(§5. Hochschild-Serre spectral sequences (つづき))

$\mathbb{K}$: field of char. 0.

(§5.3. spectral sequence relative to a reductive subalgebra (つづき))

$n \geqq 3$

$\mathfrak{so}_n(\mathbb{K}) \overset{\text{def}}{=} \{X \in \mathfrak{gl}_n(\mathbb{K}) ; {}^tX = -X\}$

$\mathfrak{so}_n(\mathbb{R}) = \mathrm{Lie}\, SO_n$

$\mathfrak{so}_n(\mathbb{K})$: semi-simple

(∵) reductive

(∵ ${}^t\mathfrak{so}_n(\mathbb{K}) = \mathfrak{so}_n(\mathbb{K})$)

$H^1(\mathfrak{so}_n(\mathbb{K}) ; \mathbb{K}) = 0$

(∵) $\mathbb{K} = \mathbb{R}$ について示せばよい.

$n \geqq 3$ より

$\pi_1(SO_n) = \mathbb{Z}/2$

(∵) $\underline{n=3}$ $SO_3 = SU_2/\{\pm 1\} = S^3/\{\pm 1\}$

$\underline{n \geqq 4}$ $SO_{n-1} \to SO_n \to S^{n-1}$ fiber bundle.

$\pi_1(S^{n-1}) = \pi_2(S^{n-1}) = 0$ (∵ $n \geqq 4$)

$\pi_1(SO_{n-1}) \cong \pi_1(SO_n)$ //

$H^1(\mathfrak{so}_n(\mathbb{R}) ; \mathbb{R}) = H^1_{DR}(SO_n) = \mathrm{Hom}(\pi_1(SO_n), \mathbb{R}) = 0$ //

レポート問題10 次を示せ

$$\pi_3(SO_n) = \begin{cases} \mathbb{Z} & \text{if } n=3,\ n \geqq 5 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n=4. \end{cases}$$

(四元数を使うのも一つのやり方. $\widetilde{SO}_3 = S^3$, $\widetilde{SO}_4 = S^3 \times S^3$, $\widetilde{SO}_5 = Sp_2$)

Theorem 5.31 $m \geqq 1$

$$H^*(\mathfrak{so}_{2m+1}(\mathbb{K}) ; \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_3, u_7, \cdots, u_{4m-1}).$$

$$\deg(u_{4i-1}) = 4i-1 \quad (1 \leq i \leq m)$$

proof induction on m

$\underline{2m+1=3}$ $\mathbb{K} = \mathbb{R}$ について示せばよい

$$H^*(\mathfrak{so}_3(\mathbb{R}) ; \mathbb{R}) = H^*_{DR}(SO_3) = H^*_{DR}(S^3/\{\pm 1\}) = \Lambda^*_{\mathbb{R}}(u_3)$$

$2m+1 \geq 5$

$\mathfrak{so}_{2m-1}(\mathbb{K}) < \mathfrak{so}_{2m+1}(\mathbb{K})$ reductive subalgebra

(∵ $\mathfrak{so}_{2m-1}(\mathbb{K})$: semi-simple)

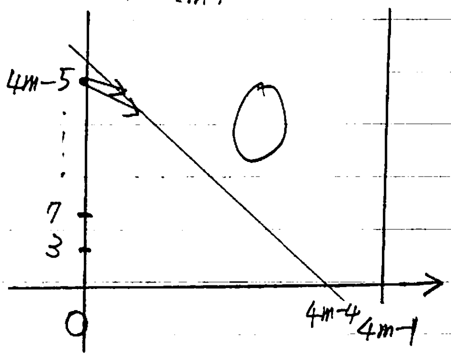
$$H^*(\mathfrak{so}_{2m+1}(\mathbb{K}), \mathfrak{so}_{2m-1}(\mathbb{K}); \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } * = 0, 4m-1. \\ 0, & \text{otherwise} \end{cases}$$

