(§3 Cohomology of reductive Lie algebras (つづき))

(§3.2. Semi-simple Lie algebras (つづき))

$\mathbb{K}$: field of char. 0.

$\mathfrak{g}$: fin. dim. Lie algebra/$\mathbb{K}$

Definition $\mathfrak{g}$: simple

$\overset{\text{def}}{\Longleftrightarrow}$ (0) $\mathfrak{g} \not\cong \mathbb{K}$ abelian

(1) $0 \subsetneq \nexists \mathfrak{h} \subsetneq \mathfrak{g}$ ideal

Theorem 3.5. ([B] §6. n°1, Thm 1, §6 n°2. Thm 2 および Prop 2)

$\mathfrak{g}$: fin-dim Lie algebra/$\mathbb{K}$ について: 次は同値である (semi-simple)

(a) $\mathfrak{g}$: semi-simple

i.e. $\forall M$: fin. dim. $\mathfrak{g}$-module: completely reducible

(b) $\mathfrak{g}$ は有限個の単純 Lie 代数の直積に同型である

(c). $\beta_{\mathfrak{g}}$ (Killing form): non-degenerate

Lemma 3.6. $\mathfrak{g}$: semi-simple, M: fin. dim. $\mathfrak{g}$-module

$$\Longrightarrow H^*(C^*(\mathfrak{g}:M)^{\mathfrak{g}}) \cong H^*(\mathfrak{g}:M)$$

(pf) $\forall p \geq 0$ $C^p(\mathfrak{g}:M)$: fin. dim. $\mathfrak{g}$-module $\Rightarrow$ completely reducible

$H^*(C^*(\mathfrak{g}:M)^{\mathfrak{g}}) \longrightarrow H^*(\mathfrak{g}:M)$ inclusion homomorphism

surjective

$dC^{p-1}(\mathfrak{g}:M) \subset Z^p(\mathfrak{g}:M) := \mathrm{Ker}(d|_{C^p(\mathfrak{g}:M)}) \subset C^p(\mathfrak{g}:M)$

$\mathfrak{g}$-submodule (∵) $\forall X \in \mathfrak{g}$ $d\mathcal{L}_X = \mathcal{L}_X d$)

$\exists V \subset Z^p(\mathfrak{g}:M)$ $\mathfrak{g}$-submodule

$Z^p(\mathfrak{g}:M) = V \oplus dC^{p-1}(\mathfrak{g}:M)$ (∵) $Z^p(\mathfrak{g}:M)$ completely reducible)

$V \xrightarrow{\cong} H^p(\mathfrak{g}:M)$

$\mathfrak{g}\cdot V = 0$ (∵) $\forall z \in Z^p(\mathfrak{g}:M)$ $\forall X \in \mathfrak{g}$, $\mathcal{L}_X z = d i_X z \in dC^{p-1}$)

$V \subset C^p(\mathfrak{g}:M)^{\mathfrak{g}} \cap Z^p(\mathfrak{g}:M)$

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<u>injective</u> $d\,C^{p-1}(\mathfrak{g}:M) \cap C^p(\mathfrak{g}:M)^{\mathfrak{g}} = d(C^{p-1}(\mathfrak{g}:M)^{\mathfrak{g}})$

$\Big($ (∵) $C^{p-1}(\mathfrak{g}:M)$, $C^p(\mathfrak{g}:M)$: completely reducible

Schur's Lemma // //

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$\mathfrak{g}$: semi-simple, M : fin. dim. $\mathfrak{g}$-module

$$\Longrightarrow \quad H^*(\mathfrak{g}:M) = H^*(\mathfrak{g}:\mathbb{K}) \otimes M^{\mathfrak{g}}$$

$$H^1(\mathfrak{g}:\mathbb{K}) = H^2(\mathfrak{g}:\mathbb{K}) = 0$$

$H^3(\mathfrak{g}:\mathbb{K}) = ?$ $\Leftarrow$ $\mathfrak{g}\otimes\overline{\mathbb{K}}$ の単純因子の個数

