(§5. Hochschild-Serre spectral sequences (つづき))

$\mathbb{K}$: field of char. 0

(§5.1. spectral sequence associated with a subalgebra (つづき))

<u>The relative cohomology of Chevalley-Eilenberg</u>

$\mathfrak{g}$: Lie algebra $/\mathbb{K}$

$\mathfrak{h} < \mathfrak{g}$: Lie subalgebra $/\mathbb{K}$

M : $\mathfrak{g}$-module

$C^*(\mathfrak{g}, \mathfrak{h} : M) \overset{\text{def}}{=} \{ f \in C^*(\mathfrak{g} : M) ; \begin{matrix} \forall Y \in \mathfrak{h} \\ i(Y) f = i(Y) df = 0 \end{matrix} \}$

$\subset C^*(\mathfrak{g} : M)$ cochain subcomplex

$H^*(\mathfrak{g}, \mathfrak{h} : M) \overset{\text{def}}{=} H^*(C^*(\mathfrak{g}, \mathfrak{h} : M))$

the relative cohomology of Chevalley-Eilenberg

(<u>Lem 5.6.</u> $E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h} : M)$ —)

<u>Proposition 5.7</u> G : compact connected Lie group

$H < G$: closed connected subgroup

($\Rightarrow G/H$: C^∞ mfd)

$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{h} = \mathrm{Lie}\, H$

$\Longrightarrow H^*(\mathfrak{g}, \mathfrak{h} : \mathbb{R}) = H^*_{DR}(G/H)$ —

<u>proof</u> $\Omega^*(G/H)$: de Rham complex of G/H

$\pi : G \to G/H$, $x \mapsto xH$, quotient map

$g \in G$, $L_g : G/H \to G/H$, $xH \mapsto gxH$,

$L_g^* \curvearrowright \Omega^*(G/H)$

$L_g^* = 1$ on $H^*_{DR}(G/H)$ ($\because$ G : connected)

$g \in G$, $R_g : G \to G$, $x \mapsto xg$.

<u>Claim 1</u> $H^*_{DR}(G/H) = H^*(\Omega^*(G/H)^G)$

($\because$) averaging operator $\Leftarrow$ G : compact

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(略) $\exists!\ dvol \in \Omega^{\dim G}(G)$

s.t. $\begin{cases} \forall g \in G,\ R_g^* dvol = dvol. & \text{(right invariant)} \\ \int_G dvol = 1 & (\because\ G: \text{compact}) \end{cases}$

$\theta \in \Omega^*(G/H)$

$A\theta := \int_{x \in G} (L_x^* \theta)\, dvol \in \Omega^*(G/H)$

$\forall g \in G\quad L_g^* A\theta = A\theta$ ($\because$ dvol : right invariant)

$A : \Omega^*(G/H) \to \Omega^*(G/H)^G$ $\begin{cases}\text{averaging operator} \\ \text{cochain map}\end{cases}$

$A_* = 1$ on $H_{DR}^*(G/H)$

$\Rightarrow H_{DR}^*(G/H) = H^*(\Omega^*(G/H)^G)$ (See §2.2) //

$e := \pi(1) = H \in G/H$.

$$\begin{array}{ccccc}
\Omega^*(G/H)^G & \xrightarrow{ev_e} & \Lambda^* T_e^* G/H & = & C^*(\mathfrak{g}/\mathfrak{h} : \mathbb{R}) \\
\pi^* \downarrow & \circlearrowleft & \downarrow \pi^* & \circlearrowleft & \downarrow \pi^* \\
\Omega^*(G) & \xrightarrow[\cong]{ev_1} & \Lambda^* T_1^* G & = & C^*(\mathfrak{g} : \mathbb{R})
\end{array}$$
isom. of cochain cpx's

$\mathrm{Im}(ev_e) \subset C^*(\mathfrak{g}, \mathfrak{h} : \mathbb{R})$

$\left(\begin{array}{l} \because\ \forall \theta \in \Omega^*(G/H)^G,\ d\theta \in \Omega^*(G/H)^G \\ \qquad \Rightarrow \forall Y \in \mathfrak{h}\quad i_Y \theta = i_Y d\theta = 0 \end{array}\right)$

$ev_e : \Omega^*(G/H)^G \hookrightarrow C^*(\mathfrak{g}, \mathfrak{h} : \mathbb{R})$ injective cochain map

Claim 2 ev_e : surjective, i.e.,

$\Omega^*(G/H)^G \cong C^*(\mathfrak{g}, \mathfrak{h} : \mathbb{R})$ isom. of cochain cpx's

