## (§2. The cohomology group of a Lie algebra (つづき))

### (§2.1. The standard cochain complex (つづき))

$\mathbb{K}$ : field of char. 0

$W_1 = \mathbb{K}[x]\frac{\partial}{\partial x}$

もう一度 $H^*(W_1;\mathbb{K})$ を計算する。(答え $\cong H^*(S^3;\mathbb{K})$)

$e_k := x^{k+1}\frac{d}{dx} \quad (k \geqq -1)$

$[e_0, e_k] = [x\frac{d}{dx}, x^{k+1}\frac{d}{dx}] = k\,x^{k+1}\frac{d}{dx} = k\,e_k$

$W_1(k) = \mathrm{Ker}(\mathrm{ad}(e_0) - k) = \begin{cases} \mathbb{K}e_k & \text{if } k \geqq -1 \\ 0 & \text{if } k \leqq -2 \end{cases}$

$\delta_\ell \in C^1(W_1;\mathbb{K}),\quad \delta_\ell(e_k) := \delta_{k,\ell}$

$(\mathcal{L}_{e_0}\delta_\ell)(e_k) = -\delta_\ell([e_0, e_k]) = -k\,\delta_{k,\ell} = -\ell\,\delta_\ell(e_k)$

$\mathcal{L}_{e_0}\delta_\ell = -\ell\,\delta_\ell$

$\mathcal{L}_{e_0}(\delta_{\ell_1}\cup\delta_{\ell_2}\cup\cdots\cup\delta_{\ell_p}) = -\left(\sum_{i=1}^{p}\ell_i\right)\delta_{\ell_1}\cup\delta_{\ell_2}\cup\cdots\cup\delta_{\ell_p}$ ($\because$ Leibniz rule)

$\forall f \in C^p(W_1;\mathbb{K}),\ \exists!\, c_{\ell_1\cdots\ell_p} \in \mathbb{K}\ (\ell_1 < \cdots < \ell_p)$

$f = \sum_{\ell_1<\cdots<\ell_p} c_{\ell_1\cdots\ell_p}\,\delta_{\ell_1}\cup\delta_{\ell_2}\cup\cdots\cup\delta_{\ell_p}$ (無限和)

$\mathcal{L}_{e_0}f = -\sum_{\ell_1<\cdots<\ell_p}\left(\sum_{i=1}^{p}\ell_i\right)c_{\ell_1\cdots\ell_p}\,\delta_{\ell_1}\cup\delta_{\ell_2}\cup\cdots\cup\delta_{\ell_p}$

$C^*(W_1;\mathbb{K}) = \prod_{\lambda\in\mathbb{Z}} C^*_{(\lambda)},\quad C^*_{(\lambda)} = C^*_{(\lambda)}(W_1;\mathbb{K}) = \mathrm{Ker}(\mathcal{L}_{E_0} - \lambda)$

$C^p_{(\lambda)}$ は $\{\delta_{\ell_1}\cup\delta_{\ell_2}\cup\cdots\cup\delta_{\ell_p} ; \ell_1 < \cdots < \ell_p,\ \lambda = -\sum_{i=1}^{p}\ell_i\}$ を基底にもつ

$d(C^*_{(\lambda)}) \subset C^*_{(\lambda)}$ ($\because$ $[W_1(\lambda), W_1(\mu)] \subset W_1(\lambda+\mu)$)

$H^*(C^*_{(\lambda)}) = 0 \quad$ if $\lambda \neq 0$

($\because$ $1_{C^*_{(\lambda)}} = \frac{1}{\lambda}\mathcal{L}_{e_0} = d(\frac{1}{\lambda}i_{e_0}) + (\frac{1}{\lambda}i_{e_0})d$)

