

§ 3. Goldman Lie algebra

交叉形式の幾何的精緻化

この§, 2, 3とは

- Goldman Lie 代数 $\mathcal{Q}\hat{\pi}$ の導入
- $\mathcal{Q}\hat{\pi}' \triangleleft \mathcal{Q}\hat{\pi}$ ideal
- $\sigma: \mathcal{Q}\hat{\pi} \rightarrow \text{Der}(\mathcal{Q}\hat{\pi})$ Lie algebra homomorphism.

注意 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000

S : oriented surface

$$\hat{\pi} = \hat{\pi}(S) := [S^1, S] \left(\stackrel{S: \text{conn'd } \exists \pi_1(S)/\text{conj}}{=} \right)$$

$p \in S, | \cdot | : \pi_1(S, p) \rightarrow \hat{\pi}(S), \alpha \mapsto |\alpha|$, 基点を忘れる写像

$\alpha, \beta \in \hat{\pi}, \alpha \ll \beta : S^1 \ll S^1 \rightarrow S$ generic immersion (253571=)

Goldman bracket

「代表元」とる

$$[\alpha, \beta] \stackrel{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi} \subset \mathcal{Q}\hat{\pi}$$

$\alpha_p \in \pi_1(S, p)$ p を基点, α は p を始点とする based loop

定理 3.1. (Goldman)

(1) $[\cdot, \cdot]$ は well-defined である. 7 個の代表元の対は異なる.

(2) $\mathcal{Q}\hat{\pi}$ は $[\cdot, \cdot]$ に関して Lie 代数となる.

$\mathcal{Q}\hat{\pi} = \mathcal{Q}\hat{\pi}(S)$: 曲面 S の Goldman Lie 代数

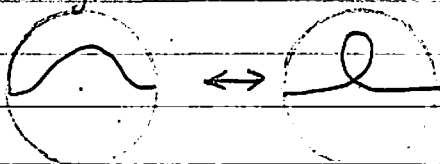
(注) (1) は君羊の本モロニニを便して代数的にも証明できる (§9)

証明 (1) 定理 1.4 を使う.

$\alpha \ll \beta$ と $\alpha' \ll \beta'$ 間の moves (w1) (w2) (w3) の関係は 1, 2, 3 と互に

$[\alpha, \beta] = [\alpha', \beta']$ であることを示す

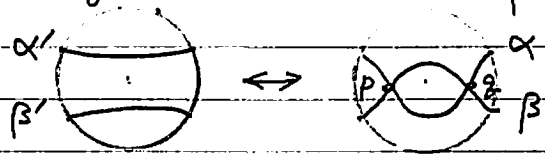
(w1) monogen



$$\alpha \wedge \beta = \alpha' \wedge \beta'$$

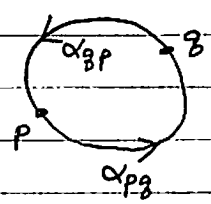
影響なし.

(W2) *bigon* と α または β ならば "影響なし"

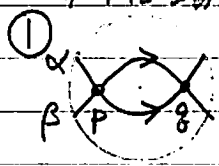


$\varepsilon(p; \alpha, \beta) = -\varepsilon(q; \alpha, \beta)$ であり $|\alpha_p \beta_p| = |\alpha_q \beta_q| \in \mathbb{Z}$ を示せばよい.

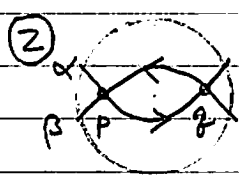
(記号) α_{pq} : α に沿って p から q までの path である
 $\alpha_p = \alpha_{pq} \alpha_{qp}$, $\alpha_q = \alpha_{qp} \alpha_{pq}$



47に場合分け

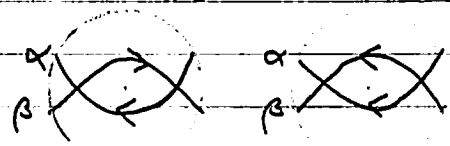


$\exists a \text{ と } z \quad \alpha_{pq} \simeq \beta_{pq} \text{ rel } \partial$
 $|\alpha_p \beta_p| = |\alpha_{pq} \alpha_{qp} \beta_{pq} \beta_{qp}| = |\beta_{pq} \alpha_{qp} \alpha_{pq} \beta_{qp}|$
 $= |\alpha_{qp} \alpha_{pq} \beta_{qp} \beta_{pq}| = |\alpha_q \beta_q|$



