§5. Hochschild-Serre spectral sequences

$\mathbb{K}$: field of char. 0.

§5.1. spectral sequence associated with a subalgebra

$\mathfrak{g}$: Lie algebra $/\mathbb{K}$
$\mathfrak{h} \subset \mathfrak{g}$: Lie subalgebra $/\mathbb{K}$.
M : $\mathfrak{g}$-module
$A^n = A^n(M) := C^n(\mathfrak{g}; M)$, $n \geq 0$
$A = A(M) = \{A^n(M)\}$: cochain complex

$A_p^n = A_p^n(M)$, $p \in \mathbb{Z}$, filtration.
$p \leqq 0$ $\quad A_p^n := A^n$
$p \geqq 1$ $\quad A_p^n := \left\{ f \in C^n(\mathfrak{g}; M) ; \begin{array}{l} f(X_1, \cdots, X_n) = 0 \text{ if } X_i \in \mathfrak{g} \text{ and} \\ \#\{i : X_i \in \mathfrak{h}\} \geqq n-p+1 \end{array} \right\}$

$p \geqq n+1$ $\quad A_p^n = 0$.
$A_p^n \supset A_{p+1}^n$ decreasing
$d(A_p^n) \subset A_p^{n+1}$.

($\because$) $f \in A_p^n$, $X_i \in \mathfrak{g}$, $0 \leq i \leq n$,
$\#\{i : X_i \in \mathfrak{h}\} \geqq n+1-p+1$
$\Rightarrow (df)(X_0, X_1, \cdots, X_n)$
$= \sum_{i=0}^{n} (-1)^i X_i f(X_0, \overset{\hat{i}}{\cdots}, X_n) + \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_0, \overset{\hat{i}\ \hat{j}}{\cdots}, X_n)$
$= 0$ $\quad$ ($\because$ $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$) //

$A_p = A_p(M) := \bigoplus_{n=0}^{\infty} A_p^n(M) \subset A$
subcomplex

$\{A_p\}_{p \in \mathbb{Z}}$: decreasing filtration

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<u>multiplicative structure</u>

M, N : $\mathfrak{g}$-modules

$\cup : A^n(M) \otimes A^m(N) \longrightarrow A^{n+m}(M \otimes N)$

cup product.

$\cup \left(A^n_p(M) \otimes A^m_s(N) \right) \subset A^{n+m}_{p+s}(M \otimes N)$

∵) $f \in A^n_p(M)$, $g \in A^m_s(N)$, $X_i \in \mathfrak{g}$, $1 \leq i \leq n+m$

$\#\{ i : X_i \in \mathfrak{h} \} \geq n+m-p-s+1 = (n-p)+(m-s)+1$

$(f \cup g)(X_1, \cdots, X_{n+m})$

$$= \frac{1}{n!\,m!} \sum_{\sigma \in \mathfrak{S}_{n+m}} (\operatorname{sign} \sigma)\, \underbrace{f(X_{\sigma(1)}, \cdots, X_{\sigma(n)}) \otimes g(X_{\sigma(n+1)}, \cdots, X_{\sigma(n+m)})}_{= \begin{cases} 0 \otimes g(\cdots) \\ \text{or} \\ f(\cdots) \otimes 0 \end{cases}}$$

//

the decreasing filtration $\{A_p\}_{p \in \mathbb{Z}}$ $\underset{\text{induces}}{\Longrightarrow}$ an approximation of $H(A)$

<u>The 0^{th} approximation</u>

$E_0^p \overset{\text{def}}{=} A_p / A_{p+1}$

$E_0^{p,q} \overset{\text{def}}{=} A_p^{p+q} / A_{p+1}^{p+q}$

$$E_0^{n,0} = A_n^n / A_{n+1}^n = A_n^n$$

$$= \{ f : \overbrace{\mathfrak{g}/\mathfrak{h} \times \cdots \times \mathfrak{g}/\mathfrak{h}}^{n} \longrightarrow M : \text{alternating } n\text{-linear map} \}$$

$$=: C^n(\mathfrak{g}/\mathfrak{h} : M) : \mathfrak{h}\text{-module}$$ (とりあえずdは考えない)

$p, q \geq 0$

$r_p : A_p^{p+q} \longrightarrow C^q(\mathfrak{h} : C^p(\mathfrak{g}/\mathfrak{h} : M))$

$f \in A_p^{p+q}$, $Y_i \in \mathfrak{h}$

$$(r_p f)(Y_1, \cdots, Y_q) := f(Y_1, \cdots, Y_q, \overbrace{\bullet, \cdots, \bullet}^{p}) \in C^p(\mathfrak{g}/\mathfrak{h} : M)$$

$\operatorname{Ker} r_p = A_{p+1}^{p+q}$ (by definition !!)

