{g,1}; H^{\otimes J_-}) \xrightarrow{\pi_!} H^{*-1}(\Gamma_{g,1}; H \otimes H^{\otimes J_-}) \to 0, \tag{2.8}$$

for $* \leq N(g) - \sharp J_-$. Compare the exact sequence (2.7) with the sequence given by applying $\otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r})$ to the sequence (2.8). Then the forgetful map ϖ induces an isomorphism

$$\varpi^* : H^*(\Gamma^s_{g,r}; H_{(s)} \otimes H_{(s)}{}^{\otimes J_-}) = H^*(\Gamma_{g,1}; H \otimes H^{\otimes J_-}) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r})$$

for $* \leq N(g) - \sharp J$. Here we use the fact the map ϖ induces an isomorphism

$$\varpi^* : H^*(\Gamma^1_{g,1}) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \cong H^*(\Gamma^{s+1}_{g,r})$$

from (1.3) and (2.1). Finally label the first H and $H_{(s)}$ the index j_0. Thus the induction proceeds.

(B). The case $I \neq \emptyset$: Then $s \geq 1$ and so choose an index $i_0 \in I$. Set $I_- := I - \{i_0\}$ and $J_0 := J \cup \{i_0\}$. The $\Gamma^s_{g,r}$-exact sequence (2.6) induces a $\Gamma^s_{g,r}$-exact sequence

$$(2.9)\quad 0 \to H_{(s-1)}{}^{\otimes I} \otimes H_{(s)}{}^{\otimes J} \to H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0} \to H_{(s-1)}{}^{\otimes I_-} \otimes \mathbb{Z}^{\otimes\{i_0\}} \otimes H_{(s)}{}^{\otimes J} \to 0.$$

By the inductive assumption applied to the index sets I_- and J_0

$$\begin{aligned} &H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0}) \\ &\qquad = (\bigoplus_{S \subset I_-} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J_0 \cup I_- - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}). \end{aligned}$$

Each element of the RHS is an image of a (uniquely determined) element of $H^*(\Gamma^s_{g,r}; H^{\otimes(I_- \cup J_0)})$. Hence the map

$$H^*(\Gamma^s_{g,r}; H^{\otimes(I_- \cup J_0)}) \to H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0})$$

has a left inverse for $* \leq N(g) - \sharp(I \cup J)$. Therefore the cohomology exact sequence induced by the $\Gamma^s_{g,r}$-exact sequence (2.9) decomposes itself into a split exact sequence

$$\begin{aligned} 0 \to H^{*-1}(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J}) &\overset{\omega^{\{i_0\}}\otimes}{\to} \\ H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I} \otimes H_{(s)}{}^{\otimes J}) &\to H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0}) \to 0 \end{aligned}$$

for $* \leq N(g) - \sharp(I \cup J)$. Thus we have

$$\begin{aligned} &H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I} \otimes H_{(s)}{}^{\otimes J}) \\ =&H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J_0}) \oplus (\omega^{\{i_0\}} \otimes H^*(\Gamma^s_{g,r}; H_{(s-1)}{}^{\otimes I_-} \otimes H_{(s)}{}^{\otimes J})) \\ =&(\bigoplus_{S \subset I_-} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J_0 \cup I_- - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \\ &\oplus (\bigoplus_{S \subset I_-} \omega^{S \cup \{i_0\}} \otimes H^*(\Gamma_{g,1}; H^{\otimes(J \cup I_- - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}) \\ =&(\bigoplus_{S \subset I} \omega^S \otimes H^*(\Gamma_{g,1}; H^{\otimes(J \cup I - S)})) \otimes_{H^*(\Gamma_{g,1})} H^*(\Gamma^s_{g,r}). \end{aligned}$$

for $* \leq N(g) - \sharp(I \cup J)$. This completes the induction. □

We have introduced two sorts of the fiber integrals or the Gysin map induced by the forgetful map $\pi : \Gamma^1_{g,1} \to \Gamma_{g,1}$ in (1.4) and (2.4). These two Gysin maps are related to each other in the following manner.

