(§5. Hochschild–Serre spectral sequences (つづき))

$\mathbb{K}$: field of char. 0.

§5.3. spectral sequence relative to a reductive subalgebra

$\mathfrak{g}$; finite dimensional Lie algebra $/\mathbb{K}$

Definition $\mathfrak{h} \le \mathfrak{g}$: reductive subalgebra $/\mathbb{K}$

$\overset{\text{def}}{\Longleftrightarrow}$ (0) $\mathfrak{h} < \mathfrak{g}$ subalgebra $/\mathbb{K}$

1) $\mathfrak{g}$: completely reducible $\mathfrak{h}$-module.

$\Longrightarrow$ $\mathfrak{h}$: reductive Lie algebra $/\mathbb{K}$

$\mathfrak{h}$: semi-simple

$\mathfrak{h} < \mathfrak{g}$ Lie subalgebra

$\Longrightarrow$ $\mathfrak{h} < \mathfrak{g}$ reductive subalgebra

complete reducibility

$\mathfrak{h}$: finite dim. Lie algebra $/\mathbb{K}$

Theorem 5.21. ([B], §6. n°5, Cor 1. Cor 3)

M, N : finite $\dim_{\mathbb{K}}$ completely reducible $\mathfrak{h}$-modules

$\Longrightarrow$ $M \otimes N$, $\mathrm{Hom}(M,N)$: completely reducible $\mathfrak{h}$-modules

Suppose

$\mathfrak{g}$: finite dim. Lie algebra $/\mathbb{K}$

$\mathfrak{h} \le \mathfrak{g}$: reductive subalgebra $/\mathbb{K}$

M : finite $\dim_{\mathbb{K}}$ $\mathfrak{g}$-module

completely reducible as $\mathfrak{h}$-module

$\Longrightarrow$ Hochschild Serre s.s.

$E_r^{p,q}$, $d_r^{p,q}$

$$E_1^{p,q} = H^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M)) \Rightarrow H^{p+q}(\mathfrak{g} ; M)$$

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$C^*(\mathfrak{g}/\mathfrak{h} : M)$ ⎫ completely reducible $\mathfrak{h}$-modules
$C^*(\mathfrak{g} : M)$ ⎭ finite dim. $/\mathbb{K}$ ($\because$ Thm 5.21)

$$\begin{tikzcd} E_1^{p,q} \arrow[r, "\cong", "\phi_1"'] \arrow[dr, bend right, "\rho_1"', "\wr\!\wr"] & H^q(\mathfrak{h} : C^p(\mathfrak{g}/\mathfrak{h} : M)) \\ & \cong_{\text{Thm 3.12}} H^q(\mathfrak{h} : \mathbb{K}) \otimes C^p(\mathfrak{g}/\mathfrak{h} : M)^{\mathfrak{h}} \\ & = H^q(\mathfrak{h} : \mathbb{K}) \otimes C^p(\mathfrak{g}, \mathfrak{h} : M) \end{tikzcd}$$

$\delta := (-1)^q 1 \otimes d_{C^p} : H^q(\mathfrak{h} : \mathbb{K}) \otimes C^p(\mathfrak{g}, \mathfrak{h} : M)$
$\longrightarrow H^q(\mathfrak{h} : \mathbb{K}) \otimes C^{p+1}(\mathfrak{g}, \mathfrak{h} : M) \xleftarrow[\cong]{\rho_1} E_1^{p+1,q}$

<u>Theorem 5.22</u> $\rho_1 \circ d_1 = \delta \circ \rho_1$

<u>pf</u>
$$\begin{array}{ccc} A_p^{p+q}/A_{p+1}^{p+q} & \xrightarrow[\cong]{\phi} & C^q(\mathfrak{h} : C^p(\mathfrak{g}/\mathfrak{h} : M)) \\ \cup & \circlearrowleft & \cup \\ Z^{p+q}(A_p/A_{p+1}) & \xrightarrow{\cong} & Z^q(\mathfrak{h} : C^p(\mathfrak{g}/\mathfrak{h} : M)) \\ \Uparrow & & \cup \\ A_p^{p+q} \cap d^{-1}(A_{p+1}^{p+q+1}) & & Z^q(\mathfrak{h} : C^p(\mathfrak{g}, \mathfrak{h} : M)) \end{array}$$

$\forall e \in E_1^{p,q}$
$\exists f \in A_p^{p+q} \cap d^{-1}(A_{p+1}^{p+q+1})$
s.t. $[f] = e \in E_1^{p,q}$
$\forall Y_i \in \mathfrak{h},\ f_q(Y_1, \cdots, Y_q) \in C^p(\mathfrak{g}, \mathfrak{h} : M)$
(where $f_q(Y_1, \cdots Y_q)(X_1, \cdots X_p)$ ($\because$ Thm 3.12)
$\stackrel{\text{def}}{=} f(Y_1, \cdots, Y_q, X_1, \cdots, X_p)$. $(X_j \in \mathfrak{g})$).

