LOG PLURICANONICAL REPRESENTATIONS AND ABUNDANCE CONJECTURE

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Abstract. We prove the finiteness of log pluricanonical representations for projective log canonical pairs with semi-ample log canonical divisor. As a corollary, we obtain that the log canonical divisor of a projective semi log canonical pair is semi-ample if and only if so is the log canonical divisor of its normalization. We also treat many other applications.

Contents

1. Introduction

The following theorem is one of the main results of this paper (cf. Theorem 3.15). It is a solution of the conjecture raised in [F1] (see [F1, Conjecture 3.2]). For the definition of the *log pluricanonical representation* ρ_m, see Definitions 2.11 and 2.14 below.

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Theorem 1.1 (cf. [F1, Section 3], [G2, Theorem B]). *Let (X, Δ) be a projective log canonical pair. Suppose that $m(K_X + \Delta)$ is Cartier and that $K_X + \Delta$ is semi-ample. Then $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group.*

In Theorem 1.1, we do not have to assume that $K_X+\Delta$ is semi-ample when $K_X + \Delta$ is big (cf. Theorem 3.11). As a corollary of this fact, we obtain the finiteness of $\mathrm{Bir}(X, \Delta)$ when $K_X + \Delta$ is big. It is an answer to the question raised by Cacciola and Tasin.

Theorem 1.2 (cf. Corollary 3.13). *Let (X, Δ) be a projective log canonical pair such that $K_X + \Delta$ is big. Then* $\mathrm{Bir}(X, \Delta)$ *is a finite group.*

In the framework of [F1], Theorem 1.1 will play important roles in the study of Conjecture 1.3 (see [Ft], [AFKM], [Ka3], [KMM], [F1], [F8], [G2], and so on).

Conjecture 1.3 ((Log) abundance conjecture). *Let (X, Δ) be a projective semi log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Suppose that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semi-ample.*

Theorem 1.1 was settled for surfaces in [F1, Section 3] and for the case where $K_X + \Delta \sim_{\mathbb{Q}} 0$ by [G2, Theorem B]. In this paper, to carry out the proof of Theorem 1.1, we introduce the notion of *$\widetilde{B}$-birational maps* and *$\widetilde{B}$-birational representations* for sub kawamata log terminal pairs, which is new and is indispensable for generalizing the arguments in [F1, Section 3] for higher dimensional log canonical pairs. For the details, see Section 3.

By Theorem 1.1, we obtain a key result.

Theorem 1.4 (cf. Proposition 4.3). *Let (X, Δ) be a projective semi log canonical pair. Let $\nu : X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ is semi-ample. Then $K_X + \Delta$ is semi-ample.*

By Theorem 1.4, Conjecture 1.3 is reduced to the problem for log canonical pairs. After we circulated this paper, Hacon and Xu proved a relative version of Theorem 1.4 by using Kollár's gluing theory (cf. [HX2]). For the details, see Subsection 4.1 below.

Let X be a smooth projective n-fold. By our experience on the low-dimensional abundance conjecture, we think that we need the abundance theorem for projective semi log canonical pairs in dimension $\leq n-1$ in order to prove the abundance conjecture for X. Therefore, Theorem 1.4 seems to be an important step for the inductive approach to the abundance conjecture. The general strategy for proving the abundance conjecture is explained in the introduction of [F1]. Theorem 1.4 is a complete solution of Step (v) in [F1, 0. Introduction].

As applications of Theorem 1.4 and [F5, Theorem 1.1], we have the following useful theorems.

Theorem 1.5 (cf. Theorem 4.2)**.** *Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Assume that $K_X + \Delta$ is nef and log abundant. Then $K_X + \Delta$ is semi-ample.*

It is a generalization of the well-known theorem for kawamata log terminal pairs (see, for example, [F4, Corollary 2.5]). Theorem 1.6 may be easier to understand than Theorem 1.5.

Theorem 1.6 (cf. Theorem 4.6)**.** *Let (X, Δ) be an n-dimensional projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Assume that the abundance conjecture holds for projective divisorial log terminal pairs in dimension $\leq n - 1$. Then $K_X + \Delta$ is semi-ample if and only if $K_X + \Delta$ is nef and abundant.*

We have many other applications. In this introduction, we explain only one of them. It is a generalization of [Fk2, Theorem 0.1] and [CKP, Corollary 3]. It also contains Theorem 1.5. For a further generalization, see Remark 4.17.

Theorem 1.7 (cf. Theorem 4.16)**.** *Let (X, Δ) be a projective log canonical pair and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that D is nef and log abundant with respect to (X, Δ). Assume that $K_X + \Delta \equiv D$. Then $K_X + \Delta$ is semi-ample.*

The reader can find many applications and generalizations in Section 4. In Section 5, we will discuss the relationship among the various conjectures in the minimal model program. Let us recall the following two important conjectures.

Conjecture 1.8 (Non-vanishing conjecture)**.** *Let (X, Δ) be a projective log canonical pair such that Δ is an $\mathbb{R}$-divisor. Assume that $K_X+\Delta$ is pseudo-effective. Then there exists an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

By [DHP, Section 8] and [G4], Conjecture 1.8 can be reduced to the case when X is a smooth projective variety and $\Delta = 0$ by using the global ACC conjecture and the ACC for log canonical thresholds (see [DHP, Conjecture 8.2 and Conjecture 8.4]).