$\because$) $\forall f \in C^*(\mathfrak{g}, \mathfrak{h} : \mathbb{R})$ $\exists!\ \theta \in \Omega^*(G)^G \mapsto \pi^* f$

$\forall Y \in \mathfrak{h}.\quad 0 = \mathcal{L}_Y \theta = \frac{d}{dt}\Big|_{t=0} R_{\exp tY}{}^* \theta$

$\Rightarrow \forall Y \in \mathfrak{h}\quad R_{\exp tY}{}^* \theta = \theta$

$\underset{H:\text{connected}}{\Longrightarrow}$ $\forall h \in H\quad R_h^* \theta = \theta$

$U \overset{\text{open}}{\subset} G/H$, $\sigma_0, \sigma_1 : U \to G$: C^∞ sections of π

$\sigma_1^* \theta = \sigma_0^* \theta \in \Omega^*(U)$

(∵) $\exists ! h : U \to H$ C^∞ map. $\forall x \in U$ $\sigma_1(x) = \sigma_0(x) h(x)$

$l :]-\varepsilon, \varepsilon[\to U$ C^∞ path $(0 < \varepsilon << 1)$

$\sigma_{1*}(\dot{l}(0)) = \frac{d}{dt}\Big|_{t=0} (\sigma_0(l(t)) h(l(t)))$

$= R_{h(l(0))*} \sigma_{0*} \dot{l}(0) + L_{\sigma_0(l(0))*} h_* \dot{l}(0)$

$\to R_h^* \theta = \theta$, $\to i(Y)\theta = 0$

(略)

$\forall x \in U$, $\exists Y : T_x U \to \mathfrak{h}$, $v \mapsto Y_v$, $\mathbb{R}$-linear map

s.t. $\forall v \in T_x U$ $\sigma_{1*}(v) = R_{h(x)*} \sigma_{0*}(v) + (Y_v)_{\sigma_1(x)}$

$\forall v_1, \cdots, \forall v_n \in T_x U$

$(\sigma_1^* \theta)(v_1, \cdots, v_n) = \theta(\sigma_{1*}(v_1), \cdots, \sigma_{1*}(v_n))$

$= \theta(R_{h(x)*}\sigma_{0*}(v_1) + (Y_{v_1})_{\sigma_1(x)}, \cdots, R_{h(x)*}\sigma_{0*}(v_n) + (Y_{v_n})_{\sigma_1(x)})$

$\underset{\left(\substack{i_Y \theta = 0 \\ \forall Y \in \mathfrak{h}}\right)}{=} \theta(R_{h(x)*}\sigma_{0*}(v_1), \cdots, R_{h(x)*}\sigma_{0*}(v_n))$

$= (\sigma_0^* R_{h(x)}^* \theta)(v_1, \cdots, v_n) = (\sigma_0^* \theta)(v_1, \cdots, v_n)$ //

$\exists \bar{\theta} \in \Omega^*(G/H)^G$ s.t. $\theta = \pi^* \bar{\theta}$

$ev_e \bar{\theta} = f$ // // Prop.

Examples

$\mathfrak{so}(n) := \{X \in \mathfrak{gl}_n(\mathbb{R}) ; {}^tX = -X\} = \mathrm{Lie}\, SO(n)$

$SO(n)$: compact connected Lie group

$\mathfrak{so}(n) \hookrightarrow \mathfrak{so}(n+1)$ $X \mapsto \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right)$

Corollary 5.8.

$H^p(\mathfrak{so}(n+1), \mathfrak{so}(n) ; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } p = 0, n \\ 0 & \text{otherwise} \end{cases}$

(∵ $SO(n+1)/SO(n) = S^n$)

$\mathfrak{su}(n) := \{X \in \mathfrak{sl}_n(\mathbb{C}) ; {}^t\bar{X} = -X\} = \mathrm{Lie}\, SU(n)$

$SU(n)$: compact connected Lie group

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$\mathfrak{su}(n) \hookrightarrow \mathfrak{su}(n+1) \quad X \mapsto \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right)$

Corollary 5.9 $H^p(\mathfrak{su}(n+1), \mathfrak{su}(n); \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } p = 2n+1, 0 \\ 0 & \text{otherwise.} \end{cases}$

$(\because)$ $SU(n+1)/SU(n) = S^{2n+1}$ //

Spectral sequence

Definition (A, d) : differential module

$\overset{\text{def}}{\iff}$ 0) A : $\mathbb{Z}$-module
$d : A \to A$ $\mathbb{Z}$-homom
1) $d^2 = 0 : A \to A$ $(\iff \operatorname{Im} d \subset \operatorname{Ker} d)$