$r_p : E_0^{p,q} = A_p^{p+q} / A_{p+1}^{p+q} \hookrightarrow C^q(\mathfrak{h} : C^p(\mathfrak{g}/\mathfrak{h} : M))$ <u>injective</u>

<u>Lemma 5.1</u> r_p : surjective, i.e.,

$$\phi := r_p : E_0^{p,q} \xrightarrow{\cong} C^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M)) \quad \text{isomorphism}$$

<u>proof</u> $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h},\ X \mapsto X'$, quotient map

$\mathfrak{g} \to \mathfrak{h},\ X \mapsto X^*$, $\mathbb{K}$-linear map. s.t. $\forall X \in \mathfrak{h},\ X^* = X$

$\forall g \in C^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M))$ $X_i \in \mathfrak{g},\ 1 \le i \le p+q$.

$$f(X_1, X_2, \cdots, X_{p+q})$$
$$\overset{\text{def}}{=} \frac{1}{p!\,q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (\operatorname{sign}\sigma)\, g(X^*_{\sigma(1)}, \cdots, X^*_{\sigma(q)})(X'_{\sigma(q+1)}, \cdots, X'_{\sigma(q+p)})$$
$$= \sum_{\substack{\sigma \in \mathfrak{S}_{p+q} \\ \sigma(1) < \cdots < \sigma(q) \\ \sigma(q+1) < \cdots < \sigma(q+p)}} (\operatorname{sign}\sigma)\, g(X^*_{\sigma(1)}, \cdots, X^*_{\sigma(q)})(X'_{\sigma(q+1)}, \cdots, X'_{\sigma(q+p)})$$

$$\left.\begin{array}{l} f \in A^{p+q} \\ \#\{i : X_i \in \mathfrak{h}\} \ge q+1 \Rightarrow f(X_1, \cdots, X_{p+q}) = 0 \end{array}\right\} \Rightarrow f \in A_p^{p+q}$$

$Y_i \in \mathfrak{h}$,

$$f(Y_1, \cdots, Y_q, \overbrace{\bullet, \cdots, \bullet}^{p}) = g(Y_1, \cdots, Y_q)(\overbrace{\bullet', \cdots, \bullet'}^{p}) \quad //$$

M, N : $\mathfrak{g}$-modules

$$\cup : E_0^{p,q}(M) \otimes E_0^{s,t}(N) \to E_0^{p+s, q+t}(M \otimes N)$$

well-defined

$(\because)$ $\cup(A_{p+1}(M) \otimes A_s(N) + A_p(M) \otimes A_{s+1}(N)) \subset A_{p+s+1}(M \otimes N)$ //

$$\cup : C^p(\mathfrak{g}/\mathfrak{h} ; M) \otimes C^s(\mathfrak{g}/\mathfrak{h} ; N) \to C^{p+s}(\mathfrak{g}/\mathfrak{h} ; M \otimes N)$$

cup product

$$\Rightarrow \cup : C^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M)) \otimes C^t(\mathfrak{h} ; C^s(\mathfrak{g}/\mathfrak{h} ; N))$$
$$\xrightarrow{\cup} C^{q+t}(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M) \otimes C^s(\mathfrak{g}/\mathfrak{h} ; N))$$
$$\xrightarrow{\cup_*} C^{q+t}(\mathfrak{h} ; C^{p+s}(\mathfrak{g}/\mathfrak{h} ; M \otimes N))$$

<u>Lemma 5.2.</u> $\forall e \in E_0^{p,q}(M),\ \forall e' \in E_0^{s,t}(N)$

$$\phi(e \cup e') = (-1)^{pt} \phi(e) \cup \phi(e')$$

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(略) proof $f \in A_p^{p+q}(M) \longmapsto e$

