§4. Deformation Theory of Lie algebras

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

after Nijenhuis – Richardson

- Kodaira – Spencer (– Nirenberg), Kuranishi
- Gerstenhaber
- Weil.

§4.1 Deformation equations

2種類の変形を考える

(I) $\mathfrak{g}, \mathfrak{h}$: fin. dim. Lie algebras $/\mathbb{K}$

$\exists G$: $\mathbb{K}$-Lie group s.t. $\mathfrak{h} = \mathrm{Lie}\, G$.

cf) Ado's Theorem ([B] §7, n°3, Thm 2)

$\forall \mathfrak{h}$: fin. dim. Lie algebra $/\mathbb{K}$ $\exists n \gg 1$

$\mathfrak{h} \overset{\exists}{\hookrightarrow} \mathfrak{gl}_n(\mathbb{K})$ embedding of Lie algebras

$\mathfrak{g} \to \mathfrak{h}$: Lie algebra homomorphism

(II) V : fin. dim. $\mathbb{K}$-vector space.

Lie algebra structure on V = Lie bracket on V

つまり、次の2種類の varieties の局所構造をしらべる

(I) $\mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathfrak{h}) \subset \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h})$

(II) $A(V) := \{\mu \in \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) ;\ \mu \text{ satisfies the Jacobi identity}\}$

$\subset \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V)$.

(注) 小さい n についての $A_n = A(\mathbb{C}^n)$ は次でくわしく調べられている。

A. A. Kirillov and Yu. A. Neretin

"The variety A_n of n-dimensional Lie algebra structures"

Amer. Math. Soc. Transl. (2) vol 137 (1987) pp 21–30.

ここでは群作用 (gauge transformation) を含めて考える必要がある

(I) $H \curvearrowright \mathfrak{h}$, $H \curvearrowright \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g},\mathfrak{h})$ adjoint actions

そこで moduli space

$$\mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g},\mathfrak{h})/H$$

の局所構造を考えることになる

(II) $GL(V) \curvearrowright A(V)$ adjoint action.

そこで Lie 代数構造の moduli space

$$A(V)/GL(V)$$

の局所構造を考えることになる

ex) $A_2 = A(\mathbb{C}^2)$, $\{X, Y\} \subset \mathbb{C}^2$ basis

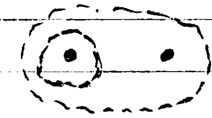
$[X, Y] = aX + bY$, $a, b \in \mathbb{C}$

自動的に Jacobi identity をみたす $(\because \Lambda^3(\mathbb{C}^2) = 0)$

$GL_2(\mathbb{C}) \curvearrowright A_2 = \mathbb{C}^2$

$T \mapsto (\det T)^{-1} T$

$\mathbb{C}^2$ は $\{0\}$ と $\{0\}$ に分かれる

$A_2/GL_2(\mathbb{C}) = \{2点\} =$ 

(I) $\varphi \in \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g},\mathfrak{h})$.

$\Longrightarrow$ $\mathfrak{h}$: $\mathfrak{g}$-module via $\mathfrak{g} \xrightarrow{\varphi} \mathfrak{h} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{h})$.

$$\alpha = \alpha_\varphi : H \to \mathrm{Hom}_{\mathrm{Lie\,alg}}(\mathfrak{g},\mathfrak{h}) \subset \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h})$$
$$h \longmapsto \mathrm{Ad}(h)\circ\varphi$$

$$(d\alpha)_e : T_e H \longrightarrow T_\varphi \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h})$$
$$T_e H = \mathfrak{h}, \quad T_\varphi \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) = C^1(\mathfrak{g};\mathfrak{h})$$

