§2. The cohomology group of a Lie algebra

$\mathbb{K}$: field of char. 0.

§2.1. The standard cochain complex

$\mathfrak{g}$: Lie algebra/$\mathbb{K}$
M : $\mathfrak{g}$-module

$\underline{n=0}$ $C^0(\mathfrak{g}:M) := M$: $\mathfrak{g}$-module

$\underline{n>0}$ $C^n(\mathfrak{g}:M) := \{ f \in \mathrm{Hom}(\mathfrak{g}^{\otimes n}, M) \mid \underbrace{X_k \in \mathfrak{g},\ \exists i < \exists j\ X_i = X_j \Rightarrow f(\cdots, X_i, \cdots, X_j, \cdots) = 0}_{\text{alternating}} \}$

$\subset \mathrm{Hom}(\mathfrak{g}^{\otimes n}, M)$
$\mathfrak{g}$-submodule

$X \in \mathfrak{g}$, $f \in C^n(\mathfrak{g}:M)$, $X_i \in \mathfrak{g}$

$$\mathcal{L}_X f := X f$$

$$(\mathcal{L}_X f)(X_1, \cdots, X_n) = X(f(X_1, \cdots, X_n)) - \sum_{i=1}^{n} f(X_1, \cdots, \overset{i}{\check{[X, X_i]}}, \cdots, X_n)$$

$\underline{\text{interior product}}$ $Y \in \mathfrak{g}$

$$(i_Y f)(X_1, \cdots, X_{n-1}) = (i(Y)f)(X_1, \cdots, X_{n-1}) := f(Y, X_1, \cdots, X_{n-1})$$

$(n = 0 \Rightarrow i(Y) f := 0)$

$\forall X, \forall Y \in \mathfrak{g}$

$$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]} : C^n(\mathfrak{g}:M) \to C^{n-1}(\mathfrak{g}:M)$$

$$\begin{aligned}
\because)\ & (\mathcal{L}_X i_Y f)(X_1, \cdots, X_{n-1}) \\
&= X((i_Y f)(X_1, \cdots, X_{n-1})) - \sum_{i=1}^{n-1} (i_Y f)(X_1, \cdots, [X, X_i], \cdots, X_{n-1}) \\
&= X(f(Y, X_1, \cdots, X_{n-1})) - \sum_{i=1}^{n-1} f(Y, X_1, \cdots, [X, X_i], \cdots, X_{n-1}) \\
&= (\mathcal{L}_X f)(Y, X_1, \cdots, X_{n-1}) + f([X, Y], X_1, \cdots, X_{n-1}) \\
&= (i_Y \mathcal{L}_X f + i_{[X,Y]} f)(X_1, \cdots, X_{n-1}) \quad //
\end{aligned}$$

$d = d_M : C^n(\mathfrak{g}:M) \to C^{n+1}(\mathfrak{g}:M)$ coboundary operator

$\underline{n=0}$ $m \in M = C^0(\mathfrak{g}:M)$, $X \in \mathfrak{g}$

$(dm)(X) := Xm$

$\underline{n \geq 1}$ $f \in C^n(\mathfrak{g}:M)$, $Y \in \mathfrak{g}$

df is defined inductively by

$$i_Y(df) = \mathcal{L}_X f - d(i_Y f)$$

$df \in C^{n+1}(\mathfrak{g}:M)$, alternating

induction on $n \geq 0$
$n=0$: clear
$n \geq 1$

∵) $X_k \in \mathfrak{g}$, $\exists i < \exists j$, $X_i = X_j$

$\underline{i \geq 2}$

$(df)(X_1,\dots,X_n) = (i(X_1)df)(X_2,\dots,X_n)$

$= (\mathcal{L}_{X_1} f - d(i(X_1)f))(X_2,\dots,X_n) \underset{\text{ind. assumption}}{=} 0$

$\underline{i = 1}$

$(df)(X_1,\dots,X_n) = (i(X_1)df)(X_2,\dots,X_n)$

$= (\mathcal{L}_{X_1} f - d\,i(X_1)f)(X_2,\dots,X_n)$

$\underset{\text{ind. assumption}}{=} (-1)^{j-2}(\mathcal{L}_{X_1} f - d\,i(X_1)f)(X_j, X_2, \overset{\hat{j}}{\dots}, X_n)$

