"Teichmüller theory : quantization and relations with physics"
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"An infinitesimal version of the Dehn-Nielsen theorem"

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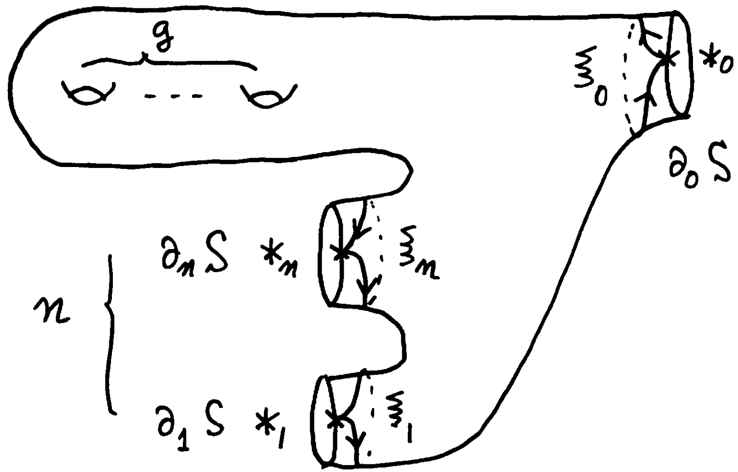
joint work with Yusuke Kuno (Tsuda College)

- arXiv : 1304.1885. (survey paper)
- http://www.ms.u-tokyo.ac.jp/~kawazumi/1304Vienna-v1.pdf
$\Downarrow$
v2

S : compact connected oriented surface with $\partial S \neq \phi$

$\Longrightarrow$ Classification Theorem

$\exists g, \exists n \geqq 0$, $S \cong \Sigma_{g,n+1} =$



$\partial S = \coprod_{j=0}^{n} \partial_j S$

$*_j \in \partial_j S$, $0 \leq j \leq n$

$E := \{*_j\}_{j=0}^{n} \subset \partial S$

$\xi_j \in \pi_1(S, *_j)$ boundary loop $0 \leq j \leq n$

$\mathcal{M}(S) := \{\varphi : S \to S$: ori. pres. diffeo, $\varphi|_{\partial S} = id_{\partial S}\} /$ isotopy fixing ∂S <u>pointwise</u>

the mapping class group of S

$\mathcal{I}^L(S) := \{\varphi \in \mathcal{M}(S)\ ;\ \varphi = id$ on $\left(H_1(S; \mathbb{Z}) \big/ \sum_{j=0}^{n} \mathbb{Z}[\xi_j] \right) \}$

the largest Torelli group of S in the sense of Putman

Dehn-Nielsen homomorphism

$\underline{n=0}$. $S=\Sigma_{g,1}$, $*_0 \in \partial_0 \Sigma_{g,1} = \partial \Sigma_{g,1}$

Theorem (Dehn-Nielsen)

$$DN : \mathcal{M}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma_{g,1}, *_0)), \quad \varphi \longmapsto \varphi_*,$$

is injective, and

$$\mathrm{Image}\, DN = \{\varphi \in \mathrm{Aut}(\pi_1(\Sigma_{g,1}, *_0))\ ;\ \varphi(\zeta_0) = \zeta_0\}$$

$\underline{n \geqq 1}$ $\quad *_0 \in \partial_0 S \subsetneqq \partial S$

$$\mathcal{M}(S) \longrightarrow \mathrm{Aut}(\pi_1(S, *_0)), \quad \varphi \longmapsto \varphi_*,$$

is <u>not</u> injective

($\because$) $1 \neq$ (Dehn twist along ζ_j) $\in$ Kernel if $j \geqq 1$)

$$\begin{pmatrix} \text{fundamental group} \\ \pi_1(S, *_0) \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{fundamental groupoid} \\ \Pi S(p,q) := [(I,0,1),(S,p,q)] \\ I=[0,1] \subset \mathbb{R}, \quad p,q \in S \end{pmatrix}$$