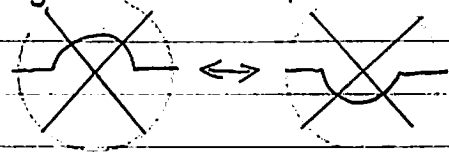
$\exists a \text{ と } z \quad \alpha_{pq} \simeq \beta_{pq}^{-1} \text{ rel } \partial$
 $|\alpha_p \beta_p| = |\alpha_{pq} \alpha_{qp} \beta_{qp} \beta_{pq}| = |\alpha_{qp} \beta_{pq}|$
 $= |\alpha_{qp} \alpha_{pq} \beta_{qp} \beta_{pq}| = |\alpha_q \beta_q|$

他の2つの場合



も同様に考へる

(W3) *jumping over a double point*



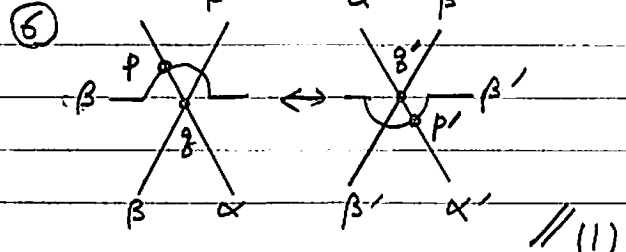
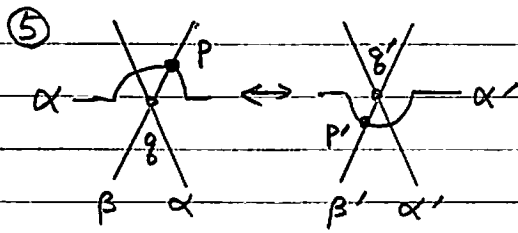
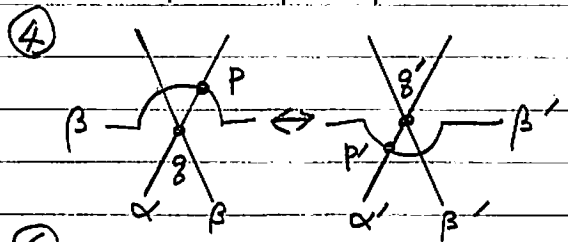
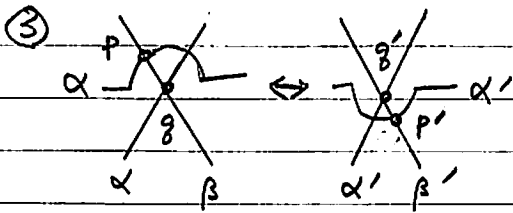
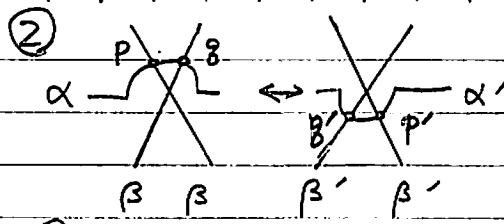
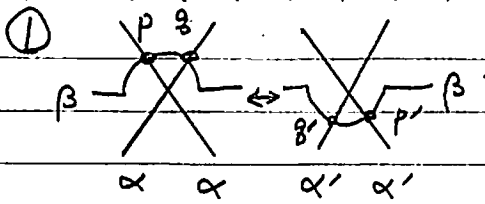
3本のうちの2本が α か? β か? $\Rightarrow 2^3 = 8$ 通り

可 \wedge 2 α または 可 \wedge 2 β の場合 \Rightarrow 影響なし

の = 11 通り

11通りのうち2本は 2本のうちの2本の値は変わらない

$\varepsilon(p; \alpha, \beta) = \varepsilon(p'; \alpha', \beta')$
 $\varepsilon(q; \alpha, \beta) = \varepsilon(q'; \alpha', \beta')$
 $|\alpha_p \beta_p| = |\alpha'_{p'} \beta'_{p'}|$
 $|\alpha_q \beta_q| = |\alpha'_{q'} \beta'_{q'}|$