$d_1 e = [df] \in E_1^{p+1,q}$
$\phi(df) \in Z^q(\mathfrak{h} : C^{p+1}(\mathfrak{g}/\mathfrak{h} : M))$
$\exists \bar{v} \in C^{q-1}(\mathfrak{h} : C^{p+1}(\mathfrak{g}/\mathfrak{h} : M))$
s.t. $\phi(df) + d\bar{v} \in Z^q(\mathfrak{h} : C^{p+1}(\mathfrak{g}, \mathfrak{h} : M))$ ($\because$ Thm 3.12)
$\exists v \in A_{p+1}^{p+q}$ $\phi(v) = \bar{v}$

$g := f + v \in A_p^{p+q} \cap d^{-1}(A_{p+1}^{p+q+1})$

$[g] = e \in E_1^{p,q}$

$g_q(Y_1,\cdots,Y_q) \in C^p(\mathfrak{g},\mathfrak{h}:M)$ $\quad (\forall Y_i \in \mathfrak{h})$

$(dg)_q(Y_1,\cdots,Y_q) \in C^{p+1}(\mathfrak{g},\mathfrak{h}:M)$

$d_1 e = [dg] \in E_1^{p+1,q}$

$(dg)_q(Y_1,\cdots,Y_q)$

$= d(g_{q-1})(Y_1,\cdots,Y_q) + (-1)^q d(g_q(Y_1,\cdots,Y_q))$ $\quad (\because$ Lem 5.5$)$

$g_{q-1} \in C^{q-1}(\mathfrak{h}: C^{p+1}(\mathfrak{g}:M))$

$\exists V \subset C^{p+1}(\mathfrak{g}:M)$ s.t. $C^{p+1}(\mathfrak{g}:M) = C^{p+1}(\mathfrak{g},\mathfrak{h}:M) \oplus V$

$(\because$ $C^{p+1}(\mathfrak{g}:M)$: completely reducible $\mathfrak{h}$-module$)$

$\exists!\, h \in C^{q-1}(\mathfrak{h}: C^{p+1}(\mathfrak{g},\mathfrak{h}:M))$

$\exists!\, k \in C^{q-1}(\mathfrak{h}:V)$

s.t. $g_{q-1} = h + k$

$dk = 0$

$\because)$ $dh(Y_1,\cdots,Y_q) \in C^{p+1}(\mathfrak{g},\mathfrak{h}:M)$

$(dh+dk)(Y_1,\cdots,Y_q) = d(g_{q-1})(Y_1,\cdots,Y_q)$

$= (dg)_q(Y_1,\cdots,Y_q) - (-1)^q d(g_q(Y_1,\cdots,Y_q))$

$\in C^{p+1}(\mathfrak{g},\mathfrak{h}:M)$

$(dk)(Y_1,\cdots,Y_q) \in V \cap C^{p+1}(\mathfrak{g},\mathfrak{h}:M) = 0$ //

$d(g_{q-1}) = dh$

$(dg_q)(Y_1,\cdots,Y_q) = (dh)(Y_1,\cdots,Y_q) + (-1)^q d(g_q(Y_1,\cdots,Y_q))$

$\rho_1(d_1 e) = [dg_q] \in H^q(\mathfrak{h}: C^{p+1}(\mathfrak{g},\mathfrak{h}:M))$

$= (-1)^q (1\otimes d)[g_q]$

$= \delta \rho_1(e)$ // Thm 5.22

Corollary 5.23 ρ_1 induces

$\rho_2 : E_2^{p,q} \xrightarrow{\cong} H^q(\mathfrak{h}:K) \oplus H^p(\mathfrak{g},\mathfrak{h}:M)$

M, N : $\mathfrak{g}$-modules as above

$$\cup : (H^q(\mathfrak{h}:\mathbb{K}) \otimes H^p(\mathfrak{g},\mathfrak{h}:M)) \times (H^t(\mathfrak{h}:\mathbb{K}) \otimes H^s(\mathfrak{g},\mathfrak{h}:N))$$
$$\longrightarrow H^{q+t}(\mathfrak{h}:\mathbb{K}) \otimes H^{p+s}(\mathfrak{g},\mathfrak{h}:M\otimes N)$$

$$(u \otimes \overset{p}{v}) \cup (\overset{t}{w} \otimes z) := (-1)^{pt}(u \cup w) \otimes (v \cup z)$$