[∵) $\mathbb{K} = \mathbb{R}$ について示せばよい

$$H^*_{DR}(SO_{2m+1}/SO_{2m-1}) = \begin{cases} \mathbb{R} & \text{if } * = 0, 4m-1 \\ 0 & \text{otherwise} \end{cases}$$

(∵ Lem 5.30) //

$$E_2^{p,q} = H^q(\mathfrak{so}_{2m-1}(\mathbb{K})) \otimes H^p(\mathfrak{so}_{2m+1}(\mathbb{K}), \mathfrak{so}_{2m-1}(\mathbb{K}))$$



$d_r(u_{4i-1}) = 0 \ (\forall r \geq 2)$

$d_r = 0 \ (\forall r \geq 2)$

(∵ multiplicativity)

$E_2^{p,q} = E_\infty^{p,q}$

$$\begin{aligned} &H^*(\mathfrak{so}_{2m+1}(\mathbb{K})) \\ &= H^*(\mathfrak{so}_{2m-1}(\mathbb{K})) \otimes H^*(\mathfrak{so}_{2m+1}(\mathbb{K}), \mathfrak{so}_{2m-1}(\mathbb{K})) \\ &= \Lambda^*_{\mathbb{K}}(u_3, u_7, \cdots, u_{4m-5}, u_{4m-1}). \end{aligned}$$

//帰納法が完成した//

Remark

$$incl^* : H^*(\mathfrak{so}_{2m+1}(\mathbb{K}); \mathbb{K}) \twoheadrightarrow H^*(\mathfrak{so}_{2m-1}(\mathbb{K}); \mathbb{K})$$

surjective

Theorem 5.32 $m \geq 2$

$$H^*(\mathfrak{so}_{2m}(\mathbb{K}); \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_3, u_7, \cdots, u_{4m-5}, v_{2m-1})$$

$\deg(u_{4i-1}) = 4i-1 \ (1 \leq i \leq m-1)$, $\deg v_{2m-1} = 2m-1$

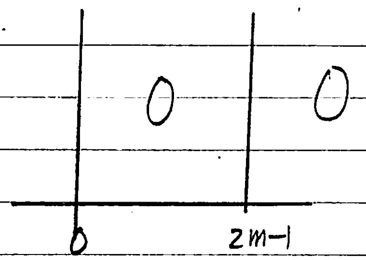
<u>proof</u> $\mathfrak{so}_{2m-1}(\mathbb{K}) \subset \mathfrak{so}_{2m}(\mathbb{K})$ reductive subalgebra

(∵ $\mathfrak{so}_{2m-1}(\mathbb{K})$: semi-simple)

$$H^*(\mathfrak{so}_{2m}(\mathbb{K}), \mathfrak{so}_{2m-1}(\mathbb{K}) ; \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } * = 0, 2m-1 \\ 0, & \text{otherwise} \end{cases}$$

(∵ $SO_{2m}/SO_{2m-1} = S^{2m-1}$)

$$E_2^{p,q} = H^q(\mathfrak{so}_{2m-1}(\mathbb{K})) \otimes H^p(\mathfrak{so}_{2m}(\mathbb{K}), \mathfrak{so}_{2m-1}(\mathbb{K}))$$



$d_r^{0,q} = 0 \quad (\forall r \geq 2)$

∵)
$$\begin{tikzcd} H^*(\mathfrak{so}_{2m+1}(\mathbb{K})) \arrow[r, "incl^*"] \arrow[dr, "surjective"'] & H^*(\mathfrak{so}_{2m}(\mathbb{K})) \arrow[d, "incl^* \text{ ; surjective}"] \\ & H^*(\mathfrak{so}_{2m-1}(\mathbb{K})) \end{tikzcd}$$
$\Downarrow$
$Z_2^{0,q} = Z_\infty^{0,q}$ //

$$E_2^{p,q} = E_\infty^{p,q}$$

$$H^*(\mathfrak{so}_{2m}(\mathbb{K})) = H^*(\mathfrak{so}_{2m-1}(\mathbb{K})) \otimes H^*(\mathfrak{so}_{2m}(\mathbb{K}), \mathfrak{so}_{2m-1}(\mathbb{K}))$$
$$= \Lambda_{\mathbb{K}}^*(u_3, u_7, \cdots, u_{4m-5}, v_{2m-1}) \quad //$$