Lemma 2.10. *For any $\Gamma_{g,1}$-module M we have*

$$\begin{tikzcd} H^p(\Gamma^1_{g,1}; M) \arrow[r, equal] \arrow[d, "\omega\cup"'] & H^p(\Gamma^1_{g,1}; M) \arrow[d, "\pi_!"'] \\ H^{p+1}(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; H \otimes M) \arrow[r, "\pi_!"] & H^p(\Gamma^1_{g,1}; H \otimes M). \end{tikzcd}$$

$(p-1)$

Here we abbreviate $H = H^1(\Sigma_{g,1};\mathbb{Z})$ as in (1.1).

Proof. Let G be a group, K a subgroup of G and M a G-module. We define the cohomology group $H^*(G,K;M)$ by that of the kernel of the restriction map

$$C^*(G,K;M) := \mathrm{Ker}(C^*(G;M) \to C^*(K;M))$$

of the normalized standard cochain complexes $C^*(\cdot;\cdot)$ [Ka]§1. Let N be a normal subgroup of G satisfying the condition: $KN = G$. In [HS] p.118 l.27ff and p.119 l.6ff two mutually equivalent filtrations (A_j) and (A^*_j) are introduced on the normalized standard cochain complex and they induce the (ordinary) Lyndon-Hochschild-Serre spectral sequence. The filtration (A^*_j) (or equivalently (A_j)) restricted to $C^*(G,K;M)$ induces the Lyndon-Hochschild-Serre spectral sequence of pairs of groups [Ka]:

$$E_2^{p,q} = H^p(G/N; H^q(N, N\cap K; M)) \Rightarrow H^{p+q}(G,K;H).$$

In our situation $G = \Gamma^1_{g,1}$, $K = \Gamma_{g,1} \times \mathbb{Z}$ and $N = \pi_1(\Sigma_{g,1}) \triangleleft \Gamma^1_{g,1}$. Since $H^{p-i}(\Gamma_{g,1}; H^i(\pi_1(\Sigma_{g,1}); M)) = 0$ for $i \geq 2$, any $u \in H^p(\Gamma^1_{g,1}; M)$ is represented by a cocycle z whose value $z(\gamma_1,\gamma_2,\dots,\gamma_p)$, $\gamma_1,\gamma_2,\dots,\gamma_p \in \Gamma^1_{g,1}$, depends only on the cosets $\gamma_2\pi_1(\Sigma_{g,1}),\dots,\gamma_p\pi_1(\Sigma_{g,1})$ and γ_1. We denote by $r_{p-1}z$ the cocycle given by restricting γ_1 into $\pi_1(\Sigma_{g,1})$ and regarding $\gamma_2,\dots,\gamma_p$ as elements of $\Gamma_{g,1} = \Gamma^1_{g,1}/\pi_1(\Sigma_{g,1})$. By definition we have $\pi_\sharp u = [r_{p-1}z] \in H^{p-1}(\Gamma_{g,1}; H\otimes M)$.

On the other hand the cocycle $\omega\cup z$ defined by

$$(\omega\cup z)(\gamma_0,\gamma_1,\dots,\gamma_p) = \omega(\gamma_0)\otimes\gamma_0(z(\gamma_1,\gamma_2,\dots,\gamma_p)), \quad \gamma_0,\gamma_1,\dots,\gamma_p \in \Gamma^1_{g,1}$$

represents the cup product $\omega\cup u \in H^{p+1}(\Gamma^1_{g,1},\Gamma_{g,1}\times\mathbb{Z}; H\otimes M)$. The value $(\omega\cup z)(\gamma_0,\gamma_1,\dots,\gamma_p)$ depends only on the cosets $\gamma_2\pi_1(\Sigma_{g,1}),\dots,\gamma_p\pi_1(\Sigma_{g,1})$ and γ_0, γ_1.

Thus, from a computation involved with [Ka] Lemma 2.3,

$$\pi_!(\omega\cup z)(\gamma_2,\dots,\gamma_p) = \sum_{i=1}^{g} -a_i\otimes z(b_i,\gamma_2\dots,\gamma_p) - b_i\otimes z({a_i}^{-1},\gamma_2\dots,\gamma_p),$$

where $\{a_1,a_2,\dots,a_g,b_1,b_2,\dots,b_g\}$ is a usual symplectic generating system of the fundamental group $\pi_1(\Sigma_{g,1})$. This implies $\pi_!(\omega\cup z) = r_{p-1}z$ and so $\pi_!(\omega\cup u) = [\pi_!(\omega\cup z)] = [r_{p-1}z] = \pi_\sharp u$, as was to be shown. □