$= (-1)^{j-2}\, i(X_1)(\mathcal{L}_{X_1} f - d\,i(X_1)f)(X_2, \overset{\hat{j}}{\dots}, X_n)$

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$i(X_1)(\mathcal{L}_{X_1} f - d\,i(X_1)f)$

$\underset{[X_1,X_1]=0}{=} \mathcal{L}_{X_1} i(X_1) f - \mathcal{L}_{X_1} \cancel{i(X_1)} f + d\,i(X_1)\,i(X_1) f \underset{f\text{ alt.}}{=} 0$ //

$\mathcal{L}_X d = d\mathcal{L}_X : C^n(\mathfrak{g}:M) \to C^{n+1}(\mathfrak{g}:M)$

∵) induction on $n \geq 0$

$\underline{n=0}$ $m \in M$

$(\mathcal{L}_X dm)(Y) = X((dm)(Y)) - (dm)[X,Y]$

$= X(Ym) - [X,Y]m = Y(Xm)$

$= d(Xm)(Y)$ //

$n \geq 1$ $f \in C^n(\mathfrak{g}:M)$, $\forall Y \in \mathfrak{g}$.

$$i(Y)\mathcal{L}_X df = \mathcal{L}_X i(Y) df - i([X,Y]) df$$

$$= \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_X d\, i(Y) f - \mathcal{L}_{[X,Y]} f + d\, i([X,Y]) f$$

$$\underset{\text{ind. assumpt.}}{=} \mathcal{L}_Y \mathcal{L}_X f - d\mathcal{L}_X i(Y) f + d\, i[X,Y] f$$

$$= \mathcal{L}_Y\mathcal{L}_X f - d\, i(Y)\mathcal{L}_X f = i(Y)\, d\mathcal{L}_X f \quad //$$

$$dd = 0 : C^n(\mathfrak{g}:M) \to C^{n+2}(\mathfrak{g}:M)$$

(∵) induction on $n \geq -1$

$n=-1$ trivial, $C^{-1}(\mathfrak{g}:M)=0$

$n\geq 0$ $f\in C^n(\mathfrak{g}:M)$, $\forall Y\in\mathfrak{g}$

$$i(Y)ddf = -d\,i(Y)df + \mathcal{L}_Y df$$

$$= +dd\,i(Y)f - d\cancel{\mathcal{L}_Y f} + \cancel{d\mathcal{L}_Y f}$$

$$\underset{\text{ind. assumpt.}}{=} 0 \quad //$$

$$(df)(X_0,\dots,X_n) = \sum_{i=0}^n (-1)^i X_i(f(X_0,\overset{\hat i}{\dots},X_n)) + \sum_{i<j}(-1)^{i+j} f([X_i,X_j],X_0,\overset{\hat i\ \hat j}{\dots},X_n)$$

((∵) induction on $n\geq 0$)

$n=0$ clear

$n\geq1$ $(df)(X_0,\dots,X_n) = (i(X_0)(df))(X_1,\dots,X_n)$

(田答)

$$= (\mathcal{L}_{X_0} f)(X_1,\dots,X_n) - d(i(X_0)f)(X_1,\dots,X_n)$$

$$= X_0(f(X_1,\dots,X_n)) - \sum_{j=1}^n f(X_1,\dots,[X_0,X_j],\dots,X_n)$$

$$+ \sum_{i=1}^n (-1)^i X_i(f(X_0,X_1,\overset{\hat i}{\dots},X_n))$$

$$- \sum_{1\leq i<j} (-1)^{i+j} f(X_0,[X_i,X_j],X_1,\overset{\hat i\ \hat j}{\dots},X_n)$$

$=$ 右辺 //

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$$C^*(\mathfrak{g}:M) = \{C^n(\mathfrak{g}:M), d\}_{n\geq 0}$$ the standard cochain complex of $\mathfrak{g}$ with values in M.

