13年度夏学期
幾何学XG＝基礎数理特別講義III
河澄響矢（東大数理）
http://www.ms.u-tokyo.ac.jp/~kawazumi/GeomXG13S.html
単位・成績：レポートによる。

"An introduction to cohomology of Lie algebras"

§1. Definition of Lie algebras

$\mathbb{K}$: field of char. 0
$\otimes = \otimes_{\mathbb{K}}$
$\mathrm{Hom} = \mathrm{Hom}_{\mathbb{K}}$.

§1.1. Lie algebras

Definition $\mathfrak{g}$: Lie algebra ($/\mathbb{K}$)
$\overset{\text{def}}{\Longleftrightarrow}$ 0) $\mathfrak{g}$: $\mathbb{K}$-vector space
$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $\mathbb{K}$-bilinear map
1) (skew) $\forall X \in \mathfrak{g}$, $[X, X] = 0$
2) (Jacobi) $\forall X, \forall Y, \forall Z \in \mathfrak{g}$
$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Remark 1) $\Leftrightarrow$ 1') $\forall X, \forall Y \in \mathfrak{g}$, $[X,Y] = -[Y,X]$
$\because$) ($\Rightarrow$) $[X+Y, X+Y] - [X,X] - [Y,Y] = [X,Y] + [Y,X]$
($\Leftarrow$) $2[X,X] = 0$, $\frac{1}{2} \in \mathbb{K}$ //

Definition $\mathfrak{h} < \mathfrak{g}$: Lie subalgebra ($/\mathbb{K}$)
$\overset{\text{def}}{\Longleftrightarrow}$ 0) $\mathfrak{h} < \mathfrak{g}$ $\mathbb{K}$-linear subspace
1) $\forall X, \forall Y \in \mathfrak{h}$, $[X,Y] \in \mathfrak{h}$

$\Longrightarrow$ $\mathfrak{h}$: Lie algebra $/\mathbb{K}$

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Examples

(1) A : associative algebra $/\mathbb{K}$

$[a,b] := ab - ba \quad (a, b \in A)$

$\Longrightarrow$ A : Lie algebra.

($\because$) 1) $[a,a] = aa - aa = 0$

2) $[a,b]c + b[a,c] = abc - b\cancel{a}c + b\cancel{a}c - bca$

$= [a, bc]$

$[a,[b,c]] = [a,bc] - [a,cb]$

$= [a,b]c + b[a,c] - [a,c]b - c[a,b]$

$= [[a,b],c] + [b,[a,c]]$ //

(2). V : $\mathbb{K}$-vector space

$\mathfrak{gl}(V) := \{T : V \to V : \mathbb{K}\text{-linear map}\}$

associative algebra $/\mathbb{K}$

$\Longrightarrow$ Lie algebra $/\mathbb{K}$.

(3) $\mu : V \times V \longrightarrow V$, $\mathbb{K}$-bilinear map

$$\mathrm{Der}(V,\mu) := \left\{ T \in \mathfrak{gl}(V) ; \forall u, \forall v \in V \quad T\mu(u,v) = \mu(Tu,v) + \mu(u,Tv) \right\}$$

(Leibniz' rule)

$< \mathfrak{gl}(V)$ Lie subalgebra

the derivation Lie algebra of (V,μ)

($\because$) $S, T \in \mathrm{Der}(V,\mu)$

$ST\mu(u,v) = S\mu(Tu,v) + S\mu(u,Tv)$

$= \mu(STu,v) + \mu(Tu,Sv) + \mu(Su,Tv) + \mu(u,STv)$

$[S,T]\mu(u,v) = \mu([S,T]u,v) + \mu(u,[S,T]v)$ //

ex). A : associative algebra $/\mathbb{K}$

$\Longrightarrow$ $\mathrm{Der}(A)$: the derivation Lie algebra of A

(4) $n \geq 1$

$A = \mathbb{K}[x_1, x_2, \cdots, x_n]$ polynomial ring

$$W_n := \mathrm{Der}(A) = \bigoplus_{i=1}^{n} \mathbb{K}[x_1, x_2, \cdots, x_n] \frac{\partial}{\partial x_i}$$

the Lie algebra of polynomial vector fields

$\because$) $\bigoplus_{i=1}^{n} \mathbb{K}[x_1, x_2, \cdots, x_n] \frac{\partial}{\partial x_i} \to W_n$

<u>injective</u> $\quad \ni X = \sum_{i=1}^{n} \xi_i(x_1, \cdots, x_n) \frac{\partial}{\partial x_i}$