(2) (anti-symmetry)

$$|\alpha_p \beta_p| = |\beta_p \alpha_p|, \quad \varepsilon(p; \alpha, \beta) = -\varepsilon(p; \beta, \alpha) \quad \text{証明}$$

(Jacobi identity)

$\alpha \cup \beta \cup \gamma; S' \cup S' \cup S' \rightarrow S$ general immersion $\forall \alpha \beta \gamma \in S$

$$[\alpha, [\beta, \gamma]] = [\alpha, \sum_{p \in \beta \cap \gamma} \varepsilon(p; \beta, \gamma) |\beta_p \gamma_p|]$$

$$= \sum_{p \in \beta \cap \gamma} \sum_{q \in \alpha \cap (\beta_p \gamma_p)} \varepsilon(p; \beta, \gamma) \varepsilon(q; \alpha, \beta_p \gamma_p) |\alpha_q (\beta_p \gamma_p)_q|$$

$$= \sum_{p \in \beta \cap \gamma} \sum_{q \in \alpha \cap \beta} \varepsilon(p; \beta, \gamma) \varepsilon(q; \alpha, \beta) |\alpha_q \beta_p \gamma_p \beta_p \gamma_p|$$

$$+ \sum_{p \in \beta \cap \gamma} \sum_{q \in \alpha \cap \gamma} \varepsilon(p; \beta, \gamma) \varepsilon(q; \alpha, \gamma) |\alpha_q \gamma_p \beta_p \gamma_p \beta_p \gamma_p|$$

$$= \sum_{p \in \beta \cap \gamma} \sum_{q \in \beta \cap \gamma} \varepsilon(p; \gamma, \alpha) \varepsilon(q; \beta, \gamma) |\beta_p \gamma_p \alpha_p \gamma_p \beta_p \gamma_p|$$

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]]$$

$$= \sum_{p \in \beta \cap \gamma} \sum_{q \in \alpha \cap \beta} \varepsilon(p; \beta, \gamma) \varepsilon(q; \alpha, \beta) |\alpha_q \beta_p \gamma_p \beta_p \gamma_p|$$

$$+ \sum_{p \in \beta \cap \gamma} \sum_{q \in \alpha \cap \gamma} \varepsilon(p; \beta, \gamma) \varepsilon(q; \alpha, \gamma) |\alpha_q \gamma_p \beta_p \gamma_p \beta_p \gamma_p|$$

$$+ \sum_{p \in \gamma \cap \alpha} \sum_{q \in \beta \cap \gamma} \varepsilon(p; \gamma, \alpha) \varepsilon(q; \beta, \gamma) |\beta_p \gamma_p \alpha_p \gamma_p \beta_p \gamma_p|$$

$$\begin{aligned}
 & + \sum_{P \in X_{n, \alpha}} \sum_{g \in \beta_{n, \alpha}} \varepsilon(p: \gamma, \alpha) \varepsilon(g: \beta, \alpha) |\beta_g \alpha_{gP} \gamma_P \alpha_{Pg}| \triangleleft \\
 & + \sum_{P \in \alpha_{n, \beta}} \sum_{g \in X_{n, \alpha}} \varepsilon(p: \alpha, \beta) \varepsilon(g: \gamma, \alpha) |\gamma_g \alpha_{gP} \beta_P \alpha_{Pg}| \triangleleft \\
 & + \sum_{P \in \alpha_{n, \beta}} \sum_{g \in X_{n, \beta}} \varepsilon(p: \alpha, \beta) \varepsilon(g: \gamma, \beta) |\gamma_g \beta_{gP} \alpha_P \beta_{Pg}| \\
 & = 0 \quad // (2) \quad // \text{Thm 3.1.}
 \end{aligned}$$

◎ homological Goldman Lie algebra との関係

$S = \Sigma_{g,1} \exists t \in \mathbb{Z} \Sigma_g$ の場合 $\Sigma_g = \underbrace{\omega \dots \omega}_g$
 $[\alpha_P, \beta_P] = [\alpha] + [\beta] \in H_{\mathbb{Z}} \quad t \text{ による}$

$\mathbb{Q}\hat{\pi} \rightarrow \mathbb{Q}H_{\mathbb{Z}}, \alpha \mapsto [[\alpha]]$ surjective Lie algebra homom.

