(§4. Deformation Theory of Lie algebras (つづき))

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

§4.3. Local structure of the moduli spaces

(I) $\mathfrak{g}, \mathfrak{h}$: fin. dim. Lie algebra/$\mathbb{K}$
($\mathcal{H}$: $\mathbb{K}$-Lie group, $\mathfrak{h} = \mathrm{Lie}\,\mathcal{H}$)
$\mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g}, \mathfrak{h})$ ($\subset \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h})$)

(II) V : fin. dim vector space/$\mathbb{K}$.
$A(V) \overset{\mathrm{def}}{=} \{\psi \in \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) : \psi$ satisfies the Jacobi identity$\}$
($\subset \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V)$)

以下, 汎用性のある (I) を中心に解説する.

(I)

Theorem 4.4. (Nijenhuis-Richardson)

$\varphi_0 \in \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g}, \mathfrak{h})$
($\Rightarrow \mathfrak{h}$: $\mathfrak{g}$-module via $\mathfrak{g} \xrightarrow{\varphi_0} \mathfrak{h} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{h})$)

$\left.\begin{array}{l} H^2 = H^2(\mathfrak{g} : \mathfrak{h}) \\ Z^1 = Z^1(C^*(\mathfrak{g} : \mathfrak{h})) = \mathrm{Ker}(d|C^1(\mathfrak{g} : \mathfrak{h})) \end{array}\right\}$ finite dim.

$\|\cdot\|$: norm on Z^1

$$\Rightarrow \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g}, \mathfrak{h}) \underset{\text{near } \varphi_0}{\underset{\mathrm{loc}}{\cong}} \{u \in Z^1 : [u \vee u] + O(\|u\|^2) = 0 \in H^2\}$$

bracket

In particular, if $H^2(\mathfrak{g} ; \mathfrak{h}) = 0$, then
$\mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g}, \mathfrak{h})$ is smooth near φ_0 and
$T_{\varphi_0} \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g}, \mathfrak{h}) = Z^1(C^*(\mathfrak{g} : \mathfrak{h}))$

bracket

where
$$\begin{array}{ccccccc} Z^1 & \to & H^1(\mathfrak{g} : \mathfrak{h}) & \to & H^2(\mathfrak{g} : \mathfrak{h} \otimes \mathfrak{h}) & \xrightarrow{[\,,\,]} & H^2(\mathfrak{g} : \mathfrak{h}) \\ u & \longmapsto & u & \longmapsto & u \vee u & \longmapsto & [u \vee u] \end{array}$$

$H^2(\mathfrak{g} : \mathfrak{h})$: obstruction to the smoothness of the moduli space

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$q \geq 0$

$C^q := C^q(\mathfrak{g} : \mathfrak{h})$

$Z^q := \mathrm{Ker}(d|_{C^q(\mathfrak{g}:\mathfrak{h})})$

$B^q := d\, C^{q-1}(\mathfrak{g} : \mathfrak{h})$

$H^q := H^q(\mathfrak{g} : \mathfrak{h})$

<u>deformation equation</u>

$u \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h}) = C^1$

$\left(\begin{array}{l} \varphi_0 + u \in \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h}) \\ \Leftrightarrow du + \frac{1}{2}[u \vee u] = 0 \in C^2 \end{array} \right.$

$\theta : C^1 \to C^2,\ u \mapsto \theta(u) := du + \frac{1}{2}[u \vee u]$

$\{\theta = 0\} = \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h})$

<u>Lemma 4.5.</u> $\forall u \in C^1$

$$[[u \vee u] \vee u] = [u \vee [u \vee u]] = 0 \in C^3(\mathfrak{g} : \mathfrak{h})$$

(pf) $\forall X, \forall Y, \forall Z \in \mathfrak{g}$

$[[u \vee u] \vee u](X, Y)$

$= 2\{[[u(X), u(Y)], u(Z)] + [[u(Y), u(Z)], u(X)] + [[u(Z), u(X)], u(Y)]\}$

$= 0$ (Jacobi identity)

$[u \vee [u \vee u]] = 0$ も同様 //

<u>Construction of a path $\varphi(t)$, $|t| \ll 1$, with $\varphi(0) = \varphi_0$ on $\mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h})$</u>

<u>under the assumption $H^2(\mathfrak{g} : \mathfrak{h}) = 0$</u>

(<u>after Kodaira - Spencer - Nirenberg</u>)