Corollary 5.24 $\forall e' \in E_2(M)$, $\forall e'' \in E_2(N)$

$$P_2(e' \cup e'') = P_2(e') \cup P_2(e'')$$

Theorem 5.25. $\mathfrak{g}$: reductive Lie algebra $/\mathbb{K}$

$\mathfrak{h} < \mathfrak{g}$: reductive subalgebra $/\mathbb{K}$

M : fin.dim.$_{\mathbb{K}}$ $\mathfrak{g}$-module

completely reducible as $\mathfrak{g}$- and $\mathfrak{h}$-module

$$\Longrightarrow H^*(\mathfrak{g},\mathfrak{h}:M) \cong H^*(\mathfrak{g},\mathfrak{h}:M^{\mathfrak{g}}) = H^*(\mathfrak{g},\mathfrak{h}:\mathbb{K}) \otimes M^{\mathfrak{g}}$$

proof Suffices to show.

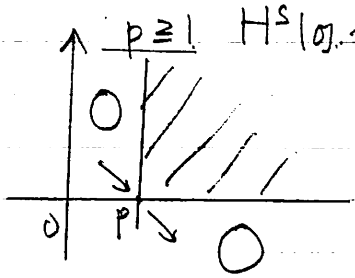
M : nontrivial simple $\mathfrak{g}$-module
$\Longrightarrow H^p(\mathfrak{g},\mathfrak{h}:M) = 0 \quad \forall p \geq 0$

$$E_2^{p,q} = H^q(\mathfrak{h}:\mathbb{K}) \otimes H^p(\mathfrak{g},\mathfrak{h}:M) \Rightarrow H^{p+q}(\mathfrak{g}:M)$$

induction on $p \geq 0$

$p=0$ $H^0(\mathfrak{g},\mathfrak{h}:M) = H^0(\mathfrak{g}:M) = 0$ ($\because$ M : nontriv, simple)

$p \geq 1$ $H^s(\mathfrak{g},\mathfrak{h}:M) = 0$ for $s \leq p-1$ ($\because$ ind. assumption)



$H^p(\mathfrak{g},\mathfrak{h}:M) = H^p(\mathfrak{g}:M) = 0$ ($\because$ Thm 3.2) //

Theorem 5.26 $\mathfrak{g}$: fin. dim. Lie algebra $/\mathbb{K}$

M : fin. dim$_{\mathbb{K}}$ $\mathfrak{g}$-module

$\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$: Lie algebras

$\mathfrak{g}$, M : completely reducible as $\mathfrak{h}$- and $\mathfrak{k}$-modules.

$\Rightarrow$ $\exists$ (multiplicative) spectral sequence

$$E_2^{p,q} = H^q(\mathfrak{h},\mathfrak{k} : \mathbb{K}) \otimes H^p(\mathfrak{g},\mathfrak{h} : M) \Rightarrow H^{p+q}(\mathfrak{g},\mathfrak{k} : M)$$

proof $A = C^*(\mathfrak{g},\mathfrak{k} : M)$

$$A_p^n := \{ f \in C^n(\mathfrak{g},\mathfrak{k} : M) ; f(X_1, \cdots, X_n) = 0 \text{ if } \#\{i : X_i \in \mathfrak{h}\} \geq n-p+1 \}$$

$$\phi : A_p^{p+q}/A_{p+1}^{p+q} \xrightarrow{\cong} C^q(\mathfrak{h}/\mathfrak{k} : C^p(\mathfrak{g}/\mathfrak{h} : M))^{\mathfrak{k}} = C^q(\mathfrak{h},\mathfrak{k} : C^p(\mathfrak{g}/\mathfrak{h} : M))$$

$$\begin{aligned} E_1 \xrightarrow[\cong]{\phi_1} & H^q(\mathfrak{h},\mathfrak{k} : C^p(\mathfrak{g}/\mathfrak{h} : M)) \\ &= H^q(\mathfrak{h},\mathfrak{k} : \mathbb{K}) \otimes C^p(\mathfrak{g}/\mathfrak{h} : M)^{\mathfrak{h}} \quad (\because \text{Thm } 5.25) \\ &= H^q(\mathfrak{h},\mathfrak{k} : \mathbb{K}) \otimes C^p(\mathfrak{g},\mathfrak{h} : M) \end{aligned}$$