$H_1(\mathbb{Z}\hat{\pi})$: 無限生成 (∵) $H_1(\mathbb{Z}\hat{\pi}) \rightarrow H_1(\mathbb{Z}H_{\mathbb{Z}})$: 無限生成

◎ Goldman bracket の由来

$S = \Sigma_g, \pi = \pi_1(\Sigma_g)$
 $G = GL_n(\mathbb{R}), GL_n(\mathbb{C}) \exists t \in \mathbb{Z} GL_n(\mathbb{H})$
 $\text{Hom}(\pi, G)/G$: symplectic manifold

$\left[\begin{aligned} \text{∵ } [\phi], T_{[\phi]}(\text{Hom}(\pi, G)/G) &= H^1(\pi; \text{Lie } G_{\text{Ad } \phi}) \\ \text{cup 積 による } \text{sympl. str. が決まる (Goldman)} \end{aligned} \right]$

→ Poisson bracket が定まる

$\alpha \in \hat{\pi}$

$f_{\alpha}: \text{Hom}(\pi, G)/G \rightarrow \mathbb{R}, [\phi] \mapsto \text{trace}(\phi(\alpha))$ trace function

Wolpert-Goldman formula $\forall \alpha, \forall \beta \in \hat{\pi}$
 $\{f_{\alpha}, f_{\beta}\} = \sum_{P \in \alpha_{n, \beta}} \varepsilon(p: \alpha, \beta) f_{[\alpha_P, \beta_P]}$

(Wolpert: 最初 $\pi = G = \text{PSL}_2(\mathbb{R})$ で類似の公式を見つけた)

$\rho: \mathbb{Q}\hat{\pi} \rightarrow C^{\infty}(\text{Hom}(\pi, G)/G)$ Lie algebra homomorphism

① 以下, S : connected とする.

$1 \in \hat{\pi}$ trivial (constant) loop

$\hat{\pi}' := \hat{\pi} - \{1\}$

補題 3.2 (Goldman)

$$[\mathcal{Q}\hat{\pi}, \mathcal{Q}\hat{\pi}] \subset \mathcal{Q}\hat{\pi}'$$

(注) Goldman の証明は誤り. 一般に $[\alpha, \alpha^{-1}] \neq 0$ である

証明 $\forall \alpha, \beta \in \hat{\pi}, [\alpha, \beta] \in \mathcal{Q}\hat{\pi}'$ を示す

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) |\alpha_p \beta_p|$$

場合分け

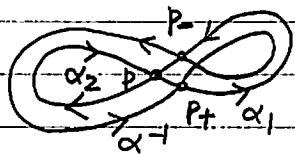
- (1) $\beta \neq \alpha^{-1}$
- (2) $\beta = \alpha^{-1}$

(1) へのとき $\forall p \in \alpha \cap \beta, |\alpha_p \beta_p| \in \hat{\pi}'$ である

(1) へのとき $\exists p \in \alpha \cap \beta, |\alpha_p \beta_p| = 1$

$\beta_p = \alpha_p^{-1} \in \pi_1(S, p)$ 仮定に矛盾 //

(2) へのとき α と generic $\beta = \alpha^{-1}$ とを仮定して押し出す



p : α の自己交叉の点

$\Rightarrow \alpha$ と α^{-1} の交点から 2 つできる

$\varepsilon(p_{\pm}; \alpha, \alpha^{-1}) = \pm 1$ となるように p_+, p_- を決める

$\alpha_1, \alpha_2 \in \pi_1(S, p)$ とする

p_+ における向き: $\alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1}$

p_- : $\alpha_2 \alpha_1 \alpha_2^{-1} \alpha_1^{-1}$

$\therefore |\alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1}| = 1 \Leftrightarrow |\alpha_2 \alpha_1 \alpha_2^{-1} \alpha_1^{-1}| = 1$ である

$$|\alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1}| - |\alpha_2 \alpha_1 \alpha_2^{-1} \alpha_1^{-1}| \in \mathcal{Q}\hat{\pi}'$$