$H(A) := \operatorname{Ker} d / \operatorname{Im} d$

Definition $\{A_p\}_{p\in\mathbb{Z}}$: decreasing filtration of (A, d)

$\overset{\text{def}}{\iff}$ 0) $\forall p \in \mathbb{Z}$ $A_p \subset A$ $\mathbb{Z}$-submodule
1) $\forall p \in \mathbb{Z}$ $d(A_p) \subset A_p$
2) (decreasing) $\forall p \in \mathbb{Z}$, $A_p \supset A_{p+1}$

$E_0^p := A_p / A_{p+1}$

$d_0^p := d : E_0^p \to E_0^p, \quad [u] \mapsto [du]$

$E_1^p := H(E_0^p) = H(A_p/A_{p+1})$

$d_1^p := \delta^* : E_1^p = H(A_p/A_{p+1}) \to E_1^{p+1} = H(A_{p+1}/A_{p+2})$
$[u] \longmapsto [du]$

connecting homomorphism associated with
$0 \to A_{p+1}/A_{p+2} \to A_p/A_{p+2} \to A_p/A_{p+1} \to 0$

$E_2^p := H^p(E_1^*)$

$d_2^p = ?$

$$E_\infty^p := \frac{\mathrm{Im}(H(A_p) \to H(A))}{\mathrm{Im}(H(A_{p+1}) \to H(A))}$$

approximation of $H(A)$ with respect to the filtration $\{A_p\}_{p\in\mathbb{Z}}$

$r \geqq 0$

$Z_r^p \overset{\text{def}}{=} \mathrm{Im}(j^*: H(A_p/A_{p+r}) \to H(A_p/A_{p+1}))$
$\quad = \mathrm{Ker}(\delta^*: H(A_p/A_{p+1}) \to H(A_{p+1}/A_{p+r}))$

$A_p \cap d^{-1}(A_{p+r}) \twoheadrightarrow Z_r^p$

$Z_\infty^p \overset{\text{def}}{=} \mathrm{Im}(j^*: H(A_p) \to H(A_p/A_{p+1}))$
$\quad = \mathrm{Ker}(\delta^*: H(A_p/A_{p+1}) \to H(A_{p+1}))$

$B_r^p \overset{\text{def}}{=} \mathrm{Im}(\delta^*: H(A_{p-r+1}/A_p) \to H(A_p/A_{p+1}))$
$\quad = \mathrm{Ker}(\iota^*: H(A_p/A_{p+1}) \to H(A_{p-r+1}/A_{p+1}))$

$B_\infty^p \overset{\text{def}}{=} \mathrm{Im}(\delta^*: H(A/A_p) \to H(A_p/A_{p+1}))$
$\quad = \mathrm{Ker}(\iota^*: H(A_p/A_{p+1}) \to H(A/A_{p+1}))$

Lemma 5.10

$0 \overset{(1)}{=} B_1^p \subset \cdots \subset B_r^p \overset{(2)}{\subset} B_{r+1}^p \subset \cdots \overset{(3)}{\subset} B_\infty^p \overset{(4)}{\subset} Z_\infty^p \overset{(5)}{\subset} \cdots$
$\cdots \subset Z_{r+1}^p \overset{(6)}{\subset} Z_r^p \subset \cdots \overset{(7)}{\subset} H(A_p/A_{p+1}) = Z_1^p$

Pf (略)

(1) $H(A_p/A_p) = 0$

(2)
$$\begin{tikzcd} H(A_{p-r+1}/A_p) \arrow[r, "\delta^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H(A_p/A_{p+1}) \\ H(A_{p-r}/A_p) \arrow[ur, "\delta^*"'] & \end{tikzcd}
\qquad
\begin{tikzcd} A_p/A_{p+1} \arrow[r, phantom, "\subset"] \arrow[dr, bend right] \arrow[dr, phantom, "\circlearrowleft"] & A_{p-r+1}/A_{p+1} \arrow[d, phantom, "\iota\ \cap"] \\ & A_{p-r}/A_{p+1} \end{tikzcd}$$

(3)
$$\begin{tikzcd} H(A_{p-r+1}/A_p) \arrow[r, "\delta^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H(A_p/A_{p+1}) \\ H(A/A_p) \arrow[ur, "\delta^*"'] & \end{tikzcd}
\qquad
\begin{tikzcd} A_p/A_{p+1} \arrow[r, phantom, "\subsetneq"] \arrow[dr, bend right] \arrow[dr, phantom, "\circlearrowleft"] & A_{p-r+1}/A_{p+1} \arrow[d] \\ & A/A_{p+1} \end{tikzcd}$$