$\Pi S|_E$: the fundamental groupoid of S restricted to $E = \{*_j\}_{j=0}^{n}$ $(\subset \partial S)$

<u>objects</u> $\mathrm{Ob}(\Pi S|_E) = E$

<u>morphisms</u> $(\Pi S|_E)(*_i, *_j) = \Pi S(*_i, *_j)$ $\quad (0 \leq i, j \leq n)$

<u>Dehn–Nielsen homomorphism</u>

$$DN : \mathcal{M}(S) \longrightarrow \mathrm{Aut}(\Pi S|_E)$$

$$\varphi \longmapsto \left(\begin{array}{l} \text{id on } E \\ \varphi_* : \Pi S(*_i, *_j) \to \Pi S(*_i, *_j),\ [\ell] \mapsto [\varphi \circ \ell] \end{array} \right.$$

(<u>Lemma</u> DN is injective —)

$\downarrow$ induces

$$\begin{tikzcd}
\widehat{DN} : \mathcal{M}(S) \arrow[r, hook] & \mathrm{Aut}(\mathbb{Q}\widehat{\Pi S}|_E) & \\
\cup \arrow[r, phantom, "\circlearrowright"] & \cup & \\
\widehat{DN}^{-1}(\boxed{////}) \arrow[r, hook] & \boxed{////} \arrow[r, "\log"] & \mathrm{Der}(\mathbb{Q}\widehat{\Pi S}|_E) \\
\cup & \circlearrowright & \\
\mathfrak{g}^L(S) \arrow[uurr] & &
\end{tikzcd}$$

we need <u>$\mathbb{Q}$-coeff</u> and <u>completion</u>

<u>Our Goal</u>: to give a geometric interpretation

a lift of the Johnson homomorphisms

(Anyway, we have to define the completion $\mathbb{Q}\widehat{\Pi S}|_E$ ---)

$\widehat{\mathbb{Q}\Pi S}|_E$: the completed "groupoid" ring (small $\mathbb{Q}$-linear category)

<u>objects</u> $\mathrm{Ob}(\widehat{\mathbb{Q}\Pi S}|_E) = E$

<u>morphisms</u> $\widehat{\mathbb{Q}\Pi S}(*_i, *_j) := \varprojlim_{m\to\infty} \mathbb{Q}\Pi S(*_i, *_j) / \gamma (I\pi_1(S,*))^m \delta \quad (0 \le i, j \le n)$

independent of the choice of

$* \in S$, $\gamma \in \Pi S(*_i, *)$ and $\delta \in \Pi S(*, *_j)$)

where $I\pi_1(S,*) := \mathrm{Ker}(\mathbb{Q}\pi_1(S,*) \to \mathbb{Q})$ augmentation ideal

$\qquad\qquad\qquad\qquad\qquad \sum a_x x \mapsto \sum a_x$

$\mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}|_E)$

$\cup$

$\boxed{} := \left\{ U = \{ U^{(*_i, *_j)} \in \mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}(*_i, *_j))\}_{0 \le i,j \le n} ; \right.$

(0) $U^{(*_i,*_j)}$ is continuous w.r.to the natural filtration on $\widehat{\mathbb{Q}\Pi S}(*_i, *_j)$ $(0 \le \forall i, \forall j \le n)$

(i) $U(uv) = (Uu)(Uv) \quad \forall u \in \widehat{\mathbb{Q}\Pi S}(*_i, *_j), \forall v \in \widehat{\mathbb{Q}\Pi S}(*_j, *_k)$

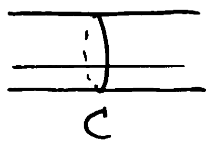
(ii) $\left. \log U := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (U-1)^m \text{ <u>converges</u>.} \right\}$

$\downarrow \log$

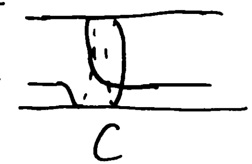
$\mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_E)$

Lemma $\widehat{DN}(\mathcal{I}^L(S)) \subset \boxed{////}$

$\widehat{DN}(t_C) \in \boxed{////}$, $^\forall C \subset \mathrm{Int}\, S$ simple closed curve.