Proof of Theorem 1.B. We prove the theorem by induction on n. When $n = 0$, the theorem is trivial, so we assume $n \geq 1$. Set $J = \{1,2,\dots,n\}$, $j_0 = 1$ and $J_- = \{2,\dots,n\}$. Recall the exact sequence (2.8) in the proof of Theorem 2.5. From Theorem 2.5 and (1.3) the Gysin map $\pi_\sharp$ restricted to

$$(H^*(\Gamma_{g,1}; H^{\otimes J_-})\otimes e\mathbb{Z}[e]) \oplus \bigoplus_{\emptyset\neq S\subset J_-} \omega^S\otimes H^*(\Gamma_{g,1}; H^{\otimes(J_- - S)})\otimes\mathbb{Z}[e]$$

is an isomorphism onto $H^{*-1}(\Gamma_{g,1}; H^{\otimes J})$ for $* \leq N(g) - n + 1$. We denote $S_+ := S\cup\{1\}$ for $S\subset J_-$. Lemma 2.10 implies

$$\pi_\sharp(e^i\omega^S) = \pi_!(e^i\omega^{S_+}) = m_{i,S_+} \in H^*(\Gamma_{g,1}; H^{\otimes S_+}).$$

Therefore from the inductive assumption we obtain

$$\begin{aligned} & H^*(\Gamma_{g,1}; H^{\otimes n}) = H^*(\Gamma_{g,1}; H^{\otimes J}) \\ = & \bigoplus_{\emptyset \neq S \subset J_-} \bigoplus_{i=0}^{\infty} m_{i,S_+} H^*(\Gamma_{g,1}; H^{\otimes (J_- - S)}) \oplus \bigoplus_{i=1}^{\infty} m_{i,\{1\}} H^*(\Gamma_{g,1}; H^{\otimes J_-}) \\ = & \bigoplus_{\widehat{P} \in \mathcal{P}_n} m_{\widehat{P}} H^*(\Gamma_{g,1}; \mathbb{Z}) \end{aligned}$$

for $* \leq N(g) - n$, which completes the induction. □

3. Stable Cohomology Algebra of Extended Mapping Class Groups.

Let $i \geq 0$, $j \geq 1$ be integers with $i + j \geq 2$. As in [Ka], we define

$$m_{i,j} := \pi_!(e^i \omega^j) \in H^{2i+j-2}(\Gamma_{g,1}; \textstyle\bigwedge^j H),$$

where $\pi_!$ is the fiber integral (1.4), and $\omega^j \in H^j(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; \bigwedge^j H)$ is the power of ω.

The n-th symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of the weighted partitions of the set $\{1, 2, \ldots, n\}$ by

$$\sigma \widehat{P} := \{(\sigma(S_1), i_1), (\sigma(S_2), i_2), \ldots, (\sigma(S_\nu), i_\nu)\},$$

where $\sigma \in \mathfrak{S}_n$ and $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \ldots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$. The $\mathfrak{S}_n$-orbits in $\mathcal{P}_n$ are parametrized by the set $\mathcal{Q}_n$ defined as follows.

Definition 3.1. *A sequence $\widehat{Q} = ((j_1, i_1), (j_2, i_2), \ldots, (j_\nu, i_\nu))$ is a weighted partition of the number n if*

(1) *The sequence $(j_1, j_2, \ldots, j_\nu)$ is a partition of the number n:*

$$j_1 + j_2 + \cdots + j_\nu = n, \quad j_1 \geq j_2 \geq \cdots \geq j_\nu \geq 1.$$

(2) *$i_1, i_2, \ldots, i_\nu \geq 0$ are non-negative integers.*
(3) *$i_a \geq i_{a+1}$ if $j_a = j_{a+1}$.*
(4) *Each (j_a, i_a) satisfies the condition: $i_a + j_a \geq 2$.*