ρ_1

$\rho_1 \circ d_1 = \delta \circ \rho_1$ similar to Thm 5.22 //

レポート問題 8: 証明の細部を詰めよ

Theorem 5.27

G : compact connected Lie group

$K \subset H \subset G$ closed connected subgroups

$\Rightarrow$ $\exists$ (multiplicative) spectral sequence

$$E_2^{p,q} = H^q_{DR}(H/K) \otimes H^p_{DR}(G/H) \Rightarrow H^{p+q}_{DR}(G/K)$$

proof Suffices to show:

Lie G : completely reducible as Lie H- and Lie K-module

Lie G : G-module via $\mathrm{Ad} : G \to \mathrm{Aut}(\mathrm{Lie}\, G)$

$\forall V \subset \mathrm{Lie}\, G$,

V : K-submodule $\Leftrightarrow$ V : Lie K-submodule ($\because$ K : conn)

H- Lie H- ($\because$ H : conn)

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Lie G : completely reducible as H- and K-module
(∵ K, H : compact $\Rightarrow$ averaging operator) //

Example

$n \geqq 2$

$\mathfrak{sl}_n(\mathbb{K}) = \{X \in \mathfrak{gl}_n(\mathbb{K}) ; \operatorname{trace} X = 0\}$

semi-simple (§3.3)

$\mathfrak{sl}_n(\mathbb{K}) \subset \mathfrak{sl}_{n+1}(\mathbb{K}) \qquad X \mapsto \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right)$

reductive subalgebra

(∵ $\mathfrak{sl}_n(\mathbb{K})$: semi-simple)

Lemma 5.28 $H^p(\mathfrak{sl}_{n+1}(\mathbb{K}), \mathfrak{sl}_n(\mathbb{K}) ; \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } p = 0, 2n+1 \\ 0, & \text{otherwise} \end{cases}$

(pf). Suffices to show the lemma for $\mathbb{K} = \mathbb{C}$.

$\mathfrak{sl}_n(\mathbb{C}) = \operatorname{Lie} SU_n \otimes \mathbb{C}$ (∵ Cor 2.3)

$H^p(\mathfrak{sl}_{n+1}(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) ; \mathbb{C})$

$= H^p(\operatorname{Lie} SU_{n+1}, \operatorname{Lie} SU_n ; \mathbb{R}) \otimes \mathbb{C}$

$= H^p_{DR}(SU_{n+1}/SU_n) \otimes \mathbb{C} = H^p_{DR}(S^{2n+1}) \otimes \mathbb{C}$ //

レポート問題9

(de Rham 同型を使わずに) 定義にもどって Lem 5.28 を示せ.

Theorem 5.29

$H^*(\mathfrak{sl}_n(\mathbb{K}) ; \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_3, u_5, \cdots, u_{2n-1})$

$\deg u_{2i-1} = 2i-1, \ (2 \leq i \leq n)$

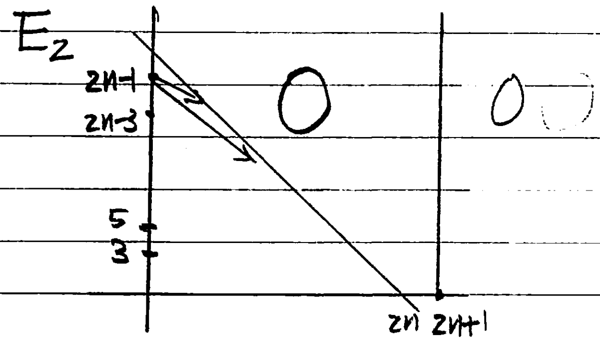
proof induction on $n \geqq 2$

$n = 2$ $\mathfrak{sl}_1(\mathbb{K}) = 0$

$H^*(\mathfrak{sl}_2(\mathbb{K}) ; \mathbb{K}) = H^*(\mathfrak{sl}_2(\mathbb{K}), \mathfrak{sl}_1(\mathbb{K}) ; \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_3)$.

$n \geqq 2$ まで示されたとする

$E_2^{p,q} = H^q(\mathfrak{sl}_n(\mathbb{K})) \otimes H^p(\mathfrak{sl}_{n+1}(\mathbb{K}), \mathfrak{sl}_n(\mathbb{K}))$



$d_r(u_i) = 0 \quad (\forall r \geq 2)$

$d_r = 0 \quad (\forall r \geq 2)$

($\because$ multiplicativity)

$E_2^{p,q} = E_\infty^{p,q}$

$$H^*(\mathfrak{sl}_{n+1}(\mathbb{K})) = H^*(\mathfrak{sl}_n(\mathbb{K})) \otimes H^*(\mathfrak{sl}_{n+1}(\mathbb{K}), \mathfrak{sl}_n(\mathbb{K}))$$
$$= \Lambda^*_{\mathbb{K}}(u_3, \cdots, u_{2n-1}, u_{2n+1}).$$ ∴帰納法が進む //