$X x_i = \xi_i(x_1, \cdots, x_n)$

<u>surjective</u> $\quad D \in \mathrm{Der}(A)$

$$X := \sum_{i=1}^{n} (D x_i) \frac{\partial}{\partial x_i}$$

$$D(x_{i_1} x_{i_2} \cdots x_{i_m}) = \sum_{k=1}^{m} (D x_{i_k}) x_{i_1} \overset{\hat{k}}{\cdots} x_{i_m} = \sum_{k=1}^{m} (X x_{i_k}) x_{i_1} \overset{\hat{k}}{\cdots} x_{i_m}$$

$$= X(x_{i_1} x_{i_2} \cdots x_{i_m}) \quad //$$

(5). ($\mathbb{K} = \mathbb{R}$)

M: C^∞ mfd.

$C^\infty(M) := \{ f : M \to \mathbb{R},\ C^\infty \text{ functions} \}$ associative algebra/$\mathbb{R}$

$\mathrm{Vect}(M) \overset{\text{def}}{=} \mathrm{Der}(C^\infty(M))$

(6) ($\mathbb{K} = \mathbb{R}$)

G: Lie group

(i.e., G: C^∞ mfd, group

$\left. \begin{array}{l} G \times G \to G, \quad (x, y) \mapsto xy \\ G \to G, \quad x \mapsto x^{-1} \end{array} \right\} C^\infty$ maps)

$G \curvearrowright \mathrm{Vect}(G) \quad \Leftarrow \quad G \curvearrowright G$ left translation

$\ \ \Psi x \qquad \Psi X \qquad\qquad f \in C^\infty(G)$

$(xX)(f) := (X(f \circ x)) \circ x^{-1}$

$xX \in \mathrm{Vect}(G)$

(略) ($\because$) $(xX)(fg) = (X((f \circ x)(g \circ x))) \circ x^{-1}$

$= ((X(f \circ x)) \circ x^{-1}) g + f ((X(g \circ x)) \circ x^{-1}) = (xX)(f) g + f (xX)(g)$)

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$\mathrm{Lie}(G) := \mathrm{Vect}(G)^G = \{X \in \mathrm{Vect}(G) ; \forall x \in G, xX = X\}$

$\mathrm{Lie}(G) < \mathrm{Vect}(G)$ Lie subalgebra

(略)
∵) $X, Y \in \mathrm{Vect}(G), x \in G, f \in C^\infty(G)$

$(xX)((xY)f) = (X(((xY)f) \circ x)) \circ x^{-1} = (X(Y(f \circ x))) \circ x^{-1}$

$[xX, xY]f = \{X(Y(f \circ x)) - Y(X(f \circ x))\} \circ x^{-1}$

$= (x[X, Y])f$ //

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(1) $e \in G$ 単位元について次の $\mathbb{R}$-線型同型を示せ

$\mathrm{Lie}(G) \cong T_e G$ (tangent space of G at e)

(2) 次の Lie 代数の同型を示せ

$\mathrm{Lie}(GL_n(\mathbb{R})) \cong \mathfrak{gl}(\mathbb{R}^n)$

§ 1.2. $\mathfrak{g}$-modules

$\mathfrak{g}$: Lie algebra / $\mathbb{K}$.

Definition M : (left) $\mathfrak{g}$-module [resp. right $\mathfrak{g}$-module]

$\overset{\text{def}}{\Longleftrightarrow}$ 0) M : $\mathbb{K}$-vector space

$\mathfrak{g} \times M \to M$, $\mathbb{K}$-bilinear map $\left[\text{resp. } M \times \mathfrak{g} \to M, \; \mathbb{K}\text{-bilinear map} \right.$

$(X, m) \mapsto Xm$ $\left. (m, X) \mapsto mX \right]$

1) $\forall X, \forall Y \in \mathfrak{g}, \forall m \in M$ [resp.