$\xrightarrow{\text{$t_C$: right-handed Dehn twist along C}}$



$\varphi \in (\widehat{DN})^{-1}(\boxed{///})$, $(\log \widehat{DN}(\varphi))(\xi_j) = 0 \quad (0 \le {}^\forall j \le n) \quad (\because\ \varphi(\xi_j) = \xi_j)$

$$\mathrm{Der}_\partial(\mathbb{Q}\widehat{\Pi S}|_E) := \left\{ \begin{array}{l} D = \{D^{(*_i,*_j)} \in \mathrm{End}(\mathbb{Q}\widehat{\Pi S}(*_i,*_j))\}_{0 \le i,j \le n} ; \\ \text{(0) } D^{(*_i,*_j)} \text{ is continuous w.r. to the natural filtration on } \mathbb{Q}\widehat{\Pi S}(*_i,*_j) \\ \qquad\qquad\qquad\qquad (0 \le {}^\forall i, {}^\forall j \le n) \\ \text{(i) } D(uv) = (Du)v + u(Dv) \quad {}^\forall u \in \mathbb{Q}\widehat{\Pi S}(*_i,*_j), {}^\forall v \in \mathbb{Q}\widehat{\Pi S}(*_j,*_k) \\ \text{(ii) } D(\xi_j) = 0 \quad (0 \le {}^\forall j \le n) \end{array} \right\}$$

complete filtered Lie algebra.

$$\log \circ \widehat{DN} : (\widehat{DN})^{-1}(\boxed{///}) \longrightarrow \mathrm{Der}_\partial(\mathbb{Q}\widehat{\Pi S}|_E).$$

[A fundamental question about the mapping class group $\mathcal{M}(S)$
What is "the Lie algebra" of $\mathcal{M}(S)$?
(an infinitesimal version of $\mathcal{M}(S)$?)
⇝ many answers (Johnson–Morita–Hain, ⋯ , Virasoro action ⋯)

A rough answer : The Goldman–Turaev Lie bialgebra $\mathbb{Z}\hat{\pi}'(S) = \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1$.

$\hat{\pi} = \hat{\pi}(S) := [S^1, S]$ the homotopy set of free loops on S

$|\ | : \pi_1(S,*) \to \hat{\pi}(S)$ forgetting the basepoint $*$

($|\ | : \mathbb{Q}\pi_1(S,*) \to \mathbb{Q}\hat{\pi}(S)$ linear extension)

$\alpha, \beta \in \hat{\pi}$ in general position

$$[\alpha,\beta] \overset{\mathrm{def}}{=} \sum_{p\in\alpha\cap\beta} \varepsilon(p;\alpha,\beta)\,|\alpha_p\beta_p| \in \mathbb{Z}\hat{\pi} \qquad \text{Goldman bracket}$$

$\varepsilon(p;\alpha,\beta) \in \{\pm1\}$ local intersection number

α_p [resp. β_p] $\in \pi_1(S,p)$: based loop along α [resp. β] with basepoint p

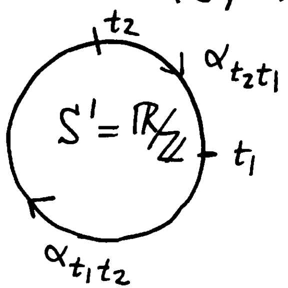
[Theorem (Goldman)
(1) $[\ ,\ ]$: well-defined
(2) $(\mathbb{Z}\hat{\pi}(S), [\ ,\ ])$: Lie algebra