$g \in A_s^{s+t}(N) \longmapsto e'$

$X_i \in \mathfrak{g}$, $1 \le i \le p+q+s+t$

$X_i \in \mathfrak{h}$ if $1 \le i \le q+t$

$$r_{p+s}(f \cup g)(X_1, \cdots, X_{q+t})(X_{q+t+1}, \cdots, X_{p+q+s+t})$$

$$= (f \cup g)(X_1, \cdots, X_{p+q+s+t})$$

$$= \sum_{(\#)\left\{\begin{array}{l}\sigma \in \mathfrak{S}_{p+q+s+t} \\ \sigma(1) < \cdots < \sigma(p+q) \\ \sigma(p+q+1) < \cdots < \sigma(p+q+s+t)\end{array}\right.} (\operatorname{sign}\sigma) f(X_{\sigma(1)}, \cdots, X_{\sigma(p+q)}) \otimes g(X_{\sigma(p+q+1)}, \cdots, X_{\sigma(p+q+s+t)})$$

$$= \sum_{\substack{(\#) \\ \sigma(q) \leqq q+t \lneqq \sigma(q+1) \\ \sigma(p+q+t) \leqq q+t \lneqq \sigma(p+q+t+1)}} (\operatorname{sign}\sigma) f(X_{\sigma(1)}, \cdots, X_{\sigma(p+q)}) \otimes g(X_{\sigma(p+q+1)}, \cdots, X_{\sigma(p+q+s+t)})$$

$$= (-1)^{pt} \sum_{\substack{\tau \in \mathfrak{S}_{q+t} \\ \rho \in \mathfrak{S}_{p+s} \\ \tau(1) < \cdots < \tau(q) \\ \tau(q+1) < \cdots < \tau(q+t) \\ \rho(1) < \cdots < \rho(p) \\ \rho(p+1) < \cdots < \rho(p+s)}} (\operatorname{sign}\tau)(\operatorname{sign}\rho) f(X_{\tau(1)}, \cdots, X_{\tau(q)}, X_{q+t+\rho(1)}, \cdots, X_{q+t+\rho(p)}) \otimes g(X_{\tau(q+1)}, \cdots, X_{\tau(q+t)}, X_{q+t+\rho(p+1)}, \cdots, X_{q+t+\rho(p+s)})$$

$$= (-1)^{pt} \sum_{\substack{\tau \in \mathfrak{S}_{q+t} \\ \tau(1) < \cdots < \tau(q) \\ \tau(q+1) < \cdots < \tau(q+t)}} (\operatorname{sign}\tau) \left[(r_p f)(X_{\tau(1)}, \cdots, X_{\tau(q)}) \cup (r_s g)(X_{\tau(q+1)}, \cdots, X_{\tau(q+t)}) \right] (X_{q+t+1}, \cdots, X_{p+q+s+t})$$

$$= (-1)^{pt} ((r_p f) \cup (r_s g))(X_1, \cdots, X_{q+t})(X_{q+t+1}, \cdots, X_{p+q+s+t}) \quad //$$

The 1^{st} approximation

$$E_1^p \overset{\text{def}}{=} H(E_0^p) = H(A_p / A_{p+1})$$

$$E_1^{p,q} \overset{\text{def}}{=} H^{p+q}(A_p / A_{p+1})$$

$$E_0^{p,q} \xrightarrow[\phi]{\cong} C^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M))$$

<u>LHS</u> A_p/A_{p+1} : cochain complex

$$d_0^{p,q} := d : E_0^{p,q} = A_p^{p+q}/A_{p+1}^{p+q} \to E_0^{p,q+1} = A_p^{p+q+1}/A_{p+1}^{p+q+1}$$

<u>RHS</u> $d_{\mathfrak{h}} : C^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M)) \to C^{q+1}(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M))$

<u>Theorem 5.3</u> $d_{\mathfrak{h}} \circ \phi = \phi \circ d_0^{p,q} : E_0^{p,q} \to C^{q+1}(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M))$

<u>Corollary 5.4</u>

$$\phi : E_1^{p,q} = H^{p+q}(A_p/A_{p+1}) \xrightarrow{\cong} H^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M))$$

<u>準備</u>

$f \in C^n(\mathfrak{g} ; M)$,

$q \geq n+1 \Rightarrow f_q := 0$

$0 \leq q \leq n$, $f_q \in C^q(\mathfrak{g} ; C^{n-q}(\mathfrak{g} ; M))$

$$f_q(X_1, \cdots X_q)(X_{q+1}, \cdots, X_n) \stackrel{\text{def}}{=} f(X_1, \cdots, X_q, X_{q+1}, \cdots X_n)$$

<u>Lemma 5.5</u> $\forall X_i \in \mathfrak{g}$

$$(df)_{q+1}(X_0, \cdots X_q) = d(f_q)(X_0, \cdots X_q) + (-1)^{q+1} d(f_{q+1}(X_0, \cdots X_q))$$

<u>proof</u> $X_i \in \mathfrak{g}$.