$X(Ym) - Y(Xm) = [X, Y]m$ $(mX)Y - (mY)X = m[X, Y]$]

左↔右 $\quad Xm = -mX$

(略)
$$\begin{array}{ccc} X(Ym) - Y(Xm) & = & (mY)X - (mX)Y \\ \| & & \| \\ [X, Y]m & & -m[X, Y] \end{array}$$

$M^{\mathfrak{g}} := \{m \in M ; \forall X \in \mathfrak{g}, Xm = 0\}$ invariants

$M_{\mathfrak{g}} := M/\mathfrak{g}M$ coinvariants

Remark 合成 $M^{\mathfrak{g}} \hookrightarrow M \twoheadrightarrow M_{\mathfrak{g}}$ は同型とは限らない

(ex) $\mathfrak{g} = \mathbb{K} \curvearrowright M = \mathbb{K}^2 \quad M^{\mathfrak{g}} = \{ \binom{*}{0} ; * \in \mathbb{K} \}$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M_{\mathfrak{g}} = M/M^{\mathfrak{g}}$

Remark M : $\mathfrak{g}$-module, $m_0 \in M$

$\mathfrak{g}_{m_0} := \{ X \in \mathfrak{g} ; X m_0 = 0 \}$ annihilator of m_0
$\subset \mathfrak{g}$ Lie subalgebra
$(\because [X, Y] m_0 = X(Y m_0) - Y(X m_0))$

M : (left) $\mathfrak{g}$-module

Definition $M' \subset M$: $\mathfrak{g}$-submodule
$\overset{\text{def}}{\Longleftrightarrow}$ 0) $M' \subset M$ $\mathbb{K}$-linear subspace
1) $\forall X \in \mathfrak{g}, \forall m' \in M', \ X m' \in M'$

$\Longrightarrow$ M', M/M' : $\mathfrak{g}$-modules

Remarks $0 \to (M')^{\mathfrak{g}} \to M^{\mathfrak{g}} \to (M/M')^{\mathfrak{g}}$ (exact)

$(M')_{\mathfrak{g}} \to M_{\mathfrak{g}} \to (M/M')_{\mathfrak{g}} \to 0$ (exact)

$(\cdot)^{\mathfrak{g}}$, $(\cdot)_{\mathfrak{g}}$ は完全函手ではない

(ex) $\mathfrak{g} = \mathbb{K} \curvearrowright M = \mathbb{K}^2 \quad M' := M^{\mathfrak{g}}$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $(M/M')^{\mathfrak{g}} = (M/M')_{\mathfrak{g}} = M/M'$

Examples

(0) V : $\mathbb{K}$-vector space
$\forall X \in \mathfrak{g}, \forall v \in V, \ X v := 0$
$\Longrightarrow$ V : <u>trivial</u> $\mathfrak{g}$-module

(1) $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
$(X, Z) \longmapsto \mathrm{ad}(X)(Z) := [X, Z]$
$\Longrightarrow$ $\mathfrak{g}$: (left) $\mathfrak{g}$-module <u>adjoint representation</u>
($\because$ Jacobi identity //)
(略) ($\mathrm{ad}(X)\mathrm{ad}(Y)(Z) - \mathrm{ad}(Y)\mathrm{ad}(X)(Z) = [X[Y, Z]] - [Y[X, Z]]$
$\overset{\text{Jacobi identity}}{=} [[X, Y], Z] = (\mathrm{ad}[X, Y])(Z)$ //

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(2). V : $\mathbb{K}$-vector space

$\mathfrak{gl}(V) \times V \longrightarrow V, \quad (T, v) \longmapsto Tv$

$\Longrightarrow$ V : (left) $\mathfrak{gl}(V)$-module.

(3) M_1, M_2 : $\mathfrak{g}$-modules

$\Longrightarrow$ $M_1 \otimes M_2$: $\mathfrak{g}$-module.

$m_1 \in M_1, \ m_2 \in M_2, \ X \in \mathfrak{g}$

$X(m_1 \otimes m_2) := (Xm_1) \otimes m_2 + m_1 \otimes (Xm_2)$. (Leibniz rule)

(略) ∵) $X(Y(m_1 \otimes m_2)) = X((Ym_1) \otimes m_2) + X(m_1 \otimes (Ym_2))$

$= (XYm_1) \otimes m_2 + (Ym_1) \otimes (Xm_2) + (Xm_1) \otimes (Ym_2) + m_1 \otimes (XYm_2)$

$X(Y(m_1 \otimes m_2)) - Y(X(m_1 \otimes m_2))$

$= ([X,Y]m_1) \otimes m_2 + m_1 \otimes ([X,Y]m_2) = [X,Y](m_1 \otimes m_2)$ //

(4) M_1, M_2 : $\mathfrak{g}$-modules

$\Longrightarrow$ $\mathrm{Hom}(M_1, M_2)$: $\mathfrak{g}$-module

$\varphi \in \mathrm{Hom}(M_1, M_2)$, $X \in \mathfrak{g}, \ m_1 \in M_1$

$(X\varphi)(m_1) = X(\varphi(m_1)) - \varphi(Xm_1)$ (Leibniz' rule)