⇝ $\mathbb{Z}\hat{\pi}(S)$: the Goldman Lie algebra of the surface S

$1 \in \hat{\pi}$ constant loop. $\mathbb{Z}1 \subset \mathrm{Center}(\mathbb{Z}\hat{\pi}(S))$

$\Longrightarrow \quad \mathbb{Z}\hat{\pi}'(S) := \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1$: Lie algebra

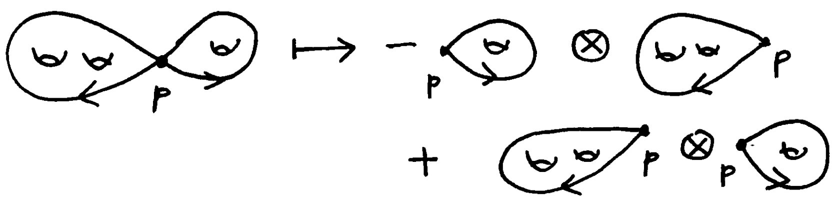
$\alpha \in \hat{\pi}(S)$ in general position



$D_\alpha := \{(t_1,t_2) \in S^1 = \mathbb{R}/\mathbb{Z} \,;\, t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$ parametrizing the double points

$$\delta(\alpha) \overset{\mathrm{def}}{=} \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in \mathbb{Z}\hat{\pi}'(S) \otimes \mathbb{Z}\hat{\pi}'(S)$$

where $|\ |' : \mathbb{Z}\pi_1(S) \xrightarrow{|\ |} \mathbb{Z}\hat{\pi}(S) \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1 = \mathbb{Z}\hat{\pi}'(S)$



Turaev cobracket

Theorem (Turaev)

(1) δ : well-defined

(2) $(\mathbb{Z}\hat{\pi}'(S), [\ ,\ ], \delta)$: Lie bialgebra

Chas — " — : involutive

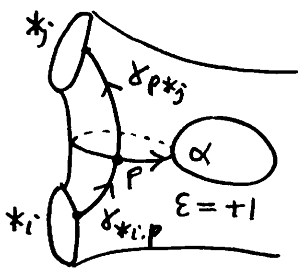
$\mathbb{Z}\hat{\pi}'(S)$: the Goldman–Turaev Lie bialgebra of the surface S

How is $\mathbb{Z}\hat{\pi}'(S)$ related to $\mathbb{Z}\Pi S|_E$?

$\Downarrow$

<u>(Co)Action of $\mathbb{Z}\hat{\pi}'(S)$ on</u> $\mathbb{Z}(\Pi S|_E)$ (Kuno-K.)

$\alpha \in \hat{\pi}(S)$, $\gamma \in \Pi S(*_i, *_j)$ in general position

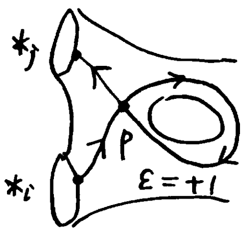


$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma)\, \gamma_{*_i p}\, \alpha_p\, \gamma_{p, *_j} \in \mathbb{Z}\Pi S(*_i, *_j)$$

$\gamma \in \Pi S(*_i, *_j)$ in general position

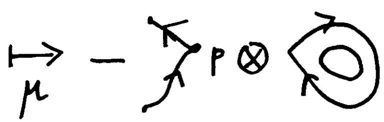
$\Gamma :=$ the set of double points of $\gamma \subset S$

$\ni p \qquad 0 \le t_1^p < t_2^p \le 1 \quad \gamma(t_1^p) = \gamma(t_2^p) = p$



$$\mu(\gamma) \overset{\text{def}}{=} -\sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\, (\gamma_{0 t_1^p}\, \gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|' \in \mathbb{Z}\Pi S(*_i, *_j) \otimes \mathbb{Z}\hat{\pi}'(S)$$

(inspired by Turaev's μ)



<u>Theorem</u> (Kuno-K.)