$H^*(W_1;\mathbb{K}) = H^*(C^*_{(0)}(W_1;\mathbb{K}))$

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$$C^*_{(0)}(W_1;\mathbb{K}) = \begin{cases} \mathbb{K} & *=0 \\ \mathbb{K}\delta_0 & =1 \\ \mathbb{K}\delta_{-1}\cup\delta_1 & =2 \\ \mathbb{K}\delta_{-1}\cup\delta_0\cup\delta_1 & =3 \\ 0 & \geqq 4 \end{cases}$$

$$d\delta_0(e_k, e_\ell) = -\delta_0([e_k, e_\ell]) = (k-\ell)\delta_{0,k+\ell}$$

$$(\delta_{-1}\cup\delta_1)(e_k, e_\ell) = \begin{cases} 1 & \text{if } (k,\ell) = (-1,1) \\ -1 & \text{if } (k,\ell) = (1,-1) \\ 0 & \text{otherwise} \end{cases}$$

$$d\delta_0 = -2\delta_{-1}\cup\delta_1$$

したがって

$$H^*(W_1;\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } *=0 \\ \mathbb{K}[\delta_{-1}\cup\delta_0\cup\delta_1] & =3 \\ 0 & \text{otherwise} \end{cases}$$

$$sl_2(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} ; a, b, c \in \mathbb{K} \right\} \hookrightarrow W_1$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto e_0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_1$$

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \longmapsto e_{-1}$$

$$C^*_{(0)}(sl_2(\mathbb{K});\mathbb{K}) = C^*_{(0)}(W_1;\mathbb{K})$$

$$H^*(sl_2(\mathbb{K});\mathbb{K}) = H^*(W_1;\mathbb{K}) = H^*(S^3;\mathbb{K})$$

直接つながらないか? $\longleftarrow$ $\left| \begin{array}{l} su_2 \otimes \mathbb{C} = sl_2(\mathbb{C}) \\ SU_2 \cong S^3 \end{array} \right.$

(回答)

$$sl_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K} = sl_2(\mathbb{K})$$

$$H^*(sl_2(\mathbb{K});\mathbb{K}) = H^*(sl_2(\mathbb{Q});\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K}$$

$$H^*(sl_2(\mathbb{Q}):\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^*(sl_2(\mathbb{C}):\mathbb{C}) = \underset{su_2\otimes\mathbb{C} = sl_2(\mathbb{C})}{H^*(su_2:\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}}$$

$$\cong H^*_{DR}(SU_2:\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^*_{DR}(S^3:\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

$SU_2$: compact connected Lie group (これを認める)

$SU_2 \cong S^3$ $C^\infty$ diffeo

## Naturality

$\mathfrak{g}, \mathfrak{h}$ : Lie algebras $/\mathbb{K}$

**Definition** $\varphi : \mathfrak{g} \to \mathfrak{h}$ : Lie algebra$_{\mathbb{K}}$ homomorphism

$\overset{\text{def}}{\iff}$ 0) $\varphi : \mathfrak{g} \to \mathfrak{h}$ : $\mathbb{K}$-linear map

1) $\forall X, \forall Y$, $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$

ex) $M$ : $\mathfrak{g}$-module

$\iff$ $\mathfrak{g} \to \mathfrak{gl}(M)$ Lie algebra homomorphism

$N$ : $\mathfrak{h}$-module

$\Rightarrow$ $N$ : $\mathfrak{g}$-module via $\varphi$.

( $n \in N$, $X \in \mathfrak{g}$. $Xn := \varphi(X)n$ )

$\varphi^* : C^*(\mathfrak{h} ; N) \to C^*(\mathfrak{g} ; N)$ cochain map

$\ni f$. $X_i \in \mathfrak{g}$

$$(\varphi^* f)(X_1, \cdots, X_n) := f(\varphi(X_1), \cdots, \varphi(X_n))$$

$\Rightarrow$ $\varphi^* : H^*(\mathfrak{h} ; N) \to H^*(\mathfrak{g} ; N)$

cup 積を保つ

とくに

$H^*( \; ; \mathbb{K})$ : (Lie alg. $/\mathbb{K}$) $\to$ ($\mathbb{K}$-graded comm. alg)

contravariant functor

$M, N$ : $\mathfrak{g}$-modules

$\psi : M \to N$ $\mathfrak{g}$-homomorphism

$\Rightarrow$ $\psi_* : C^*(\mathfrak{g} ; M) \to C^*(\mathfrak{g} ; N)$ cochain map