(回答) $(df)_{q+1}(X_0, \cdots, X_q)(X_{q+1}, \cdots, X_n)$

$$= (df)(X_0, \cdots, X_q, X_{q+1}, \cdots X_n)$$

$$= \sum_{i=0}^{q} (-1)^i X_i f(X_0, \overset{\hat{i}}{\cdots}, X_q, X_{q+1}, \cdots X_n)$$

$$+ \sum_{i=q+1}^{n} (-1)^i X_i f(X_0, \cdots, X_q, X_{q+1}, \overset{\hat{i}}{\cdots}, X_n)$$

$$+ \sum_{0 \leq i < j \leq q} (-1)^{i+j} f([X_i, X_j] X_0, \overset{\hat{i}\ \hat{j}}{\cdots}, X_q, X_{q+1}, \cdots X_n)$$

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$$+\sum_{i=0}^{q}\sum_{j=q+1}^{n}(-1)^{i+j}f([X_i,X_j],X_0,\overset{\hat{i}}{\cdots},X_q,X_{q+1},\overset{\hat{j}}{\cdots},X_n)$$

$$+\sum_{q+1\le i<j\le n}(-1)^{i+j}f([X_i,X_j],X_0,\cdots,X_q,X_{q+1},\overset{\hat{i}\ \hat{j}}{\cdots},X_n)$$

$$=\sum_{i=0}^{q}(-1)^i(X_i f_q(X_0,\overset{\hat{i}}{\cdots},X_q))(X_{q+1},\cdots,X_n)$$

$$+\underline{\sum_{i=0}^{q}\sum_{j=q+1}^{n}(-1)^i f(X_0,\overset{\hat{i}}{\cdots},X_q,X_{q+1},\cdots,\overset{\smile{j}}{[X_i,X_j]},\cdots X_n)}$$

$$+\sum_{i=q+1}^{n}(-1)^i X_i(f_{q+1}(X_0,\cdots,X_q)(X_{q+1},\overset{\hat{i}}{\cdots},X_n))$$

$$+\sum_{0\le i<j\le q}(-1)^{i+j}f_q([X_i,X_j],X_0,\overset{\hat{i}\ \hat{j}}{\cdots},X_q)(X_{q+1},\cdots X_n)$$

$$+\underline{\sum_{i=0}^{q}\sum_{j=q+1}^{n}(-1)^{i+1}f(X_0,\overset{\hat{i}}{\cdots},X_q,X_{q+1},\cdots\overset{\smile{j}}{[X_i,X_j]},\cdots X_n)}$$

cancel

$$+(-1)^{q+1}\sum_{q+1\le i<j}(-1)^{i+j}f_{q+1}(X_0,\cdots X_q)([X_i,X_j],X_{q+1},\overset{\hat{i}\ \hat{j}}{\cdots}X_n)$$

$$=d(f_q)(X_0,\cdots,X_q)(X_{q+1},\cdots,X_n)$$
$$+(-1)^{q+1}d(f_{q+1}(X_0,\cdots X_q))(X_{q+1},\cdots,X_n)\quad /\!/$$

<u>Pf of Thm 5.3</u> $f\in A_p^{p+q}$

$f_{q+1}|\mathfrak{h}=0$ $(\because f\in A_p)$

$r_p(f)=f_q|\mathfrak{h}$

$r_p(df)=(df)_{q+1}|\mathfrak{h}$.