<u>Remark</u> $n \neq 4 \Rightarrow H^3(\mathfrak{so}_n(\mathbb{K}); \mathbb{K}) = \mathbb{K}$

$\Rightarrow \mathfrak{so}_n(\mathbb{K})$: simple

$\mathfrak{so}_4(\mathbb{K}) = \mathfrak{so}_3(\mathbb{K}) \oplus \mathfrak{so}_3(\mathbb{K})$: <u>not</u> simple

A_n $\mathfrak{sl}_{n+1}$
B_n $\mathfrak{so}_{2n+1}$
C_n $\mathfrak{sp}_n$ $\leftarrow$
D_n $\mathfrak{so}_{2n}$.
(exceptional.)

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$\underline{n \geq 1}$

$$J = J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{Q}). \quad (\S 3\text{とは基底をとりかえた})$$

$$\mathfrak{sp}_n(\mathbb{K}) = \{ X \in \mathfrak{gl}_{2n}(\mathbb{K}) \, ; \, {}^tXJ + JX = 0 \}$$

semi-simple (reductive + Cor. 3.11)

<u>quaternions</u>

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

$i^2 = j^2 = k^2 = -1$ $\qquad \cup$

$ij = -ji = k$ $\qquad \mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$

$jk = -kj = i$ $\qquad \forall z, \quad jz = \bar{z}j, \; zj = j\bar{z}$

$ki = -ik = j$

$$\mathbb{H} \cong \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} ; \, z, w \in \mathbb{C} \right\}$$

$$z + jw \longmapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \quad \mathbb{R}\text{-algebra isomorphism}$$

$q = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}, \quad x_\alpha \in \mathbb{R}$

$\bar{q} = x_1 - x_2 i - x_3 j - x_4 k$

$\overline{qq'} = \overline{q'} \, \bar{q}$

明ら $\left(\begin{array}{l} q\bar{q} = \bar{q}q = x_1^2 + x_2^2 + x_3^2 + x_4^2 \in \mathbb{R}_{\geq 0} \\ q \neq 0 \Rightarrow q^{-1} = \dfrac{1}{q\bar{q}} \bar{q} \quad \text{skew-field} \end{array} \right)$

$n \geq 0 \quad \mathbb{H}^n = \left\{ \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} ; \, q_\alpha \in \mathbb{H} \right\}$ column vectors

$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} q := \begin{pmatrix} q_1 q \\ \vdots \\ q_n q \end{pmatrix} \quad \text{right } \mathbb{H}\text{-module}$$

$\mathfrak{gl}_n(\mathbb{H}) := \{ S : \mathbb{H}^n \to \mathbb{H}^n ; \text{ right } \mathbb{H}\text{-homom} \}$

associative algebra $\Rightarrow$ Lie algebra

$GL_n(\mathbb{H}) := \{ S : \mathbb{H}^n \to \mathbb{H}^n : \text{right } \mathbb{H}\text{-isom} \}$

Lie group

$\varphi : \mathbb{C}^n \times \mathbb{C}^n \xrightarrow{\cong} \mathbb{H}^n$ right $\mathbb{C}$-isom

$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto a + jb$

$\varphi^{-1}\left(\varphi\begin{pmatrix} a \\ b \end{pmatrix} j\right) = \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}$

$\left(\because\right)$ $\varphi^{-1}((a+jb)j) = \varphi^{-1}(j\bar{a} - \bar{b})$ //

$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$

$\begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = J\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}$

$\varphi^{-1}(\mathfrak{gl}_n(\mathbb{H})) = \{X \in \mathfrak{gl}_{2n}(\mathbb{C});\ J\bar{X} = XJ\}$

(田る) ($\because$) $X \in \varphi^{-1}(\mathfrak{gl}_n(\mathbb{H}))$
$\iff \forall u \in \mathbb{C}^n \times \mathbb{C}^n \quad J\overline{Xu} = XJ\bar{u}$ //