We denote by $\mathcal{Q}_n$ the set consisting of all weighted partition of the number n. Define

$$\lambda \widehat{P} := ((\sharp S_1, i_1), (\sharp S_2, i_2), \ldots, (\sharp S_\nu, i_\nu)) \in \mathcal{Q}_n$$

for $\widehat{P} = \{(S_1, i_1), (S_2, i_2), \ldots, (S_\nu, i_\nu)\} \in \mathcal{P}_n$ provided that $\sharp S_1 \geq \sharp S_2 \geq \cdots \geq \sharp S_\nu$ and $\sharp S_a = \sharp S_{a+1} \Rightarrow i_a \geq i_{a+1}$, $(1 \leq a < \nu)$. Then the map $\lambda : \mathcal{P}_n \to \mathcal{Q}_n$, $\widehat{P} \mapsto \lambda \widehat{P}$ induces a bijection $\lambda : \mathcal{P}_n / \mathfrak{S}_n = \mathcal{Q}_n$. Set

$$m_{\widehat{Q}} := m_{i_1,j_1} m_{i_2,j_2} \cdots m_{i_\nu,j_\nu} \in H^{\sum(2i_a + j_a - 1)}(\Gamma_{g,1}; \textstyle\bigwedge^n H)$$

for $\widehat{Q} = ((j_1, i_1), (j_2, i_2), \ldots, (j_\nu, i_\nu)) \in \mathcal{Q}_n$. The canonical projection $\lambda : H^{\otimes n} \to \bigwedge^n H$ maps $m_{\widehat{P}}$ to $\pm m_{\lambda \widehat{P}}$ for any $\widehat{P} \in \mathcal{P}_n$:

$$\lambda_*(m_{\widehat{P}}) = \pm m_{\lambda \widehat{P}} \in H^*(\Gamma_{g,1}; \textstyle\bigwedge^n H). \tag{3.2}$$

Theorem 3.3. *Let* $\mathbf{k}$ *be a field with* $\operatorname{ch}\mathbf{k} > n$ *or* $= 0$. *Then we have*

$$H^*(\Gamma_{g,1}; \textstyle\bigwedge^n H^1(\Sigma_{g,1};\mathbf{k})) = \displaystyle\bigoplus_{\widehat{Q}\in\mathcal{Q}_n} H^*(\Gamma_{g,1};\mathbf{k})m_{\widehat{Q}} \cong H^*(\Gamma_{g,1};\mathbf{k})^{\oplus\mathcal{Q}_n}$$

for $* \leq N(g) - n$.

As a corollary we obtain Theorem 1.C. In fact, the Lyndon-Hochschild-Serre spectral sequence of the group extension $H \to \widetilde{\Gamma_{g,1}} = H \rtimes \Gamma_{g,1} \to \Gamma_{g,1}$ is given by

$$E_2^{p,q} = H^p(\Gamma_{g,1}; \textstyle\bigwedge^q H) \Rightarrow H^{p+q}(\widetilde{\Gamma_{g,1}}),$$

and the class $m_{i,j} \in E_2^{2i+j-2,j}$ is lifted to the class $\widetilde{m_{i,j}} \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}})$.

Proof of Theorem 3.3. We fix the order in each $\widehat{P} = \{(S_1,i_1),(S_2,i_2),\dots,(S_\nu,i_\nu)\} \in \mathcal{P}_n$ as follows;

(1) $\sharp S_1 \geq \sharp S_2 \geq \cdots \geq \sharp S_\nu$.
(2) If $\sharp S_a = \sharp S_{a+1}$, $i_a \geq i_{a+1}$.
(3) If $\sharp S_a = \sharp S_{a+1}$ and $i_a = i_{a+1}$, then the minimal element of S_a is smaller than that of S_{a+1}.

Furthermore we denote

$$\tau_{\widehat{P}} = \begin{pmatrix} 12 & \cdots & n \\ S_1 & \cdots & S_\nu \end{pmatrix} \in \mathfrak{S}_n,$$

where the indices are set in numerical order inside each subset S_a. The n-th symmetric group $\mathfrak{S}_n$ acts on $H^{\otimes n}$ by

$$\sigma(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)}$$

for $\sigma \in \mathfrak{S}_n$, $u_i \in H$ $(1 \leq i \leq n)$. From the (anti-) commutativity of cup products we have $\tau_* m_{i,S} = m_{i,S} \in H^*(\Gamma_{g,1}; H^{\otimes S})$, if $S \subset \{1,2,\dots,n\}$, $\tau \in \mathfrak{S}_n$ and $\tau(S) = S$. Since $\deg m_{i,S} \equiv \sharp S \mod 2$, we have

$$\begin{pmatrix} S & T \\ T & S \end{pmatrix}_* m_{i,S}m_{j,T} = m_{j,S}m_{i,T}$$

for $S,T \subset \{1,2,\dots,n\}$, $S \cap T = \emptyset$. Hence, if $\tau \in \mathfrak{S}_n$, $\widehat{P} \in \mathcal{P}_n$ and $\tau(\widehat{P}) = \widehat{P}$, then

$$\tau_* m_{\widehat{P}} = m_{\widehat{P}} \in H^*(\Gamma_{g,1}; H^{\otimes n}). \tag{3.4}$$