$\overline{\mathbb{K}}$: algebraic closure of $\mathbb{K}$

(Rmk) $\mathfrak{g}\otimes\overline{\mathbb{K}}$: semi-simple $\Longleftrightarrow$ $\mathfrak{g}$: semi-simple

(∵) $\beta_{\mathfrak{g}\otimes\overline{\mathbb{K}}} = \beta_{\mathfrak{g}} \otimes \overline{\mathbb{K}}$)

<u>Theorem 3.7</u> $\dim_{\mathbb{K}} H^3(\mathfrak{g}:\mathbb{K}) = (\mathfrak{g}\otimes\overline{\mathbb{K}}$ における単純因子の個数$)$

<u>proof</u> 一般に $\mathfrak{g}\oplus\mathfrak{h}$: $\overset{\text{fin. dim}}{\text{Lie}}$ 代数の直積について

$$C^*(\mathfrak{g}\oplus\mathfrak{h}:\mathbb{K}) = C^*(\mathfrak{g}:\mathbb{K}) \otimes C^*(\mathfrak{h}:\mathbb{K})$$

Künneth formula より

$$H^*(\mathfrak{g}\oplus\mathfrak{h}:\mathbb{K}) = H^*(\mathfrak{g}:\mathbb{K}) \otimes H^*(\mathfrak{h}:\mathbb{K})$$

そこで $\mathbb{K} = \overline{\mathbb{K}}$, $\mathfrak{g}$: simple について

$$\dim H^3(\mathfrak{g}:\mathbb{K}) = 1$$

を示せばよい

<u>Lemma 3.8.</u> $\mathbb{K} = \overline{\mathbb{K}}$; $\mathfrak{g}$: simple Lie algebra / $\mathbb{K}$

$$\Longrightarrow \quad (\mathfrak{g}^* \otimes \mathfrak{g}^*)^{\mathfrak{g}} = \mathbb{K}\,\beta_{\mathfrak{g}} \quad (\cong \mathbb{K})$$

((Rmk) とくに $C^2(\mathfrak{g}:\mathbb{K})^{\mathfrak{g}} = 0$ である. Lem 3.6 から $H^2(\mathfrak{g}:\mathbb{K}) = 0$ の別証が得られる)

<u>pf of Lem 3.8</u> $\forall \overset{\neq 0}{c} \in (\mathfrak{g}^* \otimes \mathfrak{g}^*)^{\mathfrak{g}}$, $\exists!\, \overset{\neq 0}{\varphi} : \mathfrak{g} \to \mathfrak{g}$: $\mathfrak{g}$-homom s.t.

$\forall X\, \forall Y \in \mathfrak{g}$ $\quad c(X, Y) = \beta_{\mathfrak{g}}(X, \varphi(Y))$ $\quad$ ((∵) $\beta_{\mathfrak{g}}$: non-degen)

$\mathfrak{g}$: simple $\mathfrak{g}$-module, Schur's Lemma ヨリ φ : isom

$\mathbb{K} = \overline{\mathbb{K}}$ ヨリ $0 \neq {}^{\exists}\lambda \in \overline{\mathbb{K}}$, $0 \neq {}^{\exists}v \in \mathfrak{g}$ $\varphi(v) = \lambda v$

$0 \neq \mathrm{Ker}(\varphi - \lambda) < \mathfrak{g}$ $\mathfrak{g}$-submodule, $\mathfrak{g}$: simple $\Big\} \Rightarrow \mathrm{Ker}(\varphi - \lambda) = \mathfrak{g}$ i.e. $\varphi = \lambda$

$\therefore\ c = \lambda \beta_{\mathfrak{g}}$ //

pf of Thm 3.7

$E : C^*(\mathfrak{g} : \mathbb{K}) \to C^{*-1}(\mathfrak{g} : \mathfrak{g}^*)$ $f \longmapsto Ef$