$Y \in \mathfrak{h}$, $X \in \mathfrak{g}$

$$(d\alpha)_e(Y) = \frac{d}{dt}(\mathrm{Ad}(e^{tY})\circ\varphi)(X)\Big|_{t=0}$$
$$= \frac{d}{dt}(e^{t\,\mathrm{ad}Y}\varphi(X))\Big|_{t=0} = [Y, \varphi(X)]$$
$$= -[\varphi(X), Y] = -(dY)(X)$$

$$\begin{tikzcd} T_e\mathfrak{h} \arrow[r, "(d\alpha)_e"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & T_\varphi \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) \arrow[d, equal] \\ \mathfrak{h} \arrow[r, "-d"'] & C^1(\mathfrak{g};\mathfrak{h}) \end{tikzcd} \quad \text{--- (1)}$$

$$\beta : \underset{\varphi}{\mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h})} \longrightarrow \mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathfrak{g},\mathfrak{h})$$

$\beta(\varphi)(X,Y) := [\varphi(X),\varphi(Y)] - \varphi([X,Y]) \qquad (X,Y\in\mathfrak{g})$

$\mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g},\mathfrak{h}) = \{\varphi\in\mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) ; \beta(\varphi)=0\} = \{\beta=0\}$

$$\begin{array}{ccc} \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g},\mathfrak{h}) & \subset & \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) \\ \Psi & & \Psi \\ \varphi & & u \end{array}$$

$$\begin{aligned} &\beta(\varphi+u)(X,Y) \\ &= [(\varphi+u)(X),(\varphi+u)(Y)] - (\varphi+u)[X,Y] \\ &= [\varphi(X),u(Y)] + [u(X),\varphi(Y)] - u([X,Y]) + [u(X),u(Y)] \\ &= (du)(X,Y) + \tfrac{1}{2}[u\vee u](X,Y) \\ &= (du+\tfrac{1}{2}[u\vee u])(X,Y) \end{aligned}$$

$$\beta(\varphi+u) = du + \tfrac{1}{2}[u\vee u] \quad \text{---- (2)}$$

$$\boxed{du+\tfrac{1}{2}[u\vee u]=0}$$ deformation equation (Maurer–Cartan equation)

$$\begin{tikzcd} T_\varphi \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) \arrow[r, "(d\beta)_\varphi"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & T_0 \mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathfrak{g},\mathfrak{h}) \arrow[d, equal] \\ C^1(\mathfrak{g};\mathfrak{h}) \arrow[r, "d"'] & C^2(\mathfrak{g};\mathfrak{h}) \end{tikzcd} \quad \text{---- (3)}$$

(II). $\psi\in A(V)\subset \mathrm{Hom}_{\mathbb{K}}(\Lambda^2V,V)$

$\mathfrak{g} := (V,\psi)$; Lie algebra : $\mathfrak{g}$-module via $\mathfrak{g}\xrightarrow{ad}\mathfrak{gl}(\mathfrak{g})$

$$\begin{array}{rccc} \alpha=\alpha_\psi : & GL(V) & \longrightarrow & A(V) \\ & \Psi & & \Psi \\ & g & \longmapsto & g\circ\psi\circ g^{-1} \end{array}$$

$$\begin{tikzcd} (d\alpha)_e : T_e GL(V) \arrow[r] \arrow[d, equal] & T_\psi \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) \arrow[d, equal] \\ \mathrm{Hom}_{\mathbb{K}}(V,V) = C^1(\mathfrak{g};\mathfrak{g}) & \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) = C^2(\mathfrak{g};\mathfrak{g}) \end{tikzcd}$$

$u \in \mathrm{Hom}_{\mathbb{K}}(V,V) = T_e GL(V)$

$X, Y \in \mathfrak{g} = V$

$$(d\alpha)_e(u)(X,Y) = \frac{d}{dt}\left(e^{tu}\circ\psi\circ e^{-tu})(X,Y)\right)\Big|_{t=0}$$

$$= u(\psi(X,Y)) - \psi(u(X),Y) - \psi(X,u(Y))$$

$$= u([X,Y]) - [u(X),Y] - [X,u(Y)]$$

$$= -(du)(X,Y)$$

$$\begin{tikzcd} T_e GL(V) \arrow[r, "(d\alpha)_e"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & T_\psi \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) \arrow[d, equal] \\ C^1(\mathfrak{g};\mathfrak{g}) \arrow[r, "-d"'] & C^2(\mathfrak{g};\mathfrak{g}) \end{tikzcd} \quad \cdots\cdots (4)$$

$$\beta : \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(\Lambda^3 V, V) \qquad \text{Jacobi identity}$$
$$\psi \longmapsto \beta(\psi)$$