Especially for any $\sigma \in \mathfrak{S}_n$ the permutation $\sigma^{-1}\tau_{\sigma(\widehat{P})}\tau_{\widehat{P}}^{-1}$ fixes $m_{\widehat{P}}$, and so we have

$$\sigma_* m_{\widehat{P}} = \tau_{\sigma(\widehat{P})_*}\tau_{\widehat{P}}{}^{-1}{}_* m_{\widehat{P}} = (\operatorname{sign}\tau_{\sigma(\widehat{P})}\tau_{\widehat{P}}{}^{-1})m_{\sigma(\widehat{P})}.$$

Therefore for any $\widehat{P_0} \in \mathcal{P}_n$ the sum $\sum_{\widehat{P}\in\lambda^{-1}\lambda(\widehat{P_0})}(\operatorname{sign}\tau_{\widehat{P}})m_{\widehat{P}}$ is invariant under the $\mathfrak{S}_n$-action. This implies $m_{\lambda\widehat{P_0}} \neq 0$ in $H^*(\Gamma_{g,1}; \bigwedge^n H \otimes \mathbf{k})$.

The group $H^*(\Gamma_{g,1};H^{\otimes n})$ decomposes itself into a direct sum of $\mathfrak{S}_n$-submodules parametrized by $\mathcal{Q}_n=\mathcal{P}_n/\mathfrak{S}_n$, which implies the independency of $m_{\widehat{Q}}$'s, $\widehat{Q}\in\mathcal{Q}_n$.

From the assumption on the characteritic of the field $\mathbf{k}$ the map $\lambda_*: H^*(\Gamma_{g,1};(H\otimes\mathbf{k})^{\otimes n})\to H^*(\Gamma_{g,1};\bigwedge^n H\otimes\mathbf{k})$ is surjective. By Theorem 1.B the $H^*(\Gamma_{g,1};\mathbf{k})$-module $H^*(\Gamma_{g,1};(H\otimes\mathbf{k})^{\otimes n})$ is generated by $m_{\widehat{P}}$, $\widehat{P}\in\mathcal{P}_n$ for $*\leq N(g)-n$. Hence, from (3.2), the $H^*(\Gamma_{g,1};\mathbf{k})$-module $H^*(\Gamma_{g,1};\bigwedge^n H\otimes\mathbf{k})$ is generated by $m_{\widehat{Q}}$, $\widehat{Q}\in\mathcal{Q}_n$. □

Harer et. al. prove

$$H_1(\Gamma_{g,1};\mathbb{Z})=0 \quad \text{if } g\geq 3 \text{ [Mu1] [P] [H2]}$$
$$H_2(\Gamma_{g,1};\mathbb{Z})=\mathbb{Z} \quad \text{if } g\geq 5 \text{ [H2]},$$

which imply, when $q\leq 3$, $H^q(\Gamma_{g,1};\mathbb{Z})$ has no torsion in the stable range. So, from Theorem 3.4, $H^q(\Gamma_{g,1};\bigwedge^n H^1(\Sigma_{g,1};\mathbb{Z}))$ has no p ($>n$)-torsion for $q\leq 3$. Glover and Mislin [GM] prove $H^{4m}(\Gamma_{g,1};\mathbb{Z})$ $(m\geq 1)$ has torsion (see also [CC] and [Mis]). On the other hand Morita [Mo3] proves $H^1(\Gamma_{g,1};\bigwedge^3 H^1(\Sigma_{g,1};\mathbb{Z}))=\mathbb{Z}^2$ for $g\geq 3$ in a (completely different and) precise manner. Consequently it would be interesting to know whether $H^q(\Gamma_{g,1};\bigwedge^n H^1(\Sigma_{g,1};\mathbb{Z}))$ $(q\leq 3)$ has torsion or not.

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