Definition

$$H^p(\mathfrak{g}:M) \overset{\text{def}}{=} H^p(C^*(\mathfrak{g}:M))$$ the p^{th} cohomology group of $\mathfrak{g}$ with values in M.

cup product

N : $\mathfrak{g}$-module, $p, q \geq 0$

$f \in C^p(\mathfrak{g}:M)$, $g \in C^q(\mathfrak{g}:N)$, $X_i \in \mathfrak{g}$

$$(f \cup g)(X_1, \cdots, X_{p+q}) = \frac{1}{p!\,q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (\operatorname{sign}\sigma) f(X_{\sigma(1)}, \cdots, X_{\sigma(p)}) \otimes g(X_{\sigma(p+1)}, \cdots, X_{\sigma(p+q)})$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_{p+q} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(p+q)}} (\operatorname{sign}\sigma) f(X_{\sigma(1)}, \cdots, X_{\sigma(p)}) \otimes g(X_{\sigma(p+1)}, \cdots, X_{\sigma(p+q)})$$

$X_0 \in \mathfrak{g}$

$$i(X_0)(f \cup g) = (i(X_0)f) \cup g + (-1)^p f \cup (i(X_0) g)$$

$\because)$ $(f \cup g)(X_0, X_1, \cdots, X_{p+q-1})$

$$= \sum_{\substack{\sigma \in \mathfrak{S}\{0,1,\cdots,p+q-1\} \\ \sigma(0) < \cdots < \sigma(p-1) \\ \sigma(p) < \cdots < \sigma(p+q-1)}} (\operatorname{sign}\sigma) f(X_{\sigma(0)}, \cdots, X_{\sigma(p-1)}) \otimes g(X_{\sigma(p)}, \cdots, X_{\sigma(p+q-1)}) = \left(\sum_{\sigma(0)=0} + \sum_{\sigma(p)=0} \right)$$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_{p+q-1} \\ \sigma(1) < \cdots < \sigma(p-1) \\ \sigma(p) < \cdots < \sigma(p+q-1)}} (\operatorname{sign}\sigma) f(X_0, X_{\sigma(1)}, \cdots, X_{\sigma(p-1)}) \otimes g(X_{\sigma(p)}, \cdots, X_{\sigma(p+q-1)})$$

$$+ (-1)^p \sum_{\substack{\tau \in \mathfrak{S}_{p+q-1} \\ \tau(1) < \cdots < \tau(p) \\ \tau(p+1) < \cdots < \tau(p+q-1)}} (\operatorname{sign}\tau) f(X_{\tau(1)}, \cdots, X_{\tau(p)}) \otimes g(X_0, X_{\tau(p+1)}, \cdots, X_{\tau(p+q-1)})$$

$$\left(\sigma = \begin{pmatrix} 0 & 1 & \cdots & p+q-1 \\ 0 & \tau(1) & \cdots & \tau(p+q-1) \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & p-1 & p & p+1 & \cdots & p+q-1 \\ 1 & 2 & \cdots & p & 0 & p+1 & \cdots & p+q-1 \end{pmatrix} \right)$$

$$= \left((i(X_0) f) \cup g + (-1)^p f \cup i(X_0) g \right)(X_1, \cdots, X_{p+q-1})$$

//

$$\mathcal{L}_X(f\cup g) = (\mathcal{L}_X f)\cup g + f\cup(\mathcal{L}_X g)$$

(∵) <u>induction</u> on $p+q \geq 0$ $i_Y\mathcal{L}_X(f\cup g)$ を計算する)

<u>$p+q=0$</u> clear

<u>$p+q\geq 1$</u> $i_Y\mathcal{L}_X(f\cup g) = \mathcal{L}_X i_Y(f\cup g) - i_{[X,Y]}(f\cup g)$

(略) $= \mathcal{L}_X((i_Y f)\cup g) + (-1)^p\mathcal{L}_X(f\cup(i_Y g)) - (i_{[X,Y]}f)\cup g - (-1)^p f\cup(i_{[X,Y]}g)$

$\overset{\text{ind. assumption}}{=} \cancel{(\mathcal{L}_X i_Y f)\cup g} + (i_Y f)\cup(\mathcal{L}_X g) + (-1)^p(\mathcal{L}_X f)\cup(i_Y g) + (-1)^p \cancel{f\cup(\mathcal{L}_X i_Y g)}$
$\quad - (\cancel{\mathcal{L}_X i_Y f)\cup g} + (i_Y\mathcal{L}_X f)\cup g - \cancel{(-1)^p f\cup\mathcal{L}_X i_Y g} + (-1)^p f\cup i_Y\mathcal{L}_X g$

$= i_Y(f\cup\mathcal{L}_X g + (\mathcal{L}_X f)\cup g)$ //

$$d(f\cup g) = (df)\cup g + (-1)^p f\cup(dg)$$

(∵) <u>induction</u> on $p+q\geq 0$)