(略) ∵) $(X(Y\varphi))(m_1) = X((Y\varphi)(m_1)) - (Y\varphi)(Xm_1)$

$= X(Y(\varphi(m_1))) - X(\varphi(Ym_1)) - Y(\varphi(Xm_1)) + \varphi(YXm_1)$

$\{(X(Y\varphi)) - (Y(X\varphi))\}(m_1)$

$= [X,Y](\varphi(m_1)) - \varphi([X,Y]m_1) = ([X,Y]\varphi)(m_1)$ //

$\mathrm{Hom}(M_1, M_2)^{\mathfrak{g}} = \left\{ \varphi : M_1 \to M_2 \ ; \ \begin{array}{l} \forall X \in \mathfrak{g} \ \forall m_1 \in M_1 \\ \varphi(Xm_1) = X(\varphi(m_1)) \end{array} \right\}$

$= \mathrm{Hom}_{\mathfrak{g}}(M_1, M_2)$ $\quad$ $\mathfrak{g}$-homomorphisms

Definition $\mathfrak{k} \subset \mathfrak{g}$ ideal

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathfrak{k} \subset \mathfrak{g}$: $\mathfrak{g}$-submodule

$\Longleftrightarrow$ (0) $\mathfrak{k} \subset \mathfrak{g}$: $\mathbb{K}$-linear subspace

(1) $\forall X \in \mathfrak{g}, \ \forall Y \in \mathfrak{k} \quad [X,Y] \in \mathfrak{k}$

M : $\mathfrak{g}$-module

<u>$n=0$</u>

$$C^0(\check{\mathfrak{k}}; M) := M \quad : \mathfrak{g}\text{-module}$$

<u>$n>0$</u>

$$C^n(\check{\mathfrak{k}}; M) := \{ f \in \mathrm{Hom}(\check{\mathfrak{k}}^{\otimes n}, M) ; \underbrace{\exists i < \exists j \ Y_i = Y_j \Rightarrow f(Y_1 \otimes \cdots \otimes Y_n) = 0}_{\text{alternating}} \}$$

(<u>Notation</u> $f(Y_1, \cdots, Y_n) := f(Y_1 \otimes \cdots \otimes Y_n)$)

$C^n(\check{\mathfrak{k}}; M) \subset \mathrm{Hom}(\check{\mathfrak{k}}^{\otimes n}, M)$ $\mathfrak{g}$-submodule

[∵) $f \in C^n(\check{\mathfrak{k}}, M)$, $X \in \mathfrak{g}$, $Y_k \in \check{\mathfrak{k}}$

$\exists i < \exists j$, $Y_i = Y_j$

$$\begin{aligned}
(Xf)(Y_1, \cdots, Y_n) &= X(\underbrace{f(Y_1, \cdots, Y_n)}_{=0}) - \sum_{k \neq i,j} \underbrace{f(\cdots [X, Y_k] \cdots)}_{=0} \\
&\quad - f(\cdots [X, Y_i], \cdots, Y_j \cdots) - f(\cdots, Y_i, \cdots, [X, Y_j], \cdots) \\
&= -f(\cdots [X, Y_i], \cdots, Y_j \cdots) - f(\cdots, Y_i, \cdots, [X, Y_j] \cdots) \\
&= -f(\cdots, Y_i + [X, Y_i], \cdots, Y_j + [X, Y_j], \cdots) \\
&\quad + f(\cdots, Y_i, \cdots, Y_j, \cdots) + f(\cdots, [X, Y_i], \cdots, [X, Y_j] \cdots) \\
&= 0 \ /\!/
\end{aligned}$$

N : $\mathfrak{g}$-module, $p, q \geq 0$

$$\cup : C^p(\check{\mathfrak{k}}; M) \times C^q(\check{\mathfrak{k}}; N) \to C^{p+q}(\check{\mathfrak{k}}; M \otimes N)$$
$$(f, g) \longmapsto f \cup g$$

$Y_i \in \check{\mathfrak{k}}$

$$(f \cup g)(Y_1 \cdots Y_{p+q}) \stackrel{\text{def}}{=} \frac{1}{p!\,q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (\mathrm{sign}\,\sigma) f(Y_{\sigma(1)}, \cdots, Y_{\sigma(p)}) \otimes g(Y_{\sigma(p+1)}, \cdots, Y_{\sigma(p+q)})$$

