ON CANONICAL BUNDLE FORMULAE AND SUBADJUNCTIONS

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ABSTRACT. We consider a canonical bundle formula for generically finite proper surjective morphisms and obtain subadjunction formulae for minimal log canonical centers of log canonical pairs. We also treat related topics and applications.

CONTENTS

1. INTRODUCTION

The following lemma is one of the main results of this paper, which is missing in the literature. It is a canonical bundle formula for generically finite proper surjective morphisms.

Lemma 1.1 (Main Lemma). *Let X and Y be normal varieties and let $f : X \to Y$ be a generically finite proper surjective morphism. Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Suppose that there exists an effective $\mathbb{K}$-divisor Δ on X such that (X, Δ) is log canonical and that $K_X + \Delta \sim_{\mathbb{K},f} 0$. Then there exists an effective $\mathbb{K}$-divisor Γ on Y such that (Y, Γ) is log canonical and that*

$$K_X + \Delta \sim_{\mathbb{K}} f^*(K_Y + \Gamma).$$

Moreover, if (X, Δ) is kawamata log terminal, then we can choose Γ such that (Y, Γ) is kawamata log terminal.

As an application of Lemma 1.1, we prove a subadjunction formula for minimal lc centers. It is a generalization of Kawamata's subadjunction formula (cf. [K3, Theorem 1]). For a local version, see Theorem 7.2 below.

Theorem 1.2 (Subadjunction formula for minimal lc centers)**.** *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_\mathbb{K} K_W + D_W$$

and that the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

We summarize the contents of this paper. Section 2 is devoted to the proof of Lemma 1.1. In Section 3, we discuss Ambro's canonical bundle formula for projective kawamata log terminal pairs with a generalization for $\mathbb{R}$-divisors (cf. Theorem 3.1). It is one of the key ingredients of the proof of Theorem 1.2. Although Theorem 3.1 is sufficient for applications in subsequent sections, we treat slight generalizations of Ambro's canonical bundle formula for projective log canonical pairs. In Section 4, we prove a subadjunction formula for minimal log canonical centers (cf. Theorem 1.2), which is a generalization of Kawamata's subadjunction formula (cf. [K3, Theorem 1]). In Section 5, we treat images of log Fano varieties by generically finite surjective morphisms as an application of Lemma 1.1. Theorem 5.1 is an answer to the question raised by Professor Karl Schwede (cf. [SS, Remark 6.5]). In Section 6, we give a quick proof of the non-vanishing theorem for log canonical pairs as an application of Theorem 1.2, which is the main theorem of the first author's paper: [F4]. In Section 7, we prove a local version of our subadjunction formula for minimal log canonical centers (cf. Theorem 7.2). It is useful for local studies of singularities of pairs. This local version does not directly follow from the global version: Theorem 1.2. It is because we do not know how to compactify log canonical pairs.

We close this introduction with the following notation. We also use the standard notation in [KM].

Notation. Let $\mathbb{K}$ be the real number field $\mathbb{R}$ or the rational number field $\mathbb{Q}$.

Let X be a normal variety and let B be an effective $\mathbb{K}$-divisor such that $K_X + B$ is $\mathbb{K}$-Cartier. Then we can define the *discrepancy* $a(E, X, B) \in \mathbb{K}$ for every prime divisor E *over* X. If $a(E, X, B) \geq -1$ (resp. > -1) for every E, then (X, B) is called *log canonical* (resp. *kawamata log terminal*). We sometimes abbreviate log canonical (resp. kawamata log terminal) to *lc* (resp. *klt*).

Assume that (X, B) is log canonical. If E is a prime divisor over X such that $a(E, X, B) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X, B), where $c_X(E)$ is the closure of the image of E on X. For the basic properties of log canonical centers, see [F4, Theorem 2.4] or [F5, Section 9].

We note that $\sim_{\mathbb{K}}$ denotes $\mathbb{K}$-*linear equivalence* of $\mathbb{K}$-divisors. Let $f : X \to Y$ be a morphism between normal varieties and let D be a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on X. Then D is $\mathbb{K}$-linearly f-trivial, denoted by $D \sim_{\mathbb{K}, f} 0$, if and only if there is a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor B on Y such that $D \sim_{\mathbb{K}} f^*B$.

The base locus of the linear system Λ is denoted by BsΛ.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

2. Main lemma

In this section, we prove Lemma 1.1.