$(Ef)(X_1, \cdots, X_p)(X_{p+1}) := f(X_1, \cdots, X_p, X_{p+1})$

injective $\mathfrak{g}$-homomorphism (except $* = 0$)

$d(Ef) = E(df)$

(略)
$$\left[\begin{aligned} &\because)\ (d(Ef))(X_1, \cdots, X_p)(X_{p+1}) \\ &= \sum_{i=1}^{p} (-1)^{i+1} X_i (Ef)(X_1, \overset{\hat{i}}{\cdots}, X_p)(X_{p+1}) + \sum_{i<j} (-1)^{i+j} (Ef)([X_i, X_j], X_1, \overset{\hat{i}\ \hat{j}}{\cdots} X_p)(X_{p+1}) \\ &= \sum_{i=1}^{p} (-1)^{i} f(X_1, \overset{\hat{i}}{\cdots}, X_p, [X_i, X_{p+1}]) + \sum_{1 \leq i < j \leq p} (-1)^{i+j} f([X_i, X_j], X_1, \overset{\hat{i}\ \hat{j}}{\cdots} X_p, X_{p+1}) \\ &= (df)(X_1, \cdots, X_p, X_{p+1}) = (E\,df)(X_1, \cdots, X_p)(X_{p+1}) \quad // \end{aligned}\right.$$

$E : C^*(\mathfrak{g} : \mathbb{K})^{\mathfrak{g}} \to C^{*-1}(\mathfrak{g} : \mathfrak{g}^*)^{\mathfrak{g}}$

injective cochain map (except $* = 0$)

$D^* = \{D^p\}$, $D^p := C^p(\mathfrak{g} : \mathbb{K})^{\mathfrak{g}} / E(C^{p+1}(\mathfrak{g} : \mathbb{K})^{\mathfrak{g}})$

$$\begin{tikzcd} H^1(\mathfrak{g} : \mathfrak{g}^*) \arrow[r] \arrow[d, equal] & H^1(D^*) \arrow[r, "\delta^*", "\cong"'] & H^3(\mathfrak{g} : \mathbb{K}) \arrow[r] & H^2(\mathfrak{g} : \mathfrak{g}^*) \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

$D^0 = (\mathfrak{g}^*)^{\mathfrak{g}} \underset{\beta_{\mathfrak{g}}}{\cong} \mathfrak{g}^{\mathfrak{g}} = \mathrm{Center}(\mathfrak{g}) = 0$ ($\because$) $\mathfrak{g}$: simple)

$D^1 = C^1(\mathfrak{g} \cdot \mathfrak{g}^*)^{\mathfrak{g}} / 0 = \mathbb{K}\hat{\beta_{\mathfrak{g}}}$ ($\because$) Lem 3.9)

$\hat{\beta_{\mathfrak{g}}}(X) := \beta_{\mathfrak{g}}(X, \cdot)$ $(X \in \mathfrak{g})$

$X, Y, Z \in \mathfrak{g}$

$(d\hat{\beta_{\mathfrak{g}}})(X, Y)(Z)$

$= (X\hat{\beta_{\mathfrak{g}}}(Y))(Z) - (Y\hat{\beta_{\mathfrak{g}}}(X))(Z) - \hat{\beta_{\mathfrak{g}}}([X, Y])(Z)$

$= -\beta_{\mathfrak{g}}(Y, \cancel{[X, Z]}) + \beta_{\mathfrak{g}}(X, [Y, Z]) - \beta_{\mathfrak{g}}([X, \cancel{Y}], Z)$

$= \beta_{\mathfrak{g}}(X, [Y, Z]) = \beta_{\mathfrak{g}}([X, Y], Z) =: f_{\mathfrak{g}}(X, Y, Z)$

$f_{\mathfrak{g}} \in C^3(\mathfrak{g}:\mathbb{K})^{\mathfrak{g}} \cap Z^3(\mathfrak{g}:\mathbb{K})$

$f_{\mathfrak{g}} \neq 0 \in C^3(\mathfrak{g}:\mathbb{K})$ (∵) $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$, $\beta_{\mathfrak{g}}$: nondegen.)