$\ni f$. $X_i \in \mathfrak{g}$

$$(\psi_* f)(X_1, \cdots, X_n) := \psi(f(X_1, \cdots, X_n))$$

$\Rightarrow$ $\psi_* : H^*(\mathfrak{g} ; M) \to H^*(\mathfrak{g} ; N)$

$H_*(\mathfrak{g} ; \; )$ : ($\mathfrak{g}$-modules) $\to$ (graded $\mathbb{K}$-vector spaces)

covariant functor

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$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ exact seq. of $\mathfrak{g}$-modules

$\Longrightarrow$ $0 \to C^*(\mathfrak{g}; M') \xrightarrow{i_*} C^*(\mathfrak{g}; M) \xrightarrow{p_*} C^*(\mathfrak{g}; M'') \to 0$ (exact)

( ∵) 又は $0 \to M' \to M \to M'' \to 0$ $\mathbb{K}$-vec.sp.として split.
$\mathfrak{g}$ : $\mathbb{K}$-free )

$\Longrightarrow$ $0 \to H^0(\mathfrak{g}; M') \xrightarrow{i_*} H^0(\mathfrak{g}; M) \to \cdots$

$\cdots \xrightarrow{p_*} H^{p-1}(\mathfrak{g}; M'') \xrightarrow{\delta^*} H^p(\mathfrak{g}; M') \xrightarrow{i_*} H^p(\mathfrak{g}; M) \xrightarrow{p_*} H^p(\mathfrak{g}; M'') \to \cdots$ (exact)

## §2.2, compact connected Lie groups

$\mathbb{K} = \mathbb{R}$

$M$ : $C^\infty$ manifold, $C^\infty(M) := C^\infty(M; \mathbb{R})$

$\mathrm{Vect}(M) = \mathrm{Der}(C^\infty(M))$

$p \geq 0$

$\omega \in \Omega^p(M)$ : $p$-form on $M$

$\overset{\text{def}}{\Longleftrightarrow}$ $\omega : \mathrm{Vect}(M)^{\otimes p} \to C^\infty(M)$
alternating $C^\infty(M)$-$p$-linear map

$d\omega \in \Omega^{p+1}(M)$ exterior derivative

$$(d\omega)(X_0, X_1, \cdots, X_p) \overset{\text{def}}{=} \sum_{i=0}^{p} (-1)^i X_i(\omega(X_0, \overset{\hat{i}}{\cdots}, X_p)) \qquad \cdots (*)$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \overset{\hat{i}}{\cdots} \overset{\hat{j}}{\cdot}, X_p)$$

$dd = 0$

$\Omega^*(M) := \{\Omega^p(M), d\}_{p \geq 0}$ de Rham complex

$H^*_{DR}(M) \overset{\text{def}}{=} H^*(\Omega^*(M))$ de Rham cohomology

$\underset{\text{de Rham Isom}}{\cong} H^*(M; \mathbb{R})$ (singular cohomology)

$\Omega^*(M) \to C^*(\mathrm{Vect}(M); C^\infty(M))$ cochain map

$H^*_{DR}(M) \to H^*(\mathrm{Vect}(M); C^\infty(M))$ 全然同型ではない

$\cong$ 以下

$H^*($TMに伴うある fiber束の section space$)$

$G$ : compact connected Lie group, $d = \dim G$

$g \in G$

$L_g : G \to G,\ x \mapsto gx$, left translation

$R_g : G \to G,\ x \mapsto xg$ right 〃

$L_g^*, R_g^* : \Omega^*(G) \circlearrowleft$ cochain map

$L_g^* = R_g^* = 1 : H^*_{DR}(G) \to H^*_{DR}(G)$

( $\because$ $G$ : path-connected )

$\exists!\ dvol \in \Omega^d(G)$

s.t. $\begin{cases} \int_G dvol = 1 \\ {}^\forall g \in G,\ R_g^* dvol = dvol \end{cases}$

( $\because$ $G$ : compact )