よって $[\alpha, \alpha^{-1}] \in \mathcal{Q}\hat{\pi}'$ である //

系 3.3 $\mathcal{Q}\hat{\pi} = \mathcal{Q}\hat{\pi}' \oplus \mathcal{Q} \cdot 1$ (Lie 代数としての直和分解)

(注) $S = \Sigma_{g,1}$ または Σ_g のとき $\mathcal{Q}\hat{\pi}/\mathcal{Q}\hat{\pi}' \cong \mathcal{Q}$ と $\mathcal{Q}\hat{\pi} \rightarrow \mathcal{Q}H_2 \rightarrow \mathcal{Q}$

に注意. $\dim H_1(\mathcal{Q}\hat{\pi}) \geq 2$ である

未解決問題 $H_1(\mathcal{Q}\hat{\pi})$ は有限次元か?

◎ Goldman Lie 代数の基本群の群環への作用

$*$ $\in S$ basepoint

$\pi := \pi(S, *)$, $Q\pi$: group ring

$S^* := S \setminus \{*\}$

Lie 代数準同型

$$\sigma: Q\hat{\pi}(S^*) \rightarrow \text{Der}(Q\pi)$$

を構成する.

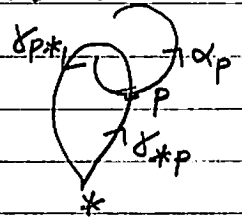
$\alpha \in \hat{\pi}(S^*)$, $\gamma \in \pi$

$\alpha \sqcup \gamma: S' \sqcup S' \rightarrow S$ generic immersion とする

$T = T \cup L$ 基点 $*$ $\in S'$ の可成り γ は immersion として $\gamma < \gamma'$ とい

(1) 基点 $*$ $\in S'$ は重みかきではない

$$\sigma(\alpha)(\gamma) \stackrel{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma) \delta_{*p} \alpha_p \delta_{p*} \in Q\pi$$



定理 3.4.

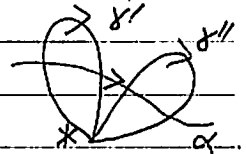
$$\sigma: Q\hat{\pi}(S^*) \rightarrow \text{Der}(Q\pi)$$

は well-defined な Lie 代数準同型である

証明 (well-defined) ... これだけ示す (これは群環のホモロジーで証明できる)

(derivation) ... 右図より明らか

(Lie algebra homom) ... Thm 3.1 (2) と同様



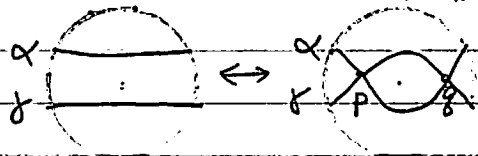
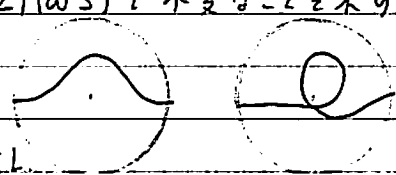
(well-defined)

定理 1.4 の 3 つの moves (w1) (w2) (w3) で不変なことを示す

(w1) monogon ... 影響なし

(w2) bigon

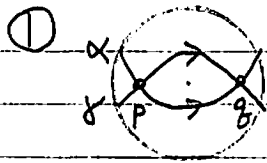
ともに α またはともに γ ならば 影響なし



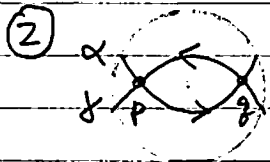
このとき α は S^* を動かしているから 基点 $*$ は disk の外にはある. \therefore に注意する

$$\varepsilon(p; \alpha, \gamma) = -\varepsilon(q; \alpha, \gamma) \text{ だから}$$

$$\delta_{*p} \alpha_p \delta_{p*} = \delta_{*q} \alpha_q \delta_{q*} \in \pi \text{ を示せばよい}$$



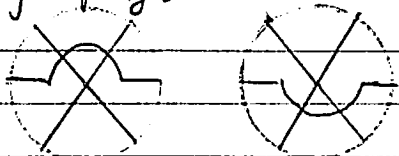
① $\alpha_{pq} \simeq \delta_{pq} \text{ rel } \partial T$ かし
 $\delta_{*p} \alpha_p \delta_{p*} = \delta_{*p} \alpha_{pq} \alpha_{qp} \delta_{pq} \delta_{q*}$
 $= \delta_{*p} \delta_{pq} \alpha_{qp} \alpha_{pq} \delta_{q*} = \delta_{*q} \alpha_q \delta_{q*}$