Taylor expansion

$$\varphi(t) = \sum_{n=0}^{\infty} t^n \varphi_n,\quad \varphi_n \in C^1$$

$$\theta\left(\sum_{n=1}^{\infty} t^n \varphi_n\right) = \sum_{n=1}^{\infty} t^n \left\{ d\varphi_n + \frac{1}{2} \sum_{a+b=n} [\varphi_a \vee \varphi_b] \right\}$$

<u>Solve the recursive equation</u>

$$d\varphi_n = -\frac{1}{2} \sum_{a=1}^{n-1} [\varphi_a \vee \varphi_{n-a}] \in C^2$$

$\underline{n=1}$ $d\varphi_1 = 0$ i.e., $\varphi_1 \in Z^1$

$\underline{n=2}$ $d[\varphi_1 \cup \varphi_1] = [\underbrace{d\varphi_1}_{\substack{\| \\ 0}} \cup \varphi_1] - [\varphi_1 \cup \underbrace{d\varphi_1}_{\substack{\| \\ 0}}] = 0$

$[\varphi_1 \cup \varphi_1] \in H^2(\mathfrak{g} : \mathfrak{h}) = 0$

$\Longrightarrow \exists\, \varphi_2 \in C^1$ s.t. $d\varphi_2 = -\frac{1}{2}[\varphi_1 \cup \varphi_1] \in C^2$

$\underline{n \geqq 3}$ $\varphi_1, \cdots, \varphi_{n-1}$

Lemma 4.5. for $u = \sum_{k=1}^{n-1} t^k \varphi_k$

$$\Longrightarrow \sum_{\substack{a+b+c=n \\ a,b,c \geqq 1}} [[\varphi_a \cup \varphi_b] \cup \varphi_c] = \sum_{\substack{a+b+c=n \\ a,b,c \geqq 1}} [\varphi_a \cup [\varphi_b \cup \varphi_c]] = 0$$

$$d\Big(\sum_{a+b=n} [\varphi_a \cup \varphi_b]\Big) = \sum_{a+b=n} [d\varphi_a \cup \varphi_b] - \sum_{a+b=n} [\varphi_a \cup d\varphi_b]$$

$$= -\frac{1}{2} \sum_{a_1+a_2+b=n} [[\varphi_{a_1} \cup \varphi_{a_2}] \cup \varphi_b] + \frac{1}{2} \sum_{a+b_1+b_2=n} [\varphi_a \cup [\varphi_{b_1} \cup \varphi_{b_2}]]$$

$$= 0$$

$$\sum_{a+b=n} [\varphi_a \cup \varphi_b] \in H^2(\mathfrak{g} : \mathfrak{h}) = 0$$

$$\Longrightarrow \exists\, \varphi \in C^1, \text{ s.t. } d\varphi_n = -\frac{1}{2} \sum_{a+b=n} [\varphi_a \cup \varphi_b] \in C^2$$

Proof of Thm 4.4.

[Reference: A. Douady "Le problème des modules pour les variétés analytiques complexes" Sém. Bourbaki, exp. nº 277, pp 7-13. (1964)

$u \in C^1$, $\theta(u) = du + \frac{1}{2}[u \cup u] \in C^2$

$\mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h}) = \{\theta = 0\}$

$$d\theta(u) = \frac{1}{2}[du \cup u] - \frac{1}{2}[u \cup du] = [du \cup u]$$

$$= [\theta(u) \cup u] - \frac{1}{2}[[u \cup u] \cup u]$$

$$\underset{\text{Lem 4.5}}{=} [\theta(u) \cup u] \in C^3$$

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$$\left[\begin{array}{l} C^3 \cong B^3 \oplus {}^{\exists}(B^3)' \quad \text{complements} \\ C^2 \cong Z^2 \oplus {}^{\exists}(Z^2)' \\ \Longrightarrow \ p_{B^3} : C^3 \twoheadrightarrow B^3 \quad \text{projections} \\ \quad\ p_{Z^2} : C^2 \twoheadrightarrow Z^2 \\ \quad \text{s.t. } \operatorname{Ker} p_{B^3} = (B^3)' \\ \qquad\ \operatorname{Ker} p_{Z^2} = (Z^2)' \end{array}\right.$$

$E := \{(u,\alpha) \in C^1 \times C^2 : d\alpha - [\alpha \cup u] \in (B^3)'\}$

vector bundle of rank $= \dim Z^2$ near $u = 0 \in C^1$

$$\left[\begin{array}{l} (\because)\ C^1 \times C^2 \longrightarrow B^3 \\ \quad (u,\alpha) \longmapsto p_{B^3}(d\alpha - [\alpha \cup u]) \\ \quad \text{surjective at } u=0 \ /\!/ \end{array}\right.$$