${}^\forall f \in C^\infty(G)$

$$\int_{x \in G} f(xg)\, dvol = \int_G (R_g^* f)\, dvol = \int_G R_g^*(f\, dvol)$$

$$= \int_G f\, dvol \quad (\because G : \text{connected より } R_g : \text{ori pres})$$

$$= \int_{x \in G} f(x)\, dvol$$

$A : \Omega^*(G) \to \Omega^*(G)$ averaging operator

$$A\theta := \int_{x \in G} (L_x^* \theta)\, dvol$$

$$L_g^*(A\theta) = \int_{x\in G} (L_g^* L_x^* \theta)\, dvol = \int_{x \in G} (L_{xg}^* \theta)\, dvol$$

$$= \int_{x \in G} (L_x^* \theta)\, dvol = A\theta \rightsquigarrow A(\Omega^*(G)) \subset \Omega^*(G)^G$$

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$Ad = dA$

$A_* = 1$ on $H^*_{DR}(G)$

($\because$ $G$ : comm.)

$\Longrightarrow$ $H^*_{DR}(G) = H^*(\Omega^*(G)^G)$

$\omega \in \Omega^p(G)^G$, $X_k \in Vect(G)^G = Lie\, G$

$e \in G$ unit, $x \in G$

$\omega_x(X_1, \cdots X_p) = \omega_x(L_{x*}(X_1|_e, \cdots, L_{x*}(X_p|_e)$

$= (L_x^*\omega)_e((X_1|_e, \cdots, (X_p|_e)$

$= \omega_e((X_1|_e, \cdots, (X_p|_e) =$ constant in $x$

$$\begin{tikzcd} \Omega^p(G)^G \arrow[r] \arrow[d, "\cong"'] & C^p(Lie\, G ; \mathbb{R}) \\ \Lambda^p T_e^* G \arrow[ur, "\cong"', "ev_e^*"] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

cochain map ($\because$ (*))

isom

**Lemma 2.2.** $G$ : compact connected Lie group

$\Longrightarrow$ $H^*(Lie\, G ; \mathbb{R}) \cong H^*_{DR}(G)$

(ex) $U_n$ : unitary group

$SU_n$ : special unitary group

$\mathbb{K}$ : field of char. 0

$\mathfrak{gl}_n(\mathbb{K}) := \mathfrak{gl}(\mathbb{K}^n)$

$\mathfrak{sl}_n(\mathbb{K}) := Ker(tr : \mathfrak{gl}(\mathbb{K}^n) \to \mathbb{K})$

$\subset \mathfrak{gl}_n(\mathbb{K})$ Lie subalgebra (ideal)

( $\because$ $X, Y \in \mathfrak{gl}(\mathbb{K}^n)$

$tr([X, Y]) = tr(XY - YX) = 0$ )

**Corollary 2.3.**

$H^*(\mathfrak{gl}_n(\mathbb{C}) ; \mathbb{C}) = H^*_{DR}(U_n) \otimes \mathbb{C}$

$H^*(\mathfrak{sl}_n(\mathbb{C}) ; \mathbb{C}) = H^*_{DR}(SU_n) \otimes \mathbb{C}$

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(pf) $\mathrm{Lie}\,U_n = \{X \in \mathfrak{gl}_n(\mathbb{C}) ; {}^t\overline{X} = -X\} \ni X$ $\quad {}^t(\overline{\sqrt{-1}X}) = \sqrt{-1}X$

$$\mathfrak{gl}_n(\mathbb{C}) = (\mathrm{Lie}\,U_n) \oplus \sqrt{-1}(\mathrm{Lie}\,U_n)$$

$$X \longmapsto \left(\tfrac{1}{2}(X - {}^t\overline{X}),\ \tfrac{1}{2}(X + {}^t\overline{X})\right)$$

$$\mathfrak{gl}_n(\mathbb{C}) = (\mathrm{Lie}\,U_n) \otimes \mathbb{C}$$

$$C^*(\mathfrak{gl}_n(\mathbb{C})) = C^*(\mathrm{Lie}\,U_n ; \mathbb{R}) \otimes \mathbb{C}$$

同様に $C^*(\mathfrak{sl}_n(\mathbb{C})) = C^*(\mathrm{Lie}\,SU_n ; \mathbb{R}) \otimes \mathbb{C}$ //

Theorem 2.4.