<u>alternating</u> $\exists i < \exists j$ $Y_i = Y_j$ ならば

$$\begin{aligned}
(f \cup g)(Y_1, \cdots, Y_{p+q}) &= (f \cup g)(\cdots, Y_j, \cdots, Y_i, \cdots) \\
&= (\mathrm{sgn}(ij))(f \cup g)(\cdots, Y_i, \cdots, Y_j \cdots) \\
&= -(f \cup g)(Y_1, \cdots, Y_{p+q}).
\end{aligned}$$

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$\cup : C^p(\mathfrak{k}:M) \otimes C^q(\mathfrak{k}:N) \to C^{p+q}(\mathfrak{k}:M\otimes N)$ $\mathfrak{g}$-homomorphism

i.e. $\cup \quad {}^\forall f \quad {}^\forall g, \quad {}^\forall X \in \mathfrak{g}$

$$X(f\cup g) = (Xf)\cup g + f\cup (Xg)$$ (次回再証明)

$\because$) $Y_i \in \mathfrak{k}$, $Y_{k,i} := \begin{cases} Y_i & \text{if } i \neq k \\ [X, Y_k] & \text{if } i = k \end{cases}$

(略)

$$\begin{aligned}
&p!q!(X(f\cup g))(Y_1,\dots,Y_{p+q}) \\
&= X\left(\sum_\sigma (\operatorname{sign}\sigma) f(Y_{\sigma(1)},\dots,Y_{\sigma(p)})\otimes g(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)})\right) \\
&\quad - \sum_{k=1}^{p+q}\sum (\operatorname{sign}\sigma) f(Y_{k,\sigma(1)},\dots,Y_{k,\sigma(p)})\otimes g(Y_{k,\sigma(p+1)},\dots,Y_{k,\sigma(p+q)}) \\
&= \sum_\sigma (\operatorname{sign}\sigma) X(f(Y_{\sigma(1)},\dots,Y_{\sigma(p)}))\otimes g(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)}) \\
&\quad + \sum_\sigma (\operatorname{sign}\sigma) f(Y_{\sigma(1)},\dots,Y_{\sigma(p)})\otimes X(g(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)})) \\
&\quad - \sum_\sigma \sum_{k\in\{\sigma(1),\dots,\sigma(p)\}} (\operatorname{sign}\sigma) f(Y_{k,\sigma(1)},\dots,Y_{k,\sigma(p)})\otimes g(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)}) \\
&\quad - \sum_\sigma \sum_{k\in\{\sigma(p+1),\dots,\sigma(p+q)\}} (\operatorname{sign}\sigma) f(Y_{\sigma(1)},\dots,Y_{\sigma(p)})\otimes g(Y_{k,\sigma(p+1)},\dots,Y_{k,\sigma(p+q)}) \\
&= \sum_\sigma (\operatorname{sign}\sigma)(Xf)(Y_{\sigma(1)},\dots,Y_{\sigma(p)})\otimes g(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)}) \\
&\quad + \sum_\sigma (\operatorname{sign}\sigma) f(Y_{\sigma(1)},\dots,Y_{\sigma(p)})\otimes (Xg)(Y_{\sigma(p+1)},\dots,Y_{\sigma(p+q)}) \\
&= p!q!((Xf)\cup g + f\cup(Xg))(Y_1,\dots,Y_{p+q}) \quad //
\end{aligned}$$

$T: M\otimes N \to N\otimes M$ $m\otimes n \mapsto n\otimes m$ switch map

$$T\circ(f\cup g) = (-1)^{pq}(g\cup f) \in C^{p+q}(\mathfrak{k}:N\otimes M)$$

$\left(\because \operatorname{sign}\begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & p+q \\ p+1 & \cdots & p+q & 1 & \cdots & p\end{pmatrix} = (-1)^{pq}\right)$

P: $\mathfrak{g}$-module, $r \geq 0$, $h \in C^r(\mathfrak{k}:P)$

$$(f\cup g)\cup h = f\cup(g\cup h) \in C^{p+q+r}(\mathfrak{k}:M\otimes N\otimes P)$$

$\because$) 両辺とも $Y_1,\dots,Y_{p+q+r}$ を代入すると

$$\frac{1}{p!q!r!}\sum_{\sigma\in\mathfrak{S}_{p+q+r}} (\operatorname{sign}\sigma) f(Y_{\sigma(1)}\cdots Y_{\sigma(p)})\otimes g(Y_{\sigma(p+1)}\cdots Y_{\sigma(p+q)})\otimes h(Y_{\sigma(p+q+1)}\cdots Y_{\sigma(p+q+r)})$$

になる //

注 shuffle を使えば char $\mathbb{K} \neq 0$ でも $\cup$ が定義できて結合的かつ次数可換