Proof of Lemma 1.1. Let

$$f : X \xrightarrow{\ g\ } Z \xrightarrow{\ h\ } Y$$

be the Stein factorization. By replacing (X, Δ) with $(Z, g_*\Delta)$, we can assume that $f : X \to Y$ is finite. Let D be a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on Y such that $K_X + \Delta \sim_{\mathbb{K}} f^*D$. We consider the following commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{\ \nu\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\ \mu\ } & Y,
\end{array}
$$

where

(i) μ is a resolution of singularities of Y,

(ii) there exists an open set $U \subseteq Y$ such that μ is isomorphic over U and f is étale over U. Moreover, $\mu^{-1}(Y - U)$ has a simple normal crossing support and $Y - U$ contains $\mathrm{Supp} f_* \Delta$, and

(iii) X' is the normalization of the irreducible component of $X \times_Y Y'$ which dominates Y'. In particular, f' is finite.

Let $\Omega = \sum_i \delta_i D_i$ be a $\mathbb{K}$-divisor on X' such that

$$K_{X'} + \Omega = \nu^*(K_X + \Delta).$$

We consider the ramification formula:

$$K_{X'} = f'^* K_{Y'} + R,$$

where $R = \sum_i (r_i - 1) D_i$ is an effective $\mathbb{Z}$-divisor such that r_i is the ramification index of D_i for every i. Note that it suffices to show the above formula outside codimension two closed subsets of X'. Then it holds that

$$(\mu \circ f')^* D \sim_{\mathbb{K}} f'^* K_{Y'} + R + \Omega.$$

By pushing forward the above formula by f', we see

$$\deg f' \cdot \mu^* D \sim_{\mathbb{K}} \deg f' \cdot K_{Y'} + f'_*(R + \Omega).$$

We set

$$\Gamma := \frac{1}{\deg f'} \mu_* f'_*(R + \Omega)$$

on Y. Then Γ is effective since

$$\mu_* f'_*(R + \Omega) = f_* \nu_*(R + \Omega) = f_*(\nu_* R + \Delta).$$

Let $Y' \setminus \mu^{-1} U = \bigcup_j E_j$ be the irreducible decomposition, where $\sum_j E_j$ is a simple normal crossing divisor. We set

$$I_j := \{i | f'(D_i) = E_j\}.$$

The coefficient of E_j in $f'_*(R + \Omega)$ is

$$\frac{\sum_{i \in I_j} (r_i + \delta_i - 1) \deg(f'|_{D_i})}{\deg f'}.$$

Since $\delta_i \leq 1$, it holds that

$$\sum_{i \in I_j} (r_i + \delta_i - 1) \deg(f'|_{D_i}) \leq \sum_{i \in I_j} r_i \deg(f'|_{D_i}) = \deg f'.$$

Thus (Y, Γ) is log canonical since $K_{Y'} + f'_*(R + \Omega) = \mu^*(K_Y + \Gamma)$. Moreover, if (X, Δ) is kawamata log terminal, then $\delta_i < 1$. Hence (Y, Γ) is kawamata log terminal. $\qquad\square$

3. Ambro's canonical bundle formula

Theorem 3.1 is Ambro's canonical bundle formula for projective klt pairs (cf. [A2, Theorem 4.1]) with a generalization for $\mathbb{R}$-divisors. We need it for the proof of our subadjunction formula: Theorem 1.2.

Theorem 3.1 (Ambro's canonical bundle formula for projective klt pairs). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a projective kawamata log terminal pair and let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y with connected fibers. Assume that*

$$K_X + B \sim_{\mathbb{K}, f} 0.$$

Then there exists an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is klt and

$$K_X + B \ \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Proof. If $\mathbb{K} = \mathbb{Q}$, then the statement is nothing but [A2, Theorem 4.1]. From now on, we assume that $\mathbb{K} = \mathbb{R}$. Let $\sum_i B_i$ be the irreducible decomposition of $\mathrm{Supp} B$. We put $V = \bigoplus_i \mathbb{R} B_i$. Then it is well known that

$$\mathcal{L} = \{\Delta \in V \,|\, (X, \Delta) \text{ is log canonical}\}$$

is a rational polytope in V. We can also check that

$$\mathcal{N} = \{\Delta \in \mathcal{L} \,|\, K_X + \Delta \text{ is } f\text{-nef}\}$$

is a rational polytope and $B \in \mathcal{N}$. We note that $\mathcal{N}$ is known as Shokurov's polytope. Therefore, we can write

$$K_X + B = \sum_{i=1}^{k} r_i(K_X + \Delta_i)$$

such that

(i) $\Delta_i \in \mathcal{N}$ is an effective $\mathbb{Q}$-divisor on X for every i,
(ii) (X, Δ_i) is klt for every i, and
(iii) $0 < r_i < 1$, $r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