$[f_{\mathfrak{g}}] \neq 0 \in H^3(\mathfrak{g}:\mathbb{K})$ (∵) $C^2(\mathfrak{g}:\mathbb{K})^{\mathfrak{g}}=0$)

$$\begin{tikzcd} H^1(D^*) \arrow[r, phantom, "="] \arrow[d, "\delta"', "\wr"] & \mathbb{K}[\hat{\beta}_{\mathfrak{g}}] \cong \mathbb{K} \arrow[d, mapsto] \\ & {[\hat{\beta}_{\mathfrak{g}}]} \arrow[d, mapsto] \\ H^3(\mathfrak{g}:\mathbb{K}) \arrow[r, phantom, "\ni"] & {[f_{\mathfrak{g}}] \neq 0} \end{tikzcd}$$

//

§3.3. Reductive Lie algebras

Theorem 3.9 - Definition ([B] §6. n°4. Prop 5)

$\mathfrak{g}$: fin.dim. Lie algebra/$\mathbb{K}$ について次の(a)-(c)は互いに同値である.

(a) $\mathfrak{g}$: completely reducible $\mathfrak{g}$-module

(b) $\mathfrak{g}$ は半単純 Lie 代数と abelian Lie algebra の直積である.

(c) $\exists M$: fin.dim$_{\mathbb{K}}$ $\mathfrak{g}$-module s.t. β_M: non-degenerate

このとき $\mathfrak{g}$: reductive という.

Corollary 3.10 (1) $\mathfrak{g}$: reductive

$\iff \mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus \mathrm{Center}(\mathfrak{g})$, $[\mathfrak{g},\mathfrak{g}]$: semi-simple

このとき $\mathfrak{g}^{abel} \cong \mathrm{Center}(\mathfrak{g})$

(2) $\mathfrak{g}$: reductive かつ $\mathfrak{g}^{abel}=0 \iff \mathfrak{g}$: semi simple.

例(1) $\mathfrak{gl}_n(\mathbb{K})$: reductive

(∵) $\mathfrak{gl}_n(\mathbb{Q})$ について示せばよい $\mathfrak{gl}_n(\mathbb{Q}) \curvearrowright \mathbb{Q}^n =: M$ vector representation

$\beta_M(X,Y) = \mathrm{trace}(XY)$, $X, Y \in \mathfrak{gl}_n(\mathbb{Q})$

$X \neq 0 \implies \beta_M(X\,{}^tX) \neq 0$, β_M: non-degen. //

例(2) $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{Q})$ Lie subalgebra

${}^t\mathfrak{g} = \mathfrak{g}$

$\implies \mathfrak{g}$: reductive.

例(3) $\mathfrak{sl}_n(\mathbb{K})$: reductive

($\because$) ${}^t\mathfrak{sl}_n(\mathbb{Q}) = \mathfrak{sl}_n(\mathbb{Q})$ //)

$\mathfrak{sl}_n(\mathbb{K})$: semi-simple

($\because$) $\mathfrak{sl}_n(\mathbb{K})^{abel} = 0$ (Lem 2.8) //)

$\mathfrak{sl}_n(\mathbb{K})$: simple

($\because$) $\dim_{\mathbb{K}} H^3(\mathfrak{sl}_n(\mathbb{K}) : \mathbb{K}) = 1$ //)

例(4) $B = \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{p} & 0 \\ 0 & \overbrace{-1 \ \ddots \ -1}^{q} \end{pmatrix}$, $p+q=n$

$\mathfrak{o}(p,q : \mathbb{K}) := \{X \in \mathfrak{gl}_n(\mathbb{K}) ; {}^tXB + BX = 0\}$ reductive

[($\because$) ${}^tXB + BX = 0 \overset{\exists B^{-1}}{\Longleftrightarrow} B\,{}^tXBB + BBXB = 0$

$\Longleftrightarrow B\,{}^tX + XB = 0$ ゆえに ${}^t\mathfrak{o}(p,q:\mathbb{Q}) = \mathfrak{o}(p,q:\mathbb{Q})$ //

レポート問題7.