② $\alpha_{pq} \simeq \delta_{qp}^{-1} \text{ rel } \partial T$ かし
 $\delta_{*p} \alpha_p \delta_{p*} = \delta_{*q} \delta_{qp} \alpha_{pq} \alpha_{qp} \delta_{p*} = \delta_{*q} \alpha_{qp} \delta_{p*}$
 $= \delta_{*q} \alpha_{qp} \alpha_{pq} \delta_{qp} \delta_{p*} = \delta_{*q} \alpha_q \delta_{q*}$

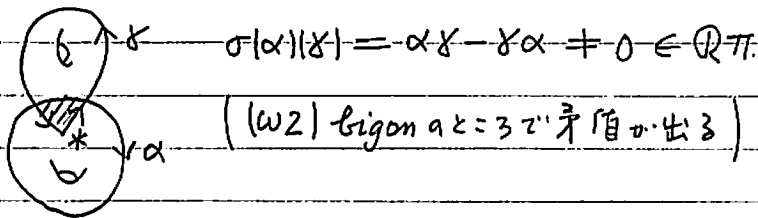
他の2つの場合も同様に考へる

(W3) jumping over a double point



Thm 3.1 と同様 (同参) // Thm 3.4

注意 (1) $\hat{\pi}(S)$ ではなく $\hat{\pi}(S^*)$ とした理由



$\sigma(\alpha|\alpha) = \alpha\alpha - \alpha\alpha \neq 0 \in \mathbb{Q}\pi$

(W2) 2-gon $n=3$ 矛盾が生じる

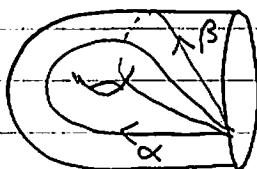
(2) 定義から明らかに $\alpha \cap \alpha = \emptyset$ ならば $\sigma(\alpha|\alpha) = 0$ である

(3) $S = \Sigma = \sum_{g \in \partial \Sigma} g =$ $\sum_{* \in \partial \Sigma} T$ かし

$\mathbb{Q}\hat{\pi}(\Sigma)$ は $\mathbb{Q}\pi_1(\Sigma, *)$ に作用する

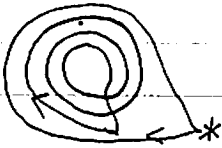
例を2つ計算する (あと2つ使う)

補題 3.5 $\forall n \geq 0$

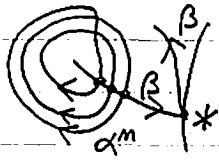


$\sigma(|\alpha^n|)|\alpha| = 0$
 $\sigma(|\alpha^n|)|\beta| = n\beta\alpha^n$

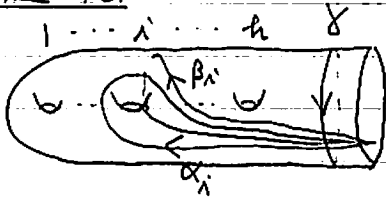
証明 (前半) $|\alpha^n|$ と α は disjoint に交わるから明らか



(後半) $|\alpha^n|$ と β は n 回 正の向きに交わり、それぞれの寄与は $\beta \alpha^n$



補題 3.6

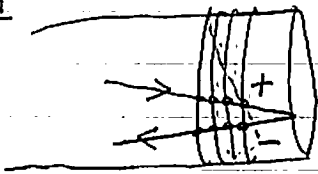


$$\forall m \geq 0$$

$$\sigma(|\gamma^m|)(\alpha_i) = -m \gamma^m \alpha_i + m \alpha_i \gamma^m$$

$$\sigma(|\gamma^m|)(\beta_i) = -m \gamma^m \beta_i + m \beta_i \gamma^m$$

証明



γ^m と α_i は

n 回 負の向きに交わり

n 回 正の向きに交わり

負の交叉の寄与 $\gamma^m \alpha_i$

正の交叉の寄与 $\alpha_i \gamma^m$

β_i についても同様 //