$\langle\ ,\ \rangle : \mathbb{H}^n \times \mathbb{H}^n \longrightarrow \mathbb{H}, \quad (\xi, \eta) \longmapsto {}^t\bar{\xi}\eta$

sesqui-linear pairing

(*) $\forall u', \forall u'' \in \mathbb{C}^n \times \mathbb{C}^n$

$\langle \varphi(u'), \varphi(u'') \rangle = {}^t\bar{u'}u'' - j({}^tu' J u'')$

$\because$) $a', b', a'', b'' \in \mathbb{C}^n$

${}^t\overline{(a'+jb')} = {}^t\bar{a}' - {}^t\bar{b}'j = {}^t\bar{a}' - j\,{}^tb'$

${}^t\overline{(a'+jb')}(a''+jb'') = ({}^t\bar{a}' - j\,{}^tb')a'' + ({}^t\bar{a}' - {}^t\bar{b}'j)jb''$

$= {}^t\bar{a}'a'' + {}^t\bar{b}'b'' + j(-{}^tb'a'' + {}^ta'b'')$ //

$\{\xi \in \mathbb{H}^n;\ \langle \xi, \xi \rangle = 1\} = S^{4n-1}$

$Sp_n \overset{\text{def}}{=} \{S \in \mathfrak{gl}_n(\mathbb{H});\ \forall \xi, \forall \eta \in \mathbb{H}^n\ \langle S\xi, S\eta \rangle = \langle \xi, \eta \rangle\}$

Lie group

$\varphi^{-1}(Sp_n) = U_{2n} \cap Sp_n(\mathbb{C})$

$\because$) $(\subset) \Leftarrow (*)$

$(\supset)$ $S \in U_{2n} \cap Sp_n(\mathbb{C})$

$\Rightarrow {}^t\bar{S}J\bar{S} = J, \quad S\,{}^t\bar{S} = 1$

$\Rightarrow SJ = J\bar{S} \Rightarrow S \in \varphi^{-1}(\mathfrak{gl}_n(\mathbb{H}))$ //

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Sp_n : compact Lie group

$Sp_{n-1} \to Sp_n \to S^{4n-1}$ fiber bundle $\forall n \geq 1$.

($\Rightarrow Sp_n$: connected)

レポート問題11 $\widetilde{SO_5} = Sp_2$ を示せ

$\mathrm{Lie}\, Sp_n = \{ X \in \mathfrak{gl}_{2n}(\mathbb{C}) ; {}^tXJ + JX = 0, {}^t\overline{X} = -X \}$

$\mathfrak{sp}_n(\mathbb{C}) = \mathrm{Lie}\, Sp_n(\mathbb{C}) = \{ X \in \mathfrak{gl}_{2n}(\mathbb{C}) ; {}^tXJ + JX = 0 \}$

$\mathfrak{sp}_n(\mathbb{C}) = (\mathrm{Lie}\, Sp_n) \otimes \mathbb{C}$

(∵) $\tau : \mathfrak{sp}_n(\mathbb{C}) \circlearrowleft \quad \tau(X) := {}^t\overline{X}, \quad \tau^2 = 1$

$\tau(\sqrt{-1}X) = -\sqrt{-1}\,\tau(X)$

$\mathfrak{sp}_n(\mathbb{C}) = \mathrm{Ker}(\tau+1) \oplus \mathrm{Ker}(\tau-1)$

$= \mathrm{Lie}\, Sp_n \oplus \sqrt{-1}\, \mathrm{Lie}\, Sp_n$ //

Lemma 5.33 $\forall n \geq 1$

$$H^*(\mathfrak{sp}_n(\mathbb{K}), \mathfrak{sp}_{n-1}(\mathbb{K}) ; \mathbb{K}) = \begin{cases} \mathbb{K} & , \text{ if } * = 0, 4n-1 \\ 0 & , \text{ otherwise} \end{cases}$$

proof $\mathbb{K} = \mathbb{C}$ で示せばよい.