$Y_i\in\mathfrak{h}$

$$r_p(df)(Y_0,\cdots Y_q)=(df)_{q+1}(Y_0,\cdots,Y_q)$$

$$\underset{\text{Lem 5.5}}{=}d(f_q)(Y_0,\cdots Y_q)+(-1)^{q+1}d(\underbrace{f_{q+1}(Y_0,\cdots Y_q)}_{=0})$$

$$=d_{\mathfrak{h}}(f_q)(Y_0,\cdots Y_q)$$

$$r_p(df)=d_{\mathfrak{h}}(f_q)=d_{\mathfrak{h}}(r_pf)\quad /\!/\ \text{Thm 5.3}$$

$d_1^{p,q} := \delta^* : E_1^{p,q} = H^{p+q}(A_p/A_{p+1}) \to E_1^{p+1,q} = H^{p+q+1}(A_{p+1}/A_{p+2})$

connecting homomorphism associated with (A_p, A_{p+1}, A_{p+2})

$E_2^{p,q} \overset{\text{def}}{=} H^p(E_1^{*,q}, d_1) \overset{!}{=} \operatorname{Ker} d_1^{p,q} / \operatorname{Im} d_1^{p-1,q}$

the E_2-term of the Hochschild–Serre spectral sequence
(the 2nd approximation of $H^*(\mathfrak{g}; M)$)

Question 1 $E_2^{p,q} = ?$

partial answers for
(i) $\mathfrak{h} < \mathfrak{g}$ ideal.
(ii) $\mathfrak{h} < \mathfrak{g}$ reductive subalgebra
($\overset{\text{def}}{\Longleftrightarrow}$ $\mathfrak{g}$: completely reducible $\mathfrak{h}$-module)

Question 2. $E_3 = H(E_2)$, $E_4 = H(E_3)$, ⋯ ?

Answer spectral sequence !!

- $E_1^{p,q} = H^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} ; M))$

- $p=0$ $E_1^{0,q} = H^q(\mathfrak{h}; M)$

- G : Lie group, $H < G$: closed subgroup
 $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{h} = \operatorname{Lie} H$.
 $X = G/H$: C^∞ mfd.
 $e := 1 \bmod H \in X$
 $C^p(\mathfrak{g}/\mathfrak{h} : M) = (\Lambda^p T_e^* X) \otimes M$ as $\mathfrak{h}$-modules
 $E_1^{p,q} = H^q(\mathfrak{h} ; C^p(\mathfrak{g}/\mathfrak{h} : M))$
 $= H^q(\mathfrak{g} ; (\text{the (formal) germs at } e \text{ of } p\text{-forms on } X) \otimes M)$ ($\because$ Shapiro Lemma)

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- $\underline{q=0}$

$$E_1^{p,0} = H^0(\mathfrak{h} : C^p(\mathfrak{g}/\mathfrak{h} : M))$$
$$= \{ f \in C^p(\mathfrak{g} : M) ;\ \forall Y \in \mathfrak{h},\ i(Y) f = \mathcal{L}(Y) f = 0 \}$$
$$= \{ f \in C^p(\mathfrak{g} : M) ;\ \forall Y \in \mathfrak{h},\ i(Y) f = i(Y) df = 0 \}$$
$$=: C^p(\mathfrak{g}, \mathfrak{h} : M)$$

$$C^*(\mathfrak{g}, \mathfrak{h} : M) = \{ C^p(\mathfrak{g}, \mathfrak{h} : M) \}_{p \geq 0}$$
$\subset C^*(\mathfrak{g} : M)$ cochain subcomplex

(∵) $f \in C^p(\mathfrak{g}, \mathfrak{h} : M)$, $Y \in \mathfrak{h}$
$i(Y) df = 0$, $i(Y) ddf = 0$ //

$$H^*(\mathfrak{g}, \mathfrak{h} : M) \stackrel{\mathrm{def}}{=} H^*(C^*(\mathfrak{g}, \mathfrak{h} : M))$$

the relative cohomology of Chevalley-Eilenberg

($\underline{\text{Rmk}}$ $H^*(\mathfrak{g}, \mathfrak{h} : M) \not\cong H^*(\text{Mapping Cone}(C^*(\mathfrak{g} : M) \to C^*(\mathfrak{h} : M)))$)

$\underline{\text{Lemma 5.6}}$ $E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h} : M)$

[∵) $E_1^{p,0} = H^p(A_p / A_{p+1}) \xrightarrow{\delta^*} E_1^{p+1,0} = H^{p+1}(A_{p+1}/A_{p+2})$
$\forall e$, $\exists f \in A_p$, s.t. $df \in A_{p+1}$, $e = [f]$
$\delta^* e = [df] \in H^{p+1}(A_{p+1}/A_{p+2})$
i.e., $d_1^{p,0} = d_{C^*(\mathfrak{g}, \mathfrak{h} : M)}$ //