(1) σ, μ : well-defined

(2) $\mathbb{Z}\Pi S(*_i, *_j)$: involutive $\mathbb{Z}\hat{\pi}'(S)$-bimodule

In particular, we have

$\sigma : \mathbb{Z}\hat{\pi}'(S) \longrightarrow \mathrm{Der}_\partial(\mathbb{Z}\Pi S|_E)$ Lie algebra homomorphism

($\because$ may choose a representative of $^\forall \alpha \in \hat{\pi}$ in $S \setminus \partial S$)

Completion

$$\mathbb{Q}\hat{\pi}(S)(m) := \mathbb{Q}1 + |(I\pi_1(S,*))^m| \subset \mathbb{Q}\hat{\pi}(S) \quad (m \geq 1)$$

independent of the choice of $* \in S$

$$\widehat{\mathbb{Q}\hat{\pi}}(S) \overset{\text{def}}{=} \varprojlim_{m\to\infty} \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(m)$$: the completed Goldman-Turaev Lie bialgebra

$\widehat{\mathbb{Q}\Pi S}(*_i, *_j)$: involutive $\widehat{\mathbb{Q}\hat{\pi}}(S)$-bimodule

In particular, we have a Lie algebra homomorphism

$$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$$

<u>Theorem</u> (Kuno-K.) (An infinitesimal Dehn-Nielsen theorem)

$$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) \quad \text{isomorphism}$$

related to <u>a tensorial description</u> of $[\ ,\ ]$ and σ

given by <u>Massuyeau-Turaev</u> and <u>Kuno-K.</u> independently

<u>Massuyeau-Turaev</u> : quiver theory

a tensorial description of the (Papakyryakopoulos-Turaev) homotopy intersection form

$\Longrightarrow [\ ,\ ], \sigma, gr(\delta), gr(\mu)$

<u>Kuno-K.</u> : twisted homology of S ("elementary string topology")

<u>Theorem</u> is a key step to prove a tensorial description of $[\ ,\ ]$ and σ

(**Remark** $\sigma : \mathbb{Q}\hat{\pi}(S) \to \mathrm{Der}_\partial(\mathbb{Q}\Pi S|_E)$ is <u>not</u> surjective ("without completion")
(ex) $\alpha \in \hat{\pi}$, $(\gamma \in \Pi S(*_i, *_j) \mapsto (\alpha \cdot \gamma)\gamma) = \sigma(\log \alpha)$, $\log\alpha \in \widehat{\mathbb{Q}\hat{\pi}} \setminus \mathbb{Q}\hat{\pi}'$))

<u>Mapping class group</u>

$$\widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow[\cong]{\sigma} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$$

$$\uparrow \log \circ \widehat{DN}$$

induces

$$\mathcal{M}(S) \qquad (\widehat{DN})^{-1}(\text{▨}) \subset \mathcal{M}(S)$$

$$\cup$$

$$\tau : (\widehat{DN})^{-1}(\text{▨}) \xrightarrow{\sigma^{-1}\circ \log \circ \widehat{DN}} \widehat{\mathbb{Q}\hat{\pi}}(S), \quad \varphi \mapsto \tau(\varphi) := \sigma^{-1}(\log \widehat{DN}(\varphi)),$$

the geometric Johnson homomorphism

Theorem (original, $\Sigma_{g,1}$ Kuno–K.; general, S Massuyeau–Turaev, Kuno–K. (indep))