$1_{C^1} \times p_{Z^2} : E \longrightarrow C^1 \times Z^2$

isom. near $u = 0 \in C^1$

$\left(\because \quad E|_{u=0} = Z^2 \ /\!/ \right)$

$(u, \theta(u)) \in E, \ \forall u \in C^1$ i.e., $\theta \in \Gamma(C^1; E)$

$\{\theta = 0\} \underset{loc.}{=} \{p_{Z^2} \circ \theta = 0\}$ near $u = 0$

$Z^2 \cong B^2 \oplus {}^{\exists}\mathcal{H}$ complement

$$\Longrightarrow \begin{array}{l} p_{B^2} : C^2 \to B^2 \\ p_{H^2} : C^2 \to H^2 \end{array} \Big\} \text{projections}$$

s.t. $\operatorname{Ker} p_{B^2} = \mathcal{H} \oplus (Z^2)'$

$\operatorname{Ker} p_{H^2} = B^2 \oplus (Z^2)'$

$\{\theta = 0\} \underset{loc.}{=} \{p_{Z^2} \circ \theta = 0\} < \{p_{B^2} \circ \theta = 0\}$

$\{p_{B^2} \circ \theta = 0\} \underset{loc}{\cong} Z^2$ near 0

i.e. $\{p_{B^2} \circ \theta = 0\}$: smooth near 0

$(\text{tang. sp. at } u = 0) = Z^1$

$(\because)$ $(p_{B^2}\circ\theta)(u) = du + p_{B^2}[u \vee u] \in B^2$
$C^1 \xrightarrow{d} B^2$ surjective //

(If $H^2=0$, then $\{\theta=0\} = \{p_{B^2}\circ\theta=0\} \overset{loc}{\cong} Z^1$)

$$\begin{array}{ccccc} Z^1 & \overset{loc.}{\cong} & \{p_{B^2}\circ\theta=0\} & \supset \{\theta=0\} = & \{p_{Z^2}\circ\theta=0\} \\ \Psi & & \Psi & & \| \\ u & \longmapsto & u+o(\|u\|) & & \{p_{B^2}\circ\theta=0,\ p_{H^2}\circ\theta=0\} \end{array}$$

$$(p_{H^2}\circ\theta)(u+o(\|u\|))$$
$$= p_{H^2}\left(d(u+o(\|u\|)) + \tfrac{1}{2}[u\vee u] + o(\|u\|^2)\right)$$
$$= \tfrac{1}{2}[u\vee u] + o(\|u\|^2) \quad (\because\ u\in Z^1 \Rightarrow [u\vee u]\in Z^2)$$

$$\mathrm{Hom}_{Lie\,alg}(\mathfrak{g},\mathfrak{h}) = \{\theta=0\} \overset{loc.}{\cong} \{u\in Z^1;\ \underbrace{[u\vee u]+o(\|u\|^2)}=0\in H^2\}$$
near $u=0$ ($\Leftrightarrow$ near φ_0) //

[Kuranishi map
$H^1 \to H^2$

If $H^2(\mathfrak{g}:\mathfrak{h})=0$ and the conjugate action $\mathfrak{h} \curvearrowright \mathrm{Hom}_{Lie\,alg}(\mathfrak{g},\mathfrak{h})$ is (free and) properly discontinuous near φ_0,
then the moduli space $\mathrm{Hom}_{Lie}(\mathfrak{g},\mathfrak{h})/\mathfrak{h}$ is a mfd near $[\varphi_0]$,
and $T_{[\varphi_0]}\mathrm{Hom}_{Lie\,alg}(\mathfrak{g},\mathfrak{h}) = H^1(\mathfrak{g}:\mathfrak{h})$

(II) V : fin.dim. vector space $/\mathbb{K}$
$\psi_0 \in A(V) \subset \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V)$
$\mathfrak{g} := (V,\psi_0)$ Lie algebra
($\Longrightarrow$ $\mathfrak{g}$: $\mathfrak{g}$-module via $\mathfrak{g} \xrightarrow{ad} \mathfrak{g}$)
$p, q \geq 1$

$$S : C^p(\mathfrak{g}:\mathfrak{g}) \times C^q(\mathfrak{g}:\mathfrak{g}) \to C^{p+q-1}(\mathfrak{g}:\mathfrak{g})$$
$$\qquad\qquad \Psi\ (v,w)$$

$$S(v,w)(X_1,\cdots,X_{p+q-1})$$
$$:= \frac{1}{(p-1)!\,q!}\sum_{\sigma\in\mathfrak{S}_{p+q-1}} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(1)},\cdots,X_{\sigma(p-1)}, w(X_{\sigma(p)},\cdots,X_{\sigma(p+q-1)}))$$