<u>$p+q=0$</u> clear ∵ Leibniz rule

<u>$p+q\geq 1$</u> $i_X d(f\cup g)$ を計算する

$i_X d(f\cup g) = \mathcal{L}_X(f\cup g) - d i_X(f\cup g)$

(略) $= (\mathcal{L}_X f)\cup g + f\cup(\mathcal{L}_X g) - d((i_X f)\cup g) - (-1)^p d(f\cup(i_X g))$

$\overset{\text{ind. assumpt.}}{=} \cancel{(d i_X f)\cup g} + (i_X df)\cup g + f\cup\cancel{(d i_X g)} + f\cup i_X dg$
$\quad - \cancel{(d i_X f)\cup g} + (-1)^p(i_X f)\cup dg - (-1)^p(df)\cup i_X g - \cancel{f\cup d i_X g}$

$= i_X((df)\cup g + (-1)^p f\cup(dg))$ //

$\cup : H^p(\mathfrak{g};M)\otimes H^q(\mathfrak{g};N) \longrightarrow H^{p+q}(\mathfrak{g};M\otimes N)$ cup product

$[f]\otimes[g] \longmapsto [f\cup g]$

(<u>well-defined</u> $d(f\cup g) = (df)\cup g + (-1)^p f\cup(dg)$)

- associative
- graded commutative
- unit $1\in\mathbb{K} = C^0(\mathfrak{g};\mathbb{K})$, $d1=0$

$M\otimes\mathbb{K} = M = \mathbb{K}\otimes M$

$f\cup 1 = f = 1\cup f \in C^m(\mathfrak{g};M)$ $\forall f\in C^m(\mathfrak{g};M)$

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とくに. $H^*(\mathfrak{g};\mathbb{K})$: graded commutative algebra / $\mathbb{K}$
$H^*(\mathfrak{g};M)$: $H^*(\mathfrak{g};\mathbb{K})$-module.

[Cartan formula
$d\, i(X) + i(X)\, d = \mathcal{L}_X$ on $C^*(\mathfrak{g};M)$.

$\Longrightarrow$ $(\mathcal{L}_X)_* = 0$ on $H^*(\mathfrak{g};M)$.

応用例 $n \geq 1$ ∞-dim

$$W_n = \bigoplus_{i=1}^{n} \mathbb{K}[x_1,\cdots,x_n]\frac{\partial}{\partial x_i}$$ the Lie algebra of polynomial vector fields

Theorem 2.1 (Gel'fand - Fuks)

$\dim_{\mathbb{K}} H^*(W_n;\mathbb{K}) < \infty$

proof $E_0 := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \in W_n$ Euler operator

$\xi = \xi(x_1,\cdots,x_n)$ homogeneous of degree λ.

$E_0 \xi = \lambda \xi$ (Euler's formula)

$[E_0, \frac{\partial}{\partial x_i}] = -\frac{\partial}{\partial x_i}$

$$W_n = \bigoplus_{\lambda=-1}^{\infty} W_n(\lambda), \quad W_n(\lambda) := \mathrm{Ker}(E_0 - \lambda : W_n \to W_n)$$

$[W_n(\lambda), W_n(\mu)] \subset W_n(\lambda+\mu)$ $(\forall\lambda, \forall\mu \in \mathbb{Z})$

$$C^*(W_n;\mathbb{K}) = \prod_{\lambda\in\mathbb{Z}} C^*_{(\lambda)}.$$

$C^*_{(\lambda)} = C^*_{(\lambda)}(W_n;\mathbb{K}) := \mathrm{Ker}(\mathcal{L}_{E_0} - \lambda)$

$d(C^*_{(\lambda)}) \subset C^*_{(\lambda)}$

$i(E_0)(C^*_{(\lambda)}) \subset C^*_{(\lambda)}$

$$\prod_{\lambda\neq 0}\left(\frac{1}{\lambda}\mathcal{L}_{E_0} \text{ on } C^*_{(\lambda)}\right) \simeq 0 \text{ on } C^*(W_n;\mathbb{K})$$

$$C^*(W_n;\mathbb{K}) \simeq C^*_{(0)}(W_n;\mathbb{K})$$

$$\dim_{\mathbb{K}} C^*_{(0)}(W_n;\mathbb{K}) \lneqq \infty$$

$$\left(\begin{array}{l} (\because)\ \dim_{\mathbb{K}} W_n(\lambda) \lneqq \infty \quad \forall \lambda \in \mathbb{Z} \\ W_n(\lambda) = 0 \text{ if } \lambda \leqq -2 \quad // \end{array} \right) \quad //$$