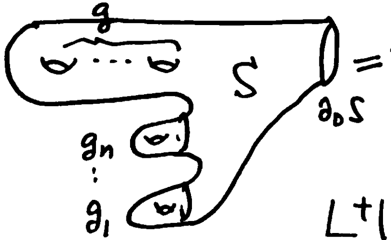
$C \subset \mathrm{Int}\, S (= S \setminus \partial S)$ simple closed curve, $C = |x|$, $x \in \pi_1(S, *)$

$$\Longrightarrow \tau(t_C) = \frac{1}{2}(\log C)^2 \left(:= \left|\frac{1}{2}(\log x)^2\right| \right) \in \widehat{\mathbb{Q}\hat{\pi}}(S)$$

i.e., $\widehat{DN}(t_C) = \exp(\sigma(\frac{1}{2}(\log C)^2)) \in \mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}|_E)$

$$L^+(S) := \{ u \in \widehat{\mathbb{Q}\pi}(S)(2) ; \ (\sigma(u)\hat{\otimes}\sigma(u))\Delta = \Delta\sigma(u), \ i(u) \in \widehat{\mathbb{Q}\pi}(\hat{S})(3) \}$$

where $\Delta : \widehat{\mathbb{Q}\Pi S|_E} \to \widehat{\mathbb{Q}\Pi S|_E} \hat{\otimes} \widehat{\mathbb{Q}\Pi S|_E}$ coproduct, $\gamma \in \Pi S|_E \mapsto \gamma \hat{\otimes} \gamma$



$= \hat{S} := S \cup_{\partial S \setminus \partial_0 S} \left(\coprod_{j=1}^{m} \Sigma_{g_j,1} \right) \ (g_j \geq 1)$

$i : S \hookrightarrow \hat{S}$ inclusion

$L^+(S)$: indep. of the choice of g_j's

$L^+(S)$: pro-nilpotent Lie subalgebra $\Rightarrow$ pro-nilpotent Lie group (by the Hausdorff series)

$\tau : \mathcal{G}^L(S) \to L^+(S)$ <u>injective</u> group homomorphism

<u>$S = \Sigma_{g,1}$</u> τ is equivalent to Massuyeau's total Johnson map.

Hence,

$$\begin{tikzcd}
\mathrm{gr}(\tau) : \mathrm{gr}(\mathcal{G}^L(\Sigma_{g,1})) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{gr}(L^+(\Sigma_{g,1})) \arrow[d, equal] \\
\text{the classical Johnson homomorphisms} : \mathrm{gr}(\mathcal{G}_{g,1}) \arrow[r] & \mathfrak{h}^+_{g,1}
\end{tikzcd}$$

w.r.to $\{\widehat{\mathbb{Q}\pi}(m)\}_{m=1}^{\infty}$

(w.r.to the Johnson filtration)

(Morita's Lie algebra)

Turaev cobracket δ ?

$\forall \varphi \in \mathcal{M}(S)$ preserves the co-action μ

$\Longrightarrow \quad \delta\tau(\varphi) = 0 \quad \forall \varphi \in \widehat{DN}^{-1}(\ )$

Theorem (Kuno-K.)

$\tau(\mathcal{J}^4(S)) \subset \mathrm{Ker}(\delta|_{L^+(S)})$

$\Downarrow$ using Massuyeau-Turaev's description of the homotopy intersection form

Theorem (Kuno-K.)

The Morita traces are recovered from the Turaev cobracket δ

(obstructions of the surjectivity of $\tau : gr(\mathcal{J}_{g,1}) \to \mathfrak{h}_{g,1}^+$)

$g=0,\ n\geq 2$

sder_n : the Lie algebra of special derivations of Free Lie $(\mathbb{Q}^n)$

(appears in Grothendieck - Ihara - Deligne theory on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$)

$$\sigma^{-1} : \mathrm{sder}_n \hookrightarrow \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{0,n+1}) \Big/ \overline{\bigoplus_{i=1}^{n} \Big(\sum_{l\in\mathbb{Z}} \mathbb{Q}|\xi_i^{\,l}| \Big)}\ \text{closure}$$

embedding of Lie algebras (through a special expansion)

$\rightsquigarrow$ No applications ?

The pullback of the Satoh traces (a refinement of the Morita traces) by the capping homomorphism $\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{0,n+1}) \to \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{n,1})$ equals the divergence cocycle (in the Kashiwara-Vergne problem) modulo low degree terms.