$X, Y, Z \in \mathfrak{g}$

$$\beta(\psi)(X,Y,Z) := \psi(X,\psi(Y,Z)) + \psi(Y,\psi(Z,X)) + \psi(Z,\psi(X,Y))$$

$$A(V) = \{\psi \in \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) \,;\, \beta(\psi) = 0\} = \{\beta = 0\}$$

$$\underset{\psi}{\underset{\cup}{A(V)}} \subset \underset{\nu}{\underset{\cup}{\mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V)}}$$

$\mathfrak{g} := (V, \psi)$: Lie algebra.

$X, Y, Z \in V = \mathfrak{g}$.

$$\beta(\psi+\nu)(X,Y,Z)$$
$$= (\psi+\nu)(X,(\psi+\nu)(Y,Z)) + (\psi+\nu)(Y,(\psi+\nu)(Z,X)) + (\psi+\nu)(Z,(\psi+\nu)(X,Y))$$
$$\underset{\beta(\psi)=0}{=} [X,\nu(Y,Z)] + [Y,\nu(Z,X)] + [Z,\nu(X,Y)]$$
$$+ \nu(X,[Y,Z]) + \nu(Y,[Z,X]) + \nu(Z,[X,Y])$$
$$\boxed{+ \nu(X,\nu(Y,Z)) + \nu(Y,\nu(Z,X)) + \nu(Z,\nu(X,Y))}$$
$$\overset{\shortparallel}{\because}\ S(\nu,\nu)(X,Y,Z), \quad S(\nu,\nu) \in \mathrm{Hom}(\Lambda^3 V, V)$$

[where $S(v,w) \in C^{p+q-1}(\mathfrak{g};\mathfrak{g})$, $v \in C^{p}(\mathfrak{g}:\mathfrak{g})$, $w \in C^{q}(\mathfrak{g}:\mathfrak{g})$

$S(v,w)(X_1,\dots,X_{p+q-1})$ $\quad (X_i \in \mathfrak{g})$

$$:= \frac{1}{(p-1)!\,q!}\sum_{\sigma\in\mathfrak{S}_{p+q-1}} (\mathrm{sign}\,\sigma)\, v(X_{\sigma(1)},\dots,X_{\sigma(p-1)}, w(X_{\sigma(p)},\dots,X_{\sigma(p+q-1)}))$$]

$$= (dv + S(v,v))(X,Y,Z)$$

$$\beta(\psi+v) = dv + S(v,v) \quad \text{---}(5)$$

$\boxed{dv + S(v,v) = 0}$ deformation equation

$$\begin{tikzcd}
T_\psi \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) \arrow[r, "(d\beta)_\psi"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & T_0 \mathrm{Hom}_{\mathbb{K}}(\Lambda^3 V, V) \arrow[d, equal] \\
C^2(\mathfrak{g}:\mathfrak{g}) \arrow[r, "d"'] & C^3(\mathfrak{g}:\mathfrak{g})
\end{tikzcd} \quad \text{---}(6)$$

§4.2. Local rigidity

ここでは非常に弱い rigidity（剛性）を考える

Definition $\mathfrak{g}, \mathfrak{h}, V$ as above.