Since $K_X + B$ is numerically f-trivial and $K_X + \Delta_i$ is f-nef for every i, $K_X + \Delta_i$ is numerically f-trivial for every i. Thus,

$$\kappa(X_\eta, (K_X + \Delta_i)_\eta) = \nu(X_\eta, (K_X + \Delta_i)_\eta) = 0$$

for every i, where η is the generic point of Y, by Nakayama (cf. [N, Chapter V 2.9. Corollary]). See also [A2, Theorem 4.2]. Therefore, $K_X + \Delta_i \sim_{\mathbb{Q}, f} 0$ for every i by [F3, Theorem 1.1]. By the case when

$\mathbb{K} = \mathbb{Q}$, we can find an effective $\mathbb{Q}$-divisor Θ_i on Y such that (Y, Θ_i) is klt and

$$K_X + \Delta_i \sim_{\mathbb{Q}} f^*(K_Y + \Theta_i)$$

for every i. By putting $B_Y = \sum_{i=1}^k r_i \Theta_i$, we obtain

$$K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y),$$

and (Y, B_Y) is klt. $\qquad\square$

Corollary 3.2 is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a log canonical pair and let $f : X \to Y$ be a projective surjective morphism between normal projective varieties. Assume that*

$$K_X + B \sim_{\mathbb{K},f} 0$$

and that every lc center of (X, B) is dominant onto Y. Then we can find an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is kawamata log terminal and that

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Proof. By taking a dlt blow-up (cf. [F5, Theorem 10.4]), we can assume that (X, B) is dlt. By replacing (X, B) with its minimal lc center and taking the Stein factorization, we can assume that (X, B) is klt and that f has connected fibers (cf. Lemma 1.1). Therefore, we can take a desired B_Y by Theorem 3.1. $\qquad\square$

From now on, we treat Ambro's canonical bundle formula for projective log canonical pairs. We note that Theorem 3.1 is sufficient for applications in subsequent sections.

3.3 (Observation). Let (X, B) be a log canonical pair and let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that $K_X + B \sim_{\mathbb{Q},f} 0$ and that (X, B) is kawamata log terminal over the generic point of Y. We can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + M_Y + \Delta_Y)$$

where M_Y is the *moduli $\mathbb{Q}$-divisor* and Δ_Y is the *discriminant $\mathbb{Q}$-divisor*. For details, see, for example, [A1]. It is conjectured that we can construct a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y, \end{array}$$

with the following properties.

   (i) ν and μ are projective birational.
  (ii) X' is normal and $K_{X'} + B_{X'} = \nu^*(K_X + B)$.
 (iii) $K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + M_{Y'} + \Delta_{Y'})$ such that Y' is smooth, the moduli $\mathbb{Q}$-divisor $M_{Y'}$ is semi-ample, and the discriminant $\mathbb{Q}$-divisor $\Delta_{Y'}$ has a simple normal crossing support.

In the above properties, the non-trivial part is the semi-ampleness of $M_{Y'}$. We know that we can construct desired commutative diagrams of $f' : X' \to Y'$ and $f : X \to Y$ when

   (1) $\dim X - \dim Y = 1$ (cf. [K2, Theorem 5] and so on),
   (2) $\dim Y = 1$ (cf. [A1, Theorem 0.1] and [A2, Theorem 3.3]),
   (3) general fibers of f are $K3$ surfaces, Abelian varieties, or smooth surfaces with $\kappa = 0$ (cf. [F2, Theorem 1.2, Theorem 6.3]),

and so on. We take a general member $D \in |mM_{Y'}|$ of the free linear system $|mM_{Y'}|$ where m is a sufficiently large and divisible integer. We put

$$K_Y + B_Y = \mu_*(K_{Y'} + \frac{1}{m}D + \Delta_{Y'}).$$

Then it is easy to see that

$$\mu^*(K_Y + B_Y) = K_{Y'} + \frac{1}{m}D + \Delta_{Y'},$$

(Y, B_Y) is log canonical, and

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

By the above observation, we have Ambro's canonical bundle formula for projective log canonical pairs under some special assumptions.