$p+q = n \geqq 3$ のとき

$H^1(\mathfrak{o}(p,q:\mathbb{K}) : \mathbb{K}) = 0$

を示せ

とくに

$\mathfrak{o}(p,q:\mathbb{K})$: semi-simple

である

例(5) $g \geqq 1$

$$J = J_g = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \mathbf{O} \\ & & \ddots & \\ \mathbf{O} & & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \quad \left(\begin{array}{l} JJ = -1_{2g} \\ \Rightarrow \exists J^{-1} \end{array} \right)$$

$\mathfrak{sp}_g(\mathbb{K}) = \{X \in \mathfrak{gl}_{2g}(\mathbb{K}) ; {}^tXJ + JX = 0\}$: reductive

[($\because$) ${}^tXJ + JX = 0 \overset{\exists J^{-1}}{\Longleftrightarrow} J\,{}^tXJJ + JJXJ = 0$

$\Longleftrightarrow J\,{}^tX + XJ = 0$ ゆえに ${}^t\mathfrak{sp}_g(\mathbb{Q}) = \mathfrak{sp}_g(\mathbb{Q})$ //

$\mathfrak{sp}_g(\mathbb{K})^{abel} = ?$

$\underset{\overset{\cup}{u,v}}{H} := \mathbb{K}^{2g}$, $\mathfrak{sp}_g(\mathbb{K}) \curvearrowright H$ vector representation

$u \cdot v := {}^tuJv$ alternating bilinear form
non-degenerate

$H \cong H^*$ 同一視

$u \longmapsto \cdot u$

$$\begin{tikzcd} \mathfrak{gl}(H) = H^* \otimes H \cong H^{\otimes 2} \\ \end{tikzcd}$$

$\underset{\overset{\psi}{X}}{\mathfrak{gl}(H)} = H^*\otimes H \cong \underset{\overset{\psi}{u_1 \otimes u_2}}{H^{\otimes 2}}$

$X \longleftarrow\!\shortmid u_1 \otimes u_2$ $\quad u_1, u_2 \in H$

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$v, v_1, v_2 \in H$

$Xv = (v\cdot u_1)u_2$

$(Xv_1)\cdot v_2 = (v_1\cdot u_1)(u_2\cdot v_2) = -(v_2\cdot u_2)(v_1\cdot u_1)$

$v_1\cdot(Xv_2) = (v_2\cdot u_1)(v_1\cdot u_2)$

${}^tXJ + JX \in \mathfrak{gl}(H) \longmapsto -u_2\otimes u_1 + u_1\otimes u_2 \in H^{\otimes 2}$

$\mathfrak{sp}_g(\mathbb{K}) = \mathfrak{sp}(H) \cong \mathrm{Sym}^2(H)$ natural isomorphism.

$u \in H \longleftrightarrow u\otimes u$

$T_u \in \mathfrak{sp}(H)$, $T_u(v) \overset{\text{def}}{=} (v\cdot u)u$.

とくに $\mathfrak{sp}(H)$ は $\mathbb{K}$-vect. sp として $\{T_u : 0 \neq u \in H\}$ によって生成される

Corollary 3.11 $\mathfrak{sp}_g(\mathbb{K})^{abel} = 0$

とくに $\mathfrak{sp}_g(\mathbb{K})$: semi-simple (実は simple)

(∵) $0 \neq \forall u \in H$, $T_u \in [\mathfrak{sp}(H), \mathfrak{sp}(H)]$ を示せばよい

$\exists v \in H$, $u\cdot v = 1$, $W := \{w \in H ; u\cdot w = v\cdot w = 0\}$

$H = \mathbb{K}u \oplus \mathbb{K}v \oplus W$

$$\begin{cases} T_u(u) = 0 \\ T_u(v) = -u \\ T_u|_W = 0 \end{cases} \qquad \begin{cases} T_v(u) = v \\ T_v(v) = 0 \\ T_v|_W = 0 \end{cases}$$

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2T_u \quad //$$

次の定理はあとで使う.