$H^*(\mathfrak{sp}_n(\mathbb{C}), \mathfrak{sp}_{n-1}(\mathbb{C}) ; \mathbb{C})$

$= H^*(\mathrm{Lie}\, Sp_n, \mathrm{Lie}\, Sp_{n-1} ; \mathbb{R}) \otimes \mathbb{C}$

$= H^*_{DR}(Sp_n / Sp_{n-1}) \otimes \mathbb{C} = H^*_{DR}(S^{4n-1}) \otimes \mathbb{C}$ //

Theorem 5.34

$H^*(\mathfrak{sp}_n(\mathbb{K}) ; \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_3, u_7, \cdots u_{4n-1})$.

$\deg u_{4i-1} = 4i-1, \quad 1 \leq i \leq n$.

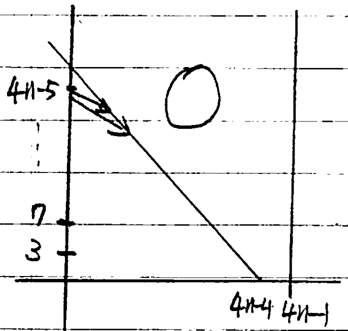
proof induction on n

$n=1$ Lem 5.33 $(n=1)$

$n \geq 2$ $\mathfrak{sp}_{n-1}(\mathbb{K}) \subset \mathfrak{sp}_n(\mathbb{K})$ reductive

(∵ $\mathfrak{sp}_{n-1}(\mathbb{K})$: semi-simple)

$$E_2^{p,q} = H^q(\mathfrak{sp}_{n-1}(\mathbb{K})) \otimes H^p(\mathfrak{sp}_n(\mathbb{K}), \mathfrak{sp}_{n-1}(\mathbb{K}))$$



$d_r(u_{4n-1}) = 0 \quad (\forall r \geq 2)$

$d_r = 0 \quad (\forall r \geq 2)$

($\because$ multiplicativity)

$E_2^{p,q} = E_\infty^{p,q}$

$$H^*(\mathfrak{sp}_n(\mathbb{K})) = H^*(\mathfrak{sp}_{n-1}(\mathbb{K})) \otimes H^*(\mathfrak{sp}_n(\mathbb{K}), \mathfrak{sp}_{n-1}(\mathbb{K}))$$
$$= \Lambda^*_{\mathbb{K}}(u_3, \cdots, u_{4n-5}, u_{4n-1}) \quad //$$

<u>Infinite-dimensional Grassmann manifolds</u>

$$BU_n \overset{\text{def}}{=} \varinjlim_{N \to \infty} U_{n+N}/U_n \times U_N$$
$$BSO_n \overset{\text{def}}{=} \varinjlim_{N \to \infty} SO_{n+N}/SO_n \times SO_N$$
$$BSp_n \overset{\text{def}}{=} \varinjlim_{N \to \infty} Sp_{n+N}/Sp_n \times Sp_N$$

<u>Thm 5.27 bis</u>

G : compact connected Lie group

$K \subset H \subset G$: closed connected subgroups

$\Longrightarrow \exists$ (multiplicative) spectral sequence

$$E_2^{p,q} = H^q_{DR}(H/K) \otimes H^p_{DR}(G/H) \Rightarrow H^{p+q}_{DR}(G/K)$$

<u>Lemma 5.35</u>

$\pi_q(SO_{n+N}/1_n \times SO_N) = 0 \quad$ if $q \leq N-2$

$\pi_q(U_{n+N}/1_n \times U_N) = 0 \quad$ if $q \leq 2N$

$\pi_q(Sp_{n+N}/1_n \times Sp_N) = 0 \quad$ if $q \leq 4N+2$

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Theorem 5.36

$H^*(U_{n+N}/U_\ell\times 1_{n-\ell}\times U_N ; \mathbb{R}) \cong \mathbb{R}[c_1,\cdots,c_\ell]$ for $*\ll 2N$

where $\deg c_i = 2i$ $(1\le i\le \ell)$

proof induction on $\ell\ge 0$

$\ell=0 \Leftarrow$ Lem 5.35

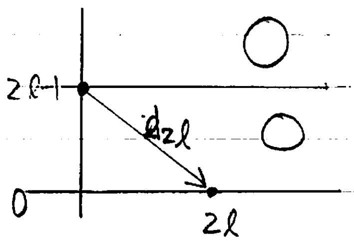
$\ell\ge 1$ $B := U_{n+N}/U_\ell\times 1_{n-\ell}\times U_N$