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Lemma 4.6 $\forall v \in C^p(\mathfrak{g};\mathfrak{g}), \forall w \in C^q(\mathfrak{g};\mathfrak{g})$

$$dS(v,w) = S(dv,w) + (-1)^{p-1} S(v,dw) + (-1)^{p-1}[v \cup w] \in C^{p+q}(\mathfrak{g};\mathfrak{g})$$

proof $C : (\mathfrak{g}\otimes\mathfrak{g}^*)\otimes\mathfrak{g} \to \mathfrak{g}$ $\mathfrak{g}$-homom

$(X\otimes f)\otimes Y \mapsto f(Y)X$

$E : C^p(\mathfrak{g};\mathfrak{g}) \to C^{p-1}(\mathfrak{g};\mathfrak{g}\otimes\mathfrak{g}^*),\ v \mapsto Ev,$

$(Ev)(X_1,\cdots,X_{p-1})(X_p) := v(X_1,\cdots,X_{p-1},X_p) \in \mathfrak{g}$

$(X_i \in \mathfrak{g})$

$$S(v,w) = C_*((Ev)\cup w)$$

$$\begin{aligned}
&(dEv)(X_1,\cdots,X_p)(X_{p+1})\\
&= \sum_{i=1}^{p}(-1)^{i+1}(X_i(Ev)(X_1,\overset{\hat{i}}{\cdots},X_p))(X_{p+1})\\
&\quad + \sum_{i<j}(-1)^{i+j}(Ev)([X_i,X_j],X_1,\overset{\hat{i}\ \hat{j}}{\cdots},X_p)(X_{p+1})\\
&= \sum_{i=1}^{p}(-1)^{i+1}X_i(v(X_1,\overset{\hat{i}}{\cdots},X_p,X_{p+1})) - \sum_{i=1}^{p}(-1)^{i+1}v(X_1,\overset{\hat{i}}{\cdots},X_p,[X_i,X_{p+1}])\\
&\quad + \sum_{i<j\le p}(-1)^{i+j}v([X_i,X_j],X_1,\overset{\hat{i}\ \hat{j}}{\cdots},X_p,X_{p+1})\\
&= \sum_{i=1}^{p}(-1)^{i+1}X_i(v(X_1,\overset{\hat{i}}{\cdots},X_i,X_{p+1}))\\
&\quad + \sum_{i<j\le p+1}(-1)^{i+j}v([X_i,X_j],X_1,\overset{\hat{i}\ \hat{j}}{\cdots},X_{p+1})\\
&= (-1)^p X_{p+1}(v(X_1,\cdots,X_p)) + (dv)(X_1,\cdots,X_{p+1})\\
&= (-1)^{p-1}[v(X_1,\cdots,X_p),X_{p+1}] + (Edv)(X_1,\cdots,X_p)(X_{p+1})
\end{aligned}$$

$$\begin{aligned}
dS(v,w) &= dC_*((Ev)\cup w)\\
&= C_*((dEv)\cup w) + (-1)^{p-1}C_*((Ev)\cup dw)\\
&= C_*((dE-Ed)v\cup w) + S(dv,w) + (-1)^{p-1}S(v,dw)
\end{aligned}$$

$$\begin{aligned}
&C_*((dE-Ed)v\cup w)(X_1,\cdots,X_{p+q})\\
&= \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}}(\mathrm{sign}\,\sigma)(dE-Ed)(v)(X_{\sigma(1)},\cdots,X_{\sigma(p)})(w(X_{\sigma(p+1)},\cdots,X_{\sigma(p+q)}))\\
&= \frac{(-1)^{p-1}}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}}(\mathrm{sign}\,\sigma)[v(X_{\sigma(1)},\cdots,X_{\sigma(p)}),w(X_{\sigma(p+1)},\cdots,X_{\sigma(p+q)})]
\end{aligned}$$