$$H^*_{DR}(U_n) = \Lambda^*_{\mathbb{R}}(u_1, u_3, \cdots, u_{2n-1})$$

$$H^*_{DR}(SU_n) = \Lambda^*_{\mathbb{R}}(u_3, u_5, \cdots, u_{2n-1})$$

where $u_1 \in H^1_{DR}(U_n)$, $u_{2i-1} \in H^{2i-1}_{DR}(SU_n)$, $2 \leq i \leq n$.

proof $H^*(\ ) = H^*_{DR}(\ ) = H^*(\ ; \mathbb{R})$

$U_n \approx U_1 \times SU_n$ homeo

$\begin{pmatrix} \lambda & & \\ & 1 & \\ & & \ddots \\ & & & 1\end{pmatrix} A \leftarrow\!\shortmid (\lambda, A)$ 　　群同型ではない

$$H^*(U_n) = H^*(U_1) \otimes H^*(SU_n)$$

$U_1 = S^1$ より $SU_n$ について示せばよい

(その1) $SU_n$ の胞体分割を使えば群構造はわかる.

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(cf) e.g. 横田一郎「群と位相」p.163)

(その2) Serre spectral sequence をつかう.

(あとで Lie alg. version で同じことをくりかえす. 今日は感じをつかむ)

$n \geq 2$ の帰納法

$n=2$ $SU_2 = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} ; z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\} = S^3$

$n \geq 2$ $SU_{n+1}/SU_n \cong S^{2n+1}$, $A \longmapsto A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

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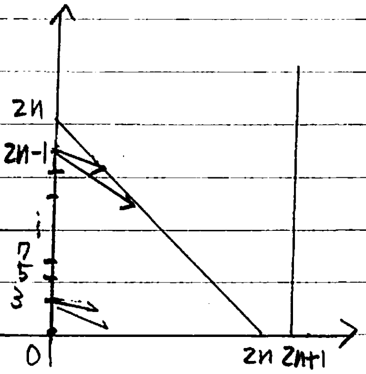
$SU_n \to SU_{n+1} \to S^{2n+1}$ principal $SU_n$-bundle

1-conn. $\Longrightarrow$ Serre fibration

$$E_2^{p,q} = H^p(S^{2n+1}) \otimes H^q(SU_n) \Rightarrow H^{p+q}(SU_{n+1})$$



$\forall i \leq 2n-1,\ \forall r \geq 2$

$d_r u_i = 0 \in E_r^{r,\, 2i-r} = 0.$

differential $d_r$ の乗法性より

$\forall r \geq 2.\quad d_r = 0$

つまり

$E_2^{p,q} = E_\infty^{p,q}$, collapses at $E_2$.

$\tilde{u}_{2n+1} \in H^{2n+1}(SU_{n+1}) \longleftarrow$ generator $\in H^{2n+1}(S^{2n+1})$

とする

$q \leqq 2n \quad H^q(SU_{n+1}) \overset{\cong}{\Rightarrow} H^q(SU_n)$

$\exists 1\ \tilde{u}_{2i-1} \longmapsto u_{2i-1}$

$$H^*(SU_{n+1}) = H^*(SU_n) \oplus H^*(SU_n)\, \tilde{u}_{2n+1}$$
$$= \Lambda^*(\tilde{u}_3, \tilde{u}_5, \cdots, \tilde{u}_{2n-1}, \tilde{u}_{2n+1})$$

帰納法が完成した //

<u>Corollary 2.5.</u> $\mathbb{K}$ : field of char. 0

$$H^*(\mathfrak{gl}_n(\mathbb{K}), \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_1, u_3, u_5, \cdots, u_{2n-1})$$

$$H^*(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) = \Lambda^*_{\mathbb{K}}(u_3, u_5, \cdots, u_{2n-1}).$$

(pf) $H^*(\mathfrak{gl}_n(\mathbb{K}) : \mathbb{K}) = H^*(\mathfrak{gl}_n(\mathbb{Q}) : \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K}.$

$H^*(\mathfrak{gl}_n(\mathbb{Q}) : \mathbb{Q}) \otimes \mathbb{C} = \Lambda^*_{\mathbb{C}}(u_1, u_3, \cdots, u_{2n-1})$ //