Theorem 3.12 $\mathfrak{g}$: (fin. dim.) reductive Lie algebra $/\mathbb{K}$

M : fin. $\dim_{\mathbb{K}}$ completely reducible $\mathfrak{g}$-module.

$\Longrightarrow H^*(\mathfrak{g}; M) = H^*(\mathfrak{g}; M^{\mathfrak{g}}) \ (= H^*(\mathfrak{g}; \mathbb{K}) \otimes M^{\mathfrak{g}})$

proof M : non-trivial simple $\mathfrak{g}$-module について

$H^*(\mathfrak{g};M)=0$ を示せばよい

$M \Rightarrow 0\neq\rho:\mathfrak{g}\to\mathfrak{gl}(M)$ Lie alg. homom.

$\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]\oplus\mathfrak{z}$, $\mathfrak{z}=$ Center $\mathfrak{g}$, $[\mathfrak{g},\mathfrak{g}]$: semi-simple

場合分け

(I) $\mathfrak{z}.M\neq 0$

(II) $\mathfrak{z}.M=0$

(I) $\mathfrak{z}.M\neq 0$ のとき

$\exists Z\in\mathfrak{z}$ $\varphi(Z)\neq 0\in\mathfrak{gl}(M)$

$\forall X\in\mathfrak{g}$ $\varphi(X)\rho(Z)-\rho(Z)\rho(X)=\varphi([X,Z])=0$

$\underset{\overset{\#}{0}}{\rho(Z)}:M\to M$ $\mathfrak{g}$-homomorphsim

M : simple, Schur's Lem より $\rho(Z)$: isom

$\rho(Z)_*:H^*(\mathfrak{g};M)\xrightarrow{\cong}H^*(\mathfrak{g};M)$ isom

$[f]\longmapsto[\rho(Z)f]$

$\forall f\in C^p(\mathfrak{g};M)$

$$(\mathcal{L}_Z f)(X_1,\cdots X_p)=\varphi(Z)f(X_1,\cdots X_p)-\sum_{i=1}^{p}f(X_1,\cdots \underset{\overset{\|}{0}\,(\because Z\in\mathfrak{z})}{\overset{i}{\smile}[Z,X_i]},\cdots X_p)$$

$$=\rho(Z)f(X_1,\cdots X_p)$$

$\varphi(Z)=\mathcal{L}_Z=di_Z+i_Zd$

$\therefore\rho(Z)_*=0$

$\therefore$ $H^*(\mathfrak{g};M)=0$ //(I)

(II) $\mathfrak{z}.M=0$ のとき

$\mathfrak{z}<\mathrm{Ker}\,\rho$, $\mathfrak{g}$: complete reducible $\mathfrak{g}$-module より

$\exists\mathfrak{h}<[\mathfrak{g},\mathfrak{g}]$ $\mathfrak{g}$-submodule s.t. $\mathfrak{g}=\mathfrak{h}\oplus\mathrm{Ker}\,\rho$

($\Updownarrow$ $\mathfrak{h}<\mathfrak{g}$ ideal)

$\mathfrak{h}$: semi-simple ($\because$ $[\mathfrak{g},\mathfrak{g}]\overset{\rho}{\twoheadrightarrow}\mathfrak{h}$)

$\mathfrak{h}\neq 0$ ($\because$ M : nontrivial)

$\beta_M|\mathfrak{h}\times\mathfrak{h}$: non-degenerate $\left(\because\ \mathfrak{h}<\mathfrak{gl}(M),\ \mathfrak{h}\text{ : semi-simple}\right.$
Thm 3.5 (E. Cartan))

ゆえに Lem 3.1 より $H^*(\mathfrak{g};M)=0$ //(II) //Thm.

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