$= (-1)^{p-1} [v \cup w](X_1, \cdots, X_{p+q})$ //

Corollay 4.7

$S : H^2(\mathfrak{g};\mathfrak{g}) \to H^3(\mathfrak{g};\mathfrak{g}),\ [v] \mapsto [S(v,v)]$, well-defined

proof $v \in Z^2$

$dS(v,v) = S(dv,v) - S(v,dv) - [\overset{2}{v} \cup \overset{2}{v}] = 0 - 0 - 0 = 0$

$f \in C^1$

$S(v+df, v+df) - S(v,v) = S(v,df) + S(df,v) + S(df,df)$

$d(-S(v,f) + S(f,v)) = S(v,df) + [\overset{2}{v} \cup \overset{1}{f}] + S(df,v) + [\overset{1}{f} \cup \overset{2}{v}]$

$= S(v,df) + S(df,v)$

$d[f \cup f] = [df \cup f] - [f \cup df] = -2[f \cup df]$

$d(S(f,df)) = S(df,df) + [f \cup df] = S(df,df) - \frac{1}{2} d[f \cup f]$ //

技術的準備

Lemma 4.8. $\forall v \in C^2(\mathfrak{g};\mathfrak{g})$

$S(S(v,v),v) = S(v, S(v,v)) \in C^4(\mathfrak{g};\mathfrak{g})$

pf $\forall X_i \in \mathfrak{g}$

$S(S(v,v),v)(X_1, X_2, X_3, X_4)$

$= \frac{1}{2!2!} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, S(v,v)(X_{\sigma(1)}, X_{\sigma(2)}, v(X_{\sigma(3)}, X_{\sigma(4)}))$

$= \frac{1}{4} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(1)}, v(X_{\sigma(2)}, v(X_{\sigma(3)}, X_{\sigma(4)})))$

$+ \frac{1}{4} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(2)}, v(v(X_{\sigma(3)}, X_{\sigma(4)}), X_{\sigma(1)}))$

$\left. + \frac{1}{4} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, v(v(X_{\sigma(3)}, X_{\sigma(4)}), v(X_{\sigma(1)}, X_{\sigma(2)})) \right] = 0$

$= \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(1)}, v(X_{\sigma(2)}, v(X_{\sigma(3)}, X_{\sigma(4)})))$

$S(v, S(v,v))(X_1, X_2, X_3, X_4)$

$= \frac{1}{1!3!} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(1)}, S(v,v)(X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}))$

$= \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_4} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(1)}, v(X_{\sigma(2)}, v(X_{\sigma(3)}, X_{\sigma(4)})))$ //

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<u>Theorem 4.9</u> (Nijenhuis - Richardson)

$\psi_0 \in A(V)$, $\mathfrak{g} := (V, \psi_0)$: Lie algebra

($\Longrightarrow$ $\mathfrak{g}$: $\mathfrak{g}$-module via $\mathfrak{g} \xrightarrow{ad} \mathfrak{gl}(\mathfrak{g})$).

$$\left.\begin{aligned} H^3 &= H^3(\mathfrak{g} : \mathfrak{g}) \\ Z^2 &= Z^2(C^*(\mathfrak{g}:\mathfrak{g})) = \mathrm{Ker}(d|_{C^2(\mathfrak{g}:\mathfrak{g})}) \end{aligned}\right\} \text{finite dim.}$$

$\|\cdot\|$: norm on Z^2

$$\Longrightarrow A(V) \underset{loc}{\cong} \{ v \in Z^2 ; \ S(v,v) + o(\|v\|^2) = 0 \in H^3 \} \text{ near } \psi_0$$

In particular, if $H^3(\mathfrak{g}:\mathfrak{g}) = 0$, then $A(V)$ is smooth near ψ_0, and

$$T_{\psi_0} A(V) = Z^2(C^*(\mathfrak{g}:\mathfrak{g}))$$

<u>proof</u> (after A. Douady)

$$C^q := C^q(\mathfrak{g}:\mathfrak{g}),\ Z^q,\ B^q,\ H^q, \quad , \quad q \geq 0$$

<u>deformation equation</u>

$v \in C^2 = \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V)$

$$\left(\begin{aligned} &\psi_0 + v \in A(V) \\ &\Longleftrightarrow dv + S(v,v) = 0 \in C^3 \end{aligned} \right.$$

$\theta : C^2 \to C^3$, $v \mapsto \theta(v)$

$\theta(v) := dv + S(v,v)$

$\{\theta = 0\} = A(V)$.