Fact (Gel'fand-Fuks)

$G_{n,2n} := \{ V \subset \mathbb{C}^{2n};$ complex linear subspace of $\dim_{\mathbb{C}} = n \}$

$\leftarrow$ Schubert cell decomposition

$B_n := (G_{n,2n})^{(2n)}$ 2n-skeleton

$P_n \to G_{n,2n}$ tautological U_n-bundle

$F_n := P_n | B_n$

$\Longrightarrow H^*(W_n;\mathbb{K}) \cong H^*(F_n;\mathbb{K})$ (singular cohomology)

$\underline{n=1}$ $G_{1,2} = \mathbb{C}P^1$

$B_1 = \mathbb{C}P^1 = G_{1,2}$

$P_1 = F_1 = S^3$

$$H^*(W_1;\mathbb{K}) = H^*(S^3;\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } * = 0,3 \\ 0 & \text{otherwise} \end{cases}$$

ここまで

定義にもどって計算する

$W_1 = \mathbb{K}[x]\frac{d}{dx}$

$e_k := x^{k+1}\frac{d}{dx}, \quad [e_k, e_\ell] = [x^{k+1}\frac{d}{dx}, x^{\ell+1}\frac{d}{dx}] = (\ell - k) e_{k+\ell}$

とくに $[e_0, e_k] = k e_k$

$\delta_\ell \in C^1(W_1;\mathbb{K}), \quad \delta_\ell(e_k) := \delta_{k,\ell}$

$\mathcal{L}_{e_0} \delta_\ell = -\ell \delta_\ell$

$$C^*_{(0)}(W_1;\mathbb{K}) = \begin{cases} \mathbb{K} & * = 0 \\ \mathbb{K}\delta_0 & = 1 \\ \mathbb{K}\delta_{-1} \cup \delta_1 & = 2 \\ \mathbb{K}\delta_{-1} \cup \delta_0 \cup \delta_1 & = 3 \\ 0 & \geqq 4 \end{cases}$$

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$$(d\delta_0)(e_k, e_\ell) = -\delta_0([e_k, e_\ell]) = (k-\ell)\,\delta_{0,k+\ell}$$

$$(\delta_{-1} \cup \delta_1)(e_k, e_\ell) = \begin{cases} 1 & \text{if } (k,\ell) = (-1,1) \\ -1 & \text{if } (k,\ell) = (1,-1) \\ 0 & \text{otherwise} \end{cases}$$

$$d\delta_0 = -2\,\delta_{-1} \cup \delta_1 \qquad \frac{1}{2} \in \mathbb{K}$$

したがって

$$H^*(W_1 ; \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } * = 0 \\ \mathbb{K}[\delta_{-1} \cup \delta_0 \cup \delta_1] & \text{if } * = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$sl_2(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} ; a, b, c \in \mathbb{K} \right\} \hookrightarrow W_1$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto e_0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_1$$

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \longmapsto e_{-1}$$

$$C^*_{(0)}(sl_2(\mathbb{K}), \mathbb{K}) = C^*_{(0)}(W_1 ; \mathbb{K})$$

$$H^*(sl_2(\mathbb{K}) ; \mathbb{K}) = H^*(W_1 ; \mathbb{K}) = H^*(S^3 ; \mathbb{K})$$

直接つながらないか?

$$sl_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K} = sl_2(\mathbb{K})$$

$$H^*(sl_2(\mathbb{K}) ; \mathbb{K}) = H^*(sl_2(\mathbb{Q}) : \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{K}$$

$$H^*(sl_2(\mathbb{C}) : \mathbb{C}) = H^*(sl_2(\mathbb{Q}) : \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

$$\| \leftarrow su_2 \otimes \mathbb{C} = sl_2(\mathbb{C})$$

$$H^*(su_2 ; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

$\wr\|$ ← SU_2 : compact connected Lie group (次回)

$$H^*_{DR}(SU_2 ; \mathbb{R}) \otimes \mathbb{C}$$

$$\| \leftarrow SU_2 \underset{\text{diffeo.}}{\cong} S^3$$

$$H^*_{DR}(S^3 ; \mathbb{C})$$

直接つながった