(I). $\varphi \in \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g},\mathfrak{h})$: locally rigid

$\overset{\text{def}}{\iff}$ $\varphi \in {}^{\exists}U \overset{\text{open}}{\subset} \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g},\mathfrak{h})$ s.t. $U \subset L_{\mathfrak{g}}\cdot\varphi$

(II) $\psi \in A(V)$: locally rigid

$\overset{\text{def}}{\iff}$ $\psi \in {}^{\exists}U \overset{\text{open}}{\subset} A(V)$ s.t. $U \subset GL(V)\cdot\psi$

つまり「少しくらい perturb しても構造は変わらない」ということ

Theorem 4.1 (Nijenhuis-Richardson) $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

(I). $\mathfrak{g}, \mathfrak{h}$: fin. dim. Lie algebras $/\mathbb{K}$

$\varphi : \mathfrak{g} \to \mathfrak{h}$: Lie algebra homomorphism

($\Longrightarrow$ $\mathfrak{h}$: $\mathfrak{g}$-module via $\mathfrak{g} \xrightarrow{\varphi} \mathfrak{h} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{h})$)

$H^1(\mathfrak{g}:\mathfrak{h}) = 0$

$\Longrightarrow$ φ : locally rigid.

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(II) V : fin. dim. $\mathbb{K}$-vector space

$\mathfrak{g}$: Lie algebra $/\mathbb{K}$, $\mathfrak{g} \cong_{\mathbb{K}} V$,

($\Rightarrow$ $\mathfrak{g}$: $\mathfrak{g}$-module via $\mathfrak{g} \xrightarrow{ad} \mathfrak{gl}(\mathfrak{g})$)

$H^2(\mathfrak{g};\mathfrak{g}) = 0$

$\Longrightarrow$ $\mathfrak{g}$: locally rigid.

(ex) $\mathfrak{g}$: semi-simple Lie algebra

$\Longrightarrow$ • $\forall$ Lie alg. homom. $\mathfrak{g} \to {}^{\forall}\mathfrak{h}$: locally rigid

• $\mathfrak{g}$: locally rigid)

定理は次の補題から従う.

Lemma 4.2 A, B, C : C^∞ manifolds

$\alpha : A \to B$, $\beta : B \to C$ C^∞ maps

$c_0 \in C$, $a_0 \in A$. $\beta\alpha(A) = \{c_0\}$

$T_{a_0}A \xrightarrow{(d\alpha)_{a_0}} T_{\alpha(a_0)}B \xrightarrow{(d\beta)_{\alpha(a_0)}} T_{c_0}C$ (exact)

$\Longrightarrow$ $\alpha(A) \overset{loc}{=} \beta^{-1}(c_0)$ (submfd of B) near $\alpha(a_0)$

(証明はあとまわし)

Pf of Thm 4.1 (I)

$$H \xrightarrow{\alpha = \alpha_\varphi} \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathfrak{g},\mathfrak{h})$$

$$\begin{array}{ccccc}
T_e H & \xrightarrow{(d\alpha)_e} & T_\varphi \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}) & \xrightarrow{(d\beta)_\varphi} & T_0 \mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathfrak{g},\mathfrak{h}) \\
\| & \circlearrowleft & \| & \circlearrowleft & \| \\
C^0(\mathfrak{g};\mathfrak{h}) & \xrightarrow{-d} & C^1(\mathfrak{g};\mathfrak{h}) & \xrightarrow{d} & C^2(\mathfrak{g};\mathfrak{h}).
\end{array}$$

exact $\Longleftrightarrow$ $H^1(\mathfrak{g};\mathfrak{h}) = 0$ //

Pf of Thm 4.2 (II) $\mathfrak{g} = (V, \psi)$, $\psi \in A(V)$

$$GL(V) \xrightarrow{\alpha = \alpha_\psi} \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{K}}(\Lambda^3 V, V)$$

$$\begin{array}{ccccccc}
T_e GL(V) & \xrightarrow{(d\alpha)_e} & T_\psi \mathrm{Hom}_{\mathbb{K}}(\Lambda^2 V, V) & \xrightarrow{(d\beta)_\psi} & T_0 \mathrm{Hom}_{\mathbb{K}}(\Lambda^3 V, V) & & \text{exact} \\
\| & \circlearrowleft & \| & \circlearrowleft & \| & & \Updownarrow \\
C^1(\mathfrak{g};\mathfrak{g}) & \xrightarrow{-d} & C^2(\mathfrak{g};\mathfrak{g}) & \xrightarrow{d} & C^3(\mathfrak{g};\mathfrak{g}) & & H^2(\mathfrak{g};\mathfrak{g}) = 0 \ //
\end{array}$$