(4)

(exact)
$$\begin{tikzcd} & & H(A_{p+1}) \\ H(A/A_p) \arrow[r, "\delta^*"] \arrow[dr] \arrow[r, phantom, shift right=3, "\circlearrowright"] & H(A_p/A_{p+1}) \arrow[ur, "\delta^*"] & \\ & H(A_p) \arrow[u] & \end{tikzcd}$$

(5)
$$\begin{tikzcd} H(A_p/A_{p+1}) \arrow[r, "\delta^*"] \arrow[dr, "\delta^*"'] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & H(A_{p+1}) \arrow[d] \\ & H(A_{p+1}/A_{p+r}) \end{tikzcd} \qquad \begin{tikzcd} A_p \arrow[r, two heads] \arrow[dr] & A_p/A_{p+r} \arrow[d] \\ & A_p/A_{p+1} \end{tikzcd}$$

(6)
$$\begin{tikzcd} H(A_p/A_{p+1}) \arrow[r, "\delta^*"] \arrow[dr, "\delta^*"'] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & H(A_{p+1}/A_{p+r+1}) \arrow[d] \\ & H(A_{p+1}/A_{p+r}) \end{tikzcd} \qquad \begin{tikzcd} A_p/A_{p+r+1} \arrow[r, two heads] \arrow[dr, two heads] & A_p/A_{p+r} \arrow[d] \\ & A_p/A_{p+1} \end{tikzcd}$$

(7) Clear

(8) $H(A_{p+1}/A_{p+1}) = 0$ //

$1 \leqq r \leqq \infty$

$$E_r^p \overset{\text{def}}{=} Z_r^p/B_r^p$$

($\underline{r=1}$ $E_1^p = Z_1^p/B_1^p = H(A_p/A_{p+1})$ (∵) Lem 5.10 (1)(8))

nested intervals!! But, in general, $E_\infty^p \neq \lim\limits_{r\to\infty} E_r^p$

$$E_{r+1}^p = H(E_r^p)$$

$\underline{\text{Lemma 5.11}}$ $Z_r^p/Z_{r+1}^p \cong B_r^{p+r}/B_{r+1}^{p+r}$ $[u] \mapsto [du]$ $(\forall u \in A_p \cap d^{-1}(A_{p+r}))$

(pf)
$$\begin{tikzcd}[column sep=small] H(A_{p+1}/A_{p+r}) \arrow[drr, "\delta^*"] \arrow[ddr, "i^*"'] & & & \\ & & H(A_{p+r}/A_{p+r+1}) \arrow[dr, "i^*"] & \\ & H(A_p/A_{p+r}) \arrow[ur, "\delta^*"] \arrow[dr, "j^*"] & & H(A_{p+1}/A_{p+r+1}) \\ H(A_p/A_{p+r+1}) \arrow[ur, "j^*"] \arrow[rr, "j^*"'] & & H(A_p/A_{p+1}) \arrow[ur, "\delta^*"'] & \end{tikzcd}$$

$$Z_r^p/Z_{r+1}^p = \delta^* j^* H(A_p/A_{p+r}) = i^*\delta^* H(A_p/A_{p+r}) = B_{r+1}^{p+r}/B_r^{p+r} \;//$$

$$d_r : E_r^p = Z_r^p/B_r^p \twoheadrightarrow Z_r^p/Z_{r+1}^p \cong B_r^{p+r}/B_{r+1}^{p+r} \hookrightarrow Z_r^{p+r}/B_r^{p+r} = E_r^{p+r}$$

$\forall u \in A_p \cap d^{-1}(A_{p+r})$ $d_r[u] = [du]$

($\underline{r=1}$ $d_1 = \delta^* : H(A_p/A_{p+1}) \to H(A_{p+1}/A_{p+2})$
connecting homomorphism)

Lemma 5.12.

(1) $\mathrm{Ker}(d_r|_{E_r^p}) = Z_{r+1}^p / B_r^p$

(2) $\mathrm{Im}(d_r|_{E_r^{p-r}}) = B_{r+1}^p / B_r^p$

(3) $d_r \circ d_r = 0 : E_r^{p-r} \to E_r^p \to E_r^{p+r}$

(4) $H(E_r^p) = \mathrm{Ker}(d_r|_{E_r^p}) \big/ \mathrm{Im}(d_r|_{E_r^{p-r}}) = E_{r+1}^p$