$\mathfrak{so}_n(\mathbb{K})$ の計算の準備

Lemma 5.30

$$H_*(SO_{n+1}/SO_{n-1} ; \mathbb{Z}) \begin{cases} \underset{n:\text{odd}}{=} \begin{cases} \mathbb{Z}, & \text{if } * = 0, n-1, n, 2n-1 \\ 0, & \text{otherwise} \end{cases} \\ \underset{n:\text{even}}{=} \begin{cases} \mathbb{Z}, & \text{if } * = 0, 2n-1 \\ \mathbb{Z}/2, & \text{if } * = n-1 \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

proof $E := SO_{n+1}/SO_{n-1} = O_{n+1}/O_{n-1}$

$$= O_{n+1} \times_{O_n} S^{n-1} \xrightarrow{\pi} O_{n+1}/O_n = S^n$$

$S^{n-1} \to E \xrightarrow{\pi} S^n$ fiber bundle

$e := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ e' := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in S^n \subset \mathbb{R}^{n+1}$

$F := \pi^{-1}(e) = S^{n-1}$

$r : S^n \to O_{n+1}$

$\qquad x \mapsto r(x) := (u \mapsto u - 2(u,x)x)$ reflection

$D^n_\pm := \{x \in S^n ; \pm(x,e) \geq 0\} \approx D^n$

$S^{n-1} = D^n_+ \cap D^n_-$

$S^n = D^n_+ \cup D^n_-$

$\varphi : (D^n_+, S^{n-1}) \to (S^n, e)$

$\qquad x \mapsto r(x)(e) = e - 2(e,x)x$

$\varphi : D^n_+/S^{n-1} \xrightarrow{\approx} S^n$: homeo

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∵) D^n_+/S^{n-1} : compact, S^n : Hausdorff 体で.

φ : bijective を示せばよい.

<u>injective</u> $e-2(e,x)x = e-2(e,x')x'$

$\Leftrightarrow (e,x)x = (e,x')x' \Rightarrow |(e,x)| = |(e,x')|$

- $(e,x)=0 \Rightarrow x, x' \in S^{n-1}$
- $(e,x)\neq 0 \Rightarrow x = \pm x' \underset{x,x'\in D^n_+\setminus S^{n-1}}{\Longrightarrow} x = x'$ //inj

<u>surjective</u> $\varphi(S^{n-1}) = \{e\}$

$\forall y \in S^n \setminus \{e\}$

$2(e,e-y) = 2-2(e,y) = \|e-y\|^2 \neq 0$

$x := \dfrac{1}{\|e-y\|}(e-y) \in D^n_+$

$\varphi(x) = e - 2\left(e, \dfrac{e-y}{\|e-y\|}\right)\dfrac{e-y}{\|e-y\|}$

$= e - \dfrac{2(e,e-y)}{\|e-y\|^2}(e-y) = e-(e-y) = y$ //surj //

$\varphi^* E \cong D^n \times S^{n-1}$ (∵ $D^n \simeq *$)

$H_*(E,F) = H_*((D^n, S^{n-1})\times S^{n-1})$

$= H_*(D^n, S^{n-1}) \otimes H_*(S^{n-1}) = \begin{cases} \mathbb{Z} & (*=n, 2n-1) \\ 0 & (\text{else}) \end{cases}$

$\hat{\varphi} : (D^n_+, S^{n-1}) \to (E,F)$

$x \longmapsto r(x) \bmod O_{n-1}$

$\hat{\varphi}_*[D^n_+] \in H_n(E,F) \cong \mathbb{Z}$ generator

$$\begin{tikzcd}
H_n(E,F) \arrow[r, "\partial_*"] & H_{n-1}(F) = H_{n-1}(S^{n-1}) \\
H_n(D^n_+, S^{n-1}) \arrow[u, "\hat{\varphi}_*", "\cong"'] \arrow[r, "\partial_*", "\cong"'] \arrow[ur, phantom, "\circlearrowleft"] & H_{n-1}(S^{n-1}) \arrow[u, "\hat{\varphi}_*"']
\end{tikzcd}$$

$\hat{\varphi}|_{S^{n-1}} : S^{n-1} \to F = S^{n-1}$

$x \longmapsto r(x)e' = e' - 2(e',x)x$

$r(-x)e' = r(x)e'$

$\deg(-1 : S^{n-1} \to S^{n-1}) = (-1)^n$

$\partial_* \hat{\varphi}_*[D^n_+] = \pm(1+(-1)^n)[S^{n-1}]$ //