Lem 4.2の準備

Lemma 4.3 M^m, N^n: C^∞ manifolds
$f: M \to N$ C^∞ map
$\mathrm{rk}(df)_p \equiv r$ const. in $p \in M$.
$\Rightarrow \forall p \in M$, $\exists \varphi$: loc. coord. of M centered at p
$\exists \psi$: loc. coord. of N centered at $f(p)$
s.t. $\psi f \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_r, 0, \cdots, 0)$ near $0 = \varphi(p) \in \mathbb{R}^m$

Pf of Lemma 4.2.
$A \overset{\text{open}}{\subset} \mathbb{R}^l$, $B \overset{\text{open}}{\subset} \mathbb{R}^m$, $C \overset{\text{open}}{\subset} \mathbb{R}^n$
$a_0 = 0$, $\alpha(a_0) = 0$, $\beta\alpha(a_0) = c_0 = 0$
としてよい. $r := \mathrm{rk}(d\alpha)_{a_0}$, $\rho := \mathrm{rk}(d\beta)_{\alpha(a_0)}$ とする.
$$r + \rho = \dim B = m$$
である.
$\exists \pi: \mathbb{R}^m = T_{c_0}C \twoheadrightarrow \mathrm{Im}(d\beta)_{\alpha(a_0)} (\cong \mathbb{R}^\rho)$ linear map
s.t. $\pi|_{\mathrm{Im}(d\beta)_{\alpha(a_0)}} = \text{identity}$
いま $\beta\alpha(A) = \{c_0\}$ より $\forall a \in A$ $\mathrm{Im}(d\alpha)_a \subset \mathrm{Ker}(d\beta)_{\alpha(a)}$ ゆえに
$$\mathrm{rk}(d\alpha)_a + \mathrm{rk}(d\beta)_{\alpha(a)} \leqq \dim B = m$$
ここで $\left.\begin{array}{l}\alpha \text{の Jacobi行列の } r \text{次小行列式} \\ \beta \text{の} \quad \text{〃} \quad \rho \quad \text{〃}\end{array}\right)$ を考えると
a_0 の nbd で
$$\mathrm{rk}(d\alpha)_a \geqq r$$
$$\mathrm{rk}(d\beta)_{\alpha(a)} \geqq \rho$$
だから. $r + \rho = m$ とあわせて a_0 の nbd で $\mathrm{rk}(d\alpha)_a = r$ (const) となる.
そこで Lem 4.3 より A, B を充分小さくとり直し, 座標変換すると
$a_0 = 0 \in A \overset{\text{open}}{\subset} \mathbb{R}^l$ の nbd で
$$\alpha(x_1, \cdots, x_r, x_{r+1}, \cdots, x_l) = (x_1, \cdots, x_r, \underbrace{0, \cdots, 0}_{m-r=\rho})$$
と表わされる.
とくに $\alpha(A)$ は $\alpha(a_0) = 0$ を通る B の r-dim. submfd である

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他方、$\pi \circ \beta : B \to \mathrm{Im}(d\beta)_{\alpha(a_0)} \cong \mathbb{R}^p$ は $\alpha(a_0) = 0 \in B$ において微分が全射である。そこで陰関数定理により、
$(\pi \circ \beta)^{-1}(0)$ は $\alpha(a_0)$ の nbd で B の $m-p(=r)$-dim submfd である

$$\alpha(A) \subset \beta^{-1}(c_0) \subset (\pi \circ \beta)^{-1}(0)$$

であって、$\alpha(a_0)$ の nbd で $\alpha(A)$ も $(\pi \circ \beta)^{-1}(0)$ も $\alpha(a_0) = 0$ を通る
r-dim, submfd である。したがって $\alpha(a_0)$ の nbd で等号が成立つ
とくに $\alpha(a_0)$ の nbd で $\alpha(A) = \beta^{-1}(c_0)$ であり、
これは B の submfd である //