$$d\theta(v) = d(S(v,v)) \overset{\text{Lem 4.6}}{=\!=} S(dv,v) - S(v,dv) - \overset{\parallel\, 0}{[v^2 \cup v^2]}$$

$$= S(\theta(v),v) - S(v,\theta(v)) - S(S(v,v),v) + S(v,S(v,v))$$

$$\underset{\text{Lem 4.8}}{=} S(\theta(v),v) - S(v,\theta(v)) \in C^4$$

$C^4 \cong B^4 \oplus {}^{\exists}(B^4)'$

$C^3 \cong Z^3 \oplus {}^{\exists}(Z^3)'$

$\Longrightarrow$ $p_{B^4} : C^4 \twoheadrightarrow B^4$ projections

$p_{Z^3} : C^3 \twoheadrightarrow Z^3$

s.t. $\mathrm{Ker}\, p_{B^4} = (B^4)'$

$\mathrm{Ker}\, p_{Z^3} = (Z^3)'$

$E := \{(v,\beta) \in C^2 \times C^3 ;\ d\beta - S(\beta, v) + S(v,\beta) \in (B^4)'\}$

vector bundle of rk $= \dim Z^3$ near $v = 0 \in C^2$

(∵) $C^2 \times C^3 \to B^4$

$(v,\beta) \mapsto p_{B^3}(d\beta - S(\beta,v) + S(v,\beta))$

surjective at $v = 0$ //

$1_{C^2} \times p_{Z^3} : E \to C^2 \times Z^3$ isom near $v = 0 \in C^2$

(∵ $E|_{v=0} = Z^3$).

$(v, \theta(v)) \in E$, $\forall v \in C^3$, i.e., $\theta \in \Gamma(C^2 ; E)$

$\{\theta = 0\} \overset{loc.}{=} \{p_{Z^3} \circ \theta = 0\}$ near $v = 0$

$Z^3 = B^3 \oplus {}^{\exists}\mathcal{H}$ complement

$\Longrightarrow$ $p_{B^3} : C^3 \twoheadrightarrow B^3$, $p_{H^3} : C^3 \twoheadrightarrow H^3$ } projections

s.t. $\operatorname{Ker} p_{B^3} = \mathcal{H} \oplus (Z^3)'$

$\operatorname{Ker} p_{H^3} = B^3 \oplus (Z^3)'$

$\mathcal{H} \cong H^3(\mathfrak{g} ; \mathfrak{h})$

$\{\theta = 0\} \overset{loc}{=} \{p_{Z^3} \circ \theta = 0\} \subset \{p_{B^3} \circ \theta = 0\}$

$\{p_{B^3} \circ \theta = 0\}$: smooth near $v = 0$

(tang. sp. at $v = 0$) $= Z^2$

(∵ $(p_{B^3} \circ \theta)(v) = dv + p_{B^3}(S(v,v)) \in B^3$

$C^2 \xrightarrow{d} B^3$ surjective //)

(If $H^3 = 0$, then $\{\theta = 0\} \underset{loc}{=} \{p_{B^3} \circ \theta = 0\} \underset{loc}{\cong} Z^2$)

$$\begin{array}{ccccc} Z^2 & \overset{loc}{\cong} & \{p_{B^3} \circ \theta = 0\} \supset \{\theta = 0\} & \overset{loc}{=} & \{p_{Z^3} \circ \theta = 0\} \\ \Psi & & \Psi & & \| \\ v & \longmapsto & v + o(\|v\|) & & \{p_{H^3} \circ \theta = 0,\ p_{B^3} \circ \theta = 0\} \end{array}$$

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$$(p_{H^3}\circ\theta)(v+o(\|v\|))$$
$$= p_{H^3}(d(v+o(\|v\|)) + S(v,v) + o(\|v\|^2))$$
$$= S(v,v)+o(\|v\|^2) \quad (\forall v\in Z^2 \Rightarrow S(v,v)\in Z^3)$$

$$A(V)=\{\theta=0\}$$
$$\overset{loc.}{=} \{v\in Z^2;\ \underbrace{S(v,v)+o(\|v\|^2)}=0\in H^3\}$$

near $v=0 \Leftrightarrow$ near ψ_0. Kuranishi map : $H^2 \to H^3$

// Thm

If $H^3(\mathfrak{g};\mathfrak{g})=0$ and the action $GL(V)\curvearrowright A(V)$ is (free and) properly discontinuous near ψ_0, then the moduli space $A(V)/GL(V)$ is a manifold near $[\psi_0]$, and

$$T_{[\psi_0]}A(V)=H^2(\mathfrak{g};\mathfrak{g})$$