Theorem 3.4. *Let (X, B) be a projective log canonical pair and let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y such that $K_X + B \sim_{\mathbb{Q}, f} 0$. Assume that $\dim Y \leq 1$ or $\dim X - \dim Y \leq 1$. Then there exists an effective $\mathbb{Q}$-divisor B_Y on Y such that (Y, B_Y) is log canonical and*

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

Proof. By taking a dlt blow-up (cf. [F5, Theorem 10.4]), we can assume that (X, B) is dlt. If necessary, by replacing (X, B) with a suitable lc center of (X, B) and by taking the Stein factorization (cf. Lemma 1.1), we can assume that $f : X \to Y$ has connected fibers and that (X, B) is kawamata log terminal over the generic point of Y. We note that we can assume that $\dim Y = 1$ or $\dim X - \dim Y = 1$. By the arguments in 3.3, we can find an effective $\mathbb{Q}$-divisor B_Y on Y such that (Y, B_Y) is log canonical and that $K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y)$. $\square$

4. Subadjunction for minimal log canonical centers

The following theorem is a generalization of Kawamata's subadjunction formula (cf. [K3, Theorem 1]). Theorem 4.1 is new even for threefolds. It is an answer to Kawamata's question (cf. [K1, Question 1.8]).

Theorem 4.1 (Subadjunction formula for minimal lc centers). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and that the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

Remark 4.2. In [K3, Theorem 1], Kawamata proved

$$(K_X + D + \varepsilon H)|_W \sim_{\mathbb{Q}} K_W + D_W,$$

where H is an ample Cartier divisor on X and ε is a positive rational number, under the extra assumption that D is a $\mathbb{Q}$-divisor and there exists an effective $\mathbb{Q}$-divisor D^o such that $D^o < D$ and that (X, D^o) is kawamata log terminal. Therefore, Kawamata's theorem claims nothing when $D = 0$.

Proof of Theorem 4.1. By taking a dlt blow-up (cf. [F5, Theorem 10.4]), we can take a projective birational morphism $f : Y \to X$ from a normal projective variety Y with the following properties.

(i) $K_Y + D_Y = f^*(K_X + D)$.
(ii) (Y, D_Y) is a $\mathbb{Q}$-factorial dlt pair.

We can take a minimal lc center Z of (Y, D_Y) such that $f(Z) = W$. We note that $K_Z + D_Z = (K_Y + D_Y)|_Z$ is klt since Z is a minimal lc center of the dlt pair (Y, D_Y). Let

$$f : Z \xrightarrow{\ g\ } V \xrightarrow{\ h\ } W$$

be the Stein factorization of $f : Z \to W$. By the construction, we can write

$$K_Z + D_Z \sim_{\mathbb{K}} f^*A$$

where A is a $\mathbb{K}$-divisor on W such that $A \sim_{\mathbb{K}} (K_X + D)|_W$. We note that W is normal (cf. [F4, Theorem 2.4 (4)]). Since (Z, D_Z) is klt, we can take an effective $\mathbb{K}$-divisor D_V on V such that

$$K_V + D_V \sim_{\mathbb{K}} h^*A$$

and that (V, D_V) is klt by Theorem 3.1. By Lemma 1.1, we can find an effective $\mathbb{K}$-divisor D_W on W such that

$$K_W + D_W \sim_\mathbb{K} A \sim_\mathbb{K} (K_X + D)|_W$$

and that (W, D_W) is klt. $\qquad\square$

5. Log Fano varieties

In this section, we give an easy application of Lemma 1.1. Theorem 5.1 is an answer to the question raised by Karl Schwede (cf. [SS, Remark 6.5]). For related topics, see [FG, Section 3].