$E := U_{n+N}/U_{\ell-1}\times 1_{n-\ell+1}\times U_N$

$U_\ell/U_{\ell-1} = S^{2\ell-1}$

$H^*(E;\mathbb{R}) \underset{\text{ind. assumption}}{\cong} \mathbb{R}[c_1,\cdots,c_{\ell-1}]$ for $*\ll 2N$

$E_2^{p,q} = H^q(S^{2\ell-1})\otimes H^p(B) \Rightarrow H^{p+q}(E)$ (∵ Thm 5.27)



$\pi^*: H^j(B) \xrightarrow{\cong} H^j(E)$ isom. if $j<2\ell-1$

$(\pi^*)^{-1}(c_i) \mapsto c_i$ $(\forall i\le \ell-1)$

$\Rightarrow \pi^*: H^*(B)\to H^*(E)$

surjective for $*\ll 2n$.

$$\begin{array}{lll}
E_2^{0,2\ell-1} = H^{2\ell-1}(S^{2\ell-1})\otimes H^0(B) & \xrightarrow[\cong]{\frown[S^{2\ell-1}]} & H^0(B) = \mathbb{R} \\
d_{2\ell}\downarrow \qquad\qquad\circlearrowleft & & \quad\ \rotatebox{90}{$\in$} \\
E_2^{2\ell,0} = H^{2\ell}(B) \ni c_\ell & \longleftarrow & \quad\ 1
\end{array}$$

c_ℓ と定める

$0\to H^{*-2\ell}(B)\xrightarrow{\cup c_\ell} H^*(B)\xrightarrow{\pi^*} H^*(E)\to 0$ (exact)

(for $*\ll 2N$)

これを

$0\to \mathbb{R}[c_1,\cdots c_\ell]\xrightarrow{c_\ell}\mathbb{R}[c_1,\cdots,c_\ell]\to\mathbb{R}[c_1,\cdots c_{\ell-1}]\to 0$ (exact)

とを比較して 5-lemma を使うと

$H^*(B)\cong\mathbb{R}[c_1,\cdots,c_\ell]$ for $*\ll 2N$

帰納法が完成した //

(レポート問題 12

細部をつめよ. とくに同型が成立つ範囲を具体的に与えよ.

Corollary 5.37

$H^*(BU_n ; \mathbb{R}) = \mathbb{R}[c_1, c_2, \cdots, c_n]$

$c_i \in H^{2i}(BU_n ; \mathbb{R})$ the i^{th} Chern class

同様に

Theorem 5.38.

(1) $H^*(BSO_{2m+1} ; \mathbb{R}) = \mathbb{R}[p_1, \cdots, p_m]$

$p_i \in H^{4i}(BSO_{2m+1} ; \mathbb{R})$ the i^{th} Pontrjagin class

(2) $H^*(BSO_{2m} ; \mathbb{R}) = \mathbb{R}[p_1, \cdots, p_{m-1}, e]$

$p_i \in H^{4i}(BSO_{2m} ; \mathbb{R})$ the i^{th} Pontrjagin class

$e \in H^{2m}(BSO_{2m} ; \mathbb{R})$ the Euler class

(3) $H^*(BSp_n ; \mathbb{R}) = \mathbb{R}[q_1, \cdots, q_n]$

$q_i \in H^{4i}(BSp_n ; \mathbb{R})$ the i^{th} symplectic Pontrjagin class

レポート問題 13 これらを証明せよ.

No.

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