[(∵) (3) $B_{r+1}^p \subset Z_{r+1}^p$ //

Lemma 5.13

$$E_\infty^p = Z_\infty^p / B_\infty^p \cong \frac{\mathrm{Im}(H(A_p) \xrightarrow{i^*} H(A))}{\mathrm{Im}(H(A_{p+1}) \xrightarrow{i^*} H(A))} =: gr_p H(A)$$

(pf)

$$\begin{tikzcd}
H(A_{p+1}) \arrow[r, "i^*"] \arrow[d] & H(A) \arrow[r, "j^*"] & H(A/A_{p+1}) \\
H(A_p) \arrow[ur, "i^*"] \arrow[r, "j^*"] & H(A_p/A_{p+1}) \arrow[ur, "i^*"'] & \\
H(A/A_p) \arrow[u, "\delta^*"] \arrow[ur, "\delta^*"'] & &
\end{tikzcd}$$

$$Z_\infty^p / B_\infty^p = i^* j^* H(A_p) = j^* i^* H(A_p) = gr_p H(A)$$ //

$$E_\infty^p \underset{?}{\ominus} \lim_{r\to\infty} E_r^p$$

$A = \bigoplus_{n=0}^{\infty} A^n$, $d(A^n) \subset A^{n+1}$ (i.e., (A, d) : cochain complex)

$A_p = \bigoplus_{n=0}^{\infty} A_p^n$, $A_p^n := A^n \cap A_p$, homogeneous ($\forall p \in \mathbb{Z}$)

$1 \leq r \leq p$

$Z_r^{p,q} := Z_r^p \cap H^{p+q}(A_p/A_{p+1})$

$B_r^{p,q} := B_r^p \cap H^{p+q}(A_p/A_{p+1})$

$E_r^{p,q} := Z_r^{p,q} / B_r^{p,q}$ $\left(\Rightarrow E_r^p = \bigoplus_{q=-\infty}^{+\infty} E_r^{p,q}\right)$

$d_r^{p,q} := d_r|_{E_r^{p,q}} : E_r^{p,q} \to E_r^{p+r, q+r-1}$

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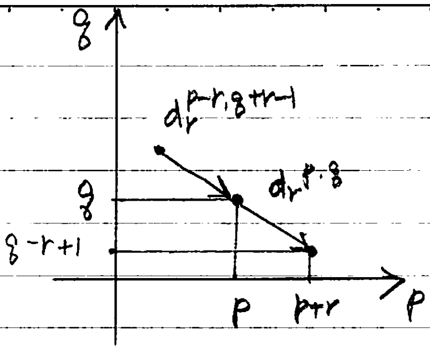
Lemma 5.14



$A_0 = A$

$A_p^n = 0$ if $p \geqq n$

$\Rightarrow$

(1) $E_r^{p,q} = 0$ if $p<0$ or $q<0$

(2) $d_r^{p,q} = 0$ if $r \geqq q+2$

$d_r^{p-r,q+r-1} = 0$ if $r \geqq p+1$

(3) $E_r^{p,q} = E_{r+1}^{p,q} = E_\infty^{p,q}$ if $r \geqq \max\{p+1, q+2\}$

proof (1) $p<0 \Rightarrow A_p = A_{p+1} = A \Rightarrow Z_r^p \subset H(A_p/A_{p+1}) = 0$

$q<0 \Rightarrow A_p^{p+q} = 0 \Rightarrow Z_r^{p,q} \subset H^{p+q}(A_p/A_{p+1}) = 0$

(2) $\Leftarrow$ (1)

(3) $r \geqq \max\{p+1, q+2\}$ とする

$E_r^{p,q} = E_{r+1}^{p,q}$ ($\because$ (2))

$A_{p+r}^{p+q-1} = A_{p+r}^{p+q} = A_{p+r}^{p+q+1} = 0$ ($\because$ $r \geqq q+2$)

$H^{p+q}(A_p/A_{p+r}) = H^{p+q}(A_p)$

$Z_r^{p,q} = \mathrm{Im}(j^*: H^{p+q}(A_p/A_{p+r}) \to H^{p+q}(A_p/A_{p+1}))$

$= \mathrm{Im}(j^*: H^{p+q}(A_p) \to H^{p+q}(A_p/A_{p+1})) = Z_\infty^{p,q}$

$H(A_{p-r+1}/A_p) = H(A/A_p)$ ($\because$ $r \geqq p+1$)

$B_r^p = \mathrm{Im}(\delta^*: H(A_{p-r+1}/A_p) \to H(A_p/A_{p+1}))$

$= \mathrm{Im}(\delta^*: H(A/A_p) \to H(A_p/A_{p+1})) = B_\infty^p$

$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} = E_\infty^{p,q}$ //