Theorem 5.1. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample. Let $f : X \to Y$ be a generically finite surjective morphism to a normal projective variety Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

Proof. Without loss of generality, we can assume that Δ is a $\mathbb{Q}$-divisor by perturbing the coefficients of Δ. Let H be a general very ample Cartier divisor on Y and let ε be a sufficiently small positive rational number. Then $K_X + \Delta + \varepsilon f^* H$ is anti-ample and $(X, \Delta + \varepsilon f^* H)$ is klt. We can take an effective $\mathbb{Q}$-divisor Θ on X such that $m\Theta$ is a general member of the free linear system $|-m(K_X + \Delta + \varepsilon f^* H)|$ where m is a sufficiently large and divisible integer. Then

$$K_X + \Delta + \varepsilon f^* H + \Theta \sim_\mathbb{Q} 0.$$

Let δ be a positive rational number such that $0 < \delta < \varepsilon$. Then

$$K_X + \Delta + (\varepsilon - \delta) f^* H + \Theta \sim_\mathbb{Q} f^*(-\delta H).$$

By Lemma 1.1, we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that

$$K_Y + \Delta_Y \sim_\mathbb{Q} -\delta H$$

and that (Y, Δ_Y) is klt. We note that

$$-(K_Y + \Delta_Y) \sim_\mathbb{Q} \delta H$$

is ample. $\qquad\square$

By combining Theorem 5.1 with [FG, Theorem 3.1], we can easily obtain the following corollary.

Corollary 5.2. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample. Let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

6. Non-vanishing theorem for log canonical pairs

The following theorem is the main result of [F4]. It is almost equivalent to the base point free theorem for log canonical pairs. For details, see [F4].

Theorem 6.1 (Non-vanishing theorem). *Let X be a normal projective variety and let B be an effective $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a nef Cartier divisor on X. Assume that $aL - (K_X + B)$ is ample for some $a > 0$. Then the base locus of the linear system $|mL|$ contains no lc centers of (X, B) for every $m \gg 0$, that is, there is a positive integer m_0 such that $\mathrm{Bs}|mL|$ contains no lc centers of (X, B) for every $m \geq m_0$.*

Here, we give a quick proof of Theorem 6.1 by using Theorem 4.1.

Proof. Let W be any minimal lc center of the pair (X, B). It is sufficient to prove that W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$. By Theorem 4.1, we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $K_W + D_W \sim_{\mathbb{Q}} (K_X + B)|_W$. Therefore, $aL|_W - (K_W + B_W) \sim_{\mathbb{Q}} (aL - (K_X + B))|_W$ is ample. By the Kawamata–Shokurov base point free theorem, $|mL|_W|$ is free for $m \gg 0$. By [F4, Theorem 2.2],

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(W, \mathcal{O}_W(mL))$$

is surjective for $m \geq a$. Therefore, W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$. $\qquad\square$

Remark 6.2. The above proof of Theorem 6.1 is shorter than the original proof in [F4]. However, the proof of Theorem 4.1 depends on very deep results such as existence of dlt blow-ups. The proof of Theorem 6.1 in [F4] only depends on various well-prepared vanishing theorems and standard techniques.

7. Subadjunction formula: local version

In this section, we give a local version of our subadjunction formula for minimal log canonical centers. Theorem 7.1 is a local version of Ambro's canonical bundle formula for kawamata log terminal pairs: Theorem 3.1. It is essentially [F1, Theorem 1.2].

Theorem 7.1. *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a kawamata log terminal pair and let $f : X \to Y$ be a proper surjective morphism onto a normal affine variety Y with connected fibers. Assume that*

$$K_X + B \sim_{\mathbb{K}, f} 0.$$

Then there exists an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is klt and

$$K_X + B \ \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Sketch of the proof. First, we assume that $\mathbb{K} = \mathbb{Q}$. In this case, the proof of [F1, Theorem 1.2] works with some minor modifications. We note that M in the proof of [F1, Theorem 1.2] is μ-nef. We also note that we can assume $H = 0$ in [F1, Theorem 1.2] since Y is affine. Next, we assume that $\mathbb{K} = \mathbb{R}$. In this case, the reduction argument in the proof of Theorem 3.1 can be applied. So, we obtain the desired formula. $\square$

By Theorem 7.1, we can obtain a local version of Theorem 4.1. The proof of Theorem 4.1 works without any modifications.

Theorem 7.2 (Subadjunction formula for minimal lc centers: local version). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal affine variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and that the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

Theorem 7.2 does not directly follow from Theorem 4.1. It is because we do not know how to compactify log canonical pairs.

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