Conjecture 1.9 (Extension conjecture for divisorial log terminal pairs (cf. [DHP, Conjecture 1.3]))**.** *Let (X, Δ) be an n-dimensional projective divisorial log terminal pair such that Δ is a $\mathbb{Q}$-divisor, $\llcorner \Delta \lrcorner = S$, $K_X + \Delta$ is nef, and $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$ where $S \subset \operatorname{Supp} D$. Then*

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$$

is surjective for all sufficiently divisible integers $m \geq 2$*.*

Note that Conjecture 1.9 holds true when $K_X + \Delta$ is semi-ample (cf. Proposition 5.11). It is an easy consequence of a cohomology injectivity theorem. We also note that Conjecture 1.9 is true if (X, Δ) is purely log terminal (cf. [DHP, Corollary 1.8]). The following theorem is one of the main results of Section 5. It is a generalization of [DHP, Theorem 1.4].

Theorem 1.10 (cf. [DHP, Theorem 1.4])**.** *Assume that* Conjecture 1.8 *and* Conjecture 1.9 *hold true in dimension* $\leq n$*. Let* (X, Δ) *be an* n*-dimensional projective divisorial log terminal pair such that* $K_X + \Delta$ *is pseudo-effective. Then* (X, Δ) *has a good minimal model. In particular, if* $K_X + \Delta$ *is nef, then* $K_X + \Delta$ *is semi-ample.*

By our inductive treatment of Theorem 1.10, Theorem 1.4 plays a crucial role. Therefore, Theorem 1.1 is indispensable for Theorem 1.10.

We summarize the contents of this paper. In Section 2, we collects some basic notations and results. Section 3 is the main part of this paper. In this section, we prove Theorem 1.1. We divide the proof into the three steps: sub kawamata log terminal pairs in 3.1, log canonical pairs with big log canonical divisor in 3.2, and log canonical pairs with semi-ample log canonical divisor in 3.3. Section 4 contains various applications of Theorem 1.1. They are related to the abundance conjecture: Conjecture 1.3. In Subsection 4.2, we generalize the main theorem in [Fk2] (cf. [CKP, Corollary 3]), the second author's result in [G1], and so on. In Section 5, we discuss the relationship among the various conjectures in the minimal model program.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will freely use the standard notations in [KM].

2. Preliminaries

In this section, we collects some basic notations and results.

2.1 (Convention)**.** Let D be a Weil divisor on a normal variety X. We sometimes simply write $H^0(X, D)$ to denote $H^0(X, \mathcal{O}_X(D))$.

2.2 ($\mathbb{Q}$-divisors)**.** For a $\mathbb{Q}$-divisor $D = \sum_{j=1}^{r} d_j D_j$ on a normal variety X such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$, where for every rational number x, $\llcorner x \lrcorner$ is the integer defined by $x - 1 < \llcorner x \lrcorner \leq x$. We put

$$D^{=1} = \sum_{d_j = 1} D_j.$$

We note that $\sim_{\mathbb{Z}}$ ($\sim$, for short) denotes the *linear equivalence* of divisors. We also note that $\sim_{\mathbb{Q}}$ (resp. $\equiv$) denotes the $\mathbb{Q}$-*linear equivalence* (resp. *numerical equivalence*) of $\mathbb{Q}$-divisors. Let $f : X \to Y$ be a morphism and let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Then $D_1 \sim_{\mathbb{Q}, Y} D_2$ means that there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor B on Y such that $D_1 \sim_{\mathbb{Q}} D_2 + f^*B$. We can also treat $\mathbb{R}$-divisors similarly.

2.3 (Log resolution)**.** Let X be a normal variety and let D be an $\mathbb{R}$-divisor on X. A *log resolution* $f : Y \to X$ means that

(i) f is a proper birational morphism,
(ii) Y is smooth, and
(iii) $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1} D$ is a simple normal crossing divisor on Y, where $\mathrm{Exc}(f)$ is the *exceptional locus* of f.

We recall the notion of singularities of pairs.

Definition 2.4 (Singularities of pairs)**.** Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\varphi : Y \to X$ be a log resolution of (X, Δ). We set

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where E_i is a prime divisor on Y for every i. The pair (X, Δ) is called

(a) *sub kawamata log terminal* (*subklt*, for short) if $a_i > -1$ for all i, or
(b) *sub log canonical* (*sublc*, for short) if $a_i \geq -1$ for all i.

If Δ is effective and (X, Δ) is subklt (resp. sublc), then we simply call it *klt* (resp. *lc*).

Let (X, Δ) be an lc pair. If there is a log resolution $\varphi : Y \to X$ of (X, Δ) such that $\mathrm{Exc}(\varphi)$ is a divisor and that $a_i > -1$ for every φ-exceptional divisor E_i, then the pair (X, Δ) is called *divisorial log terminal* (*dlt*, for short).

Let us recall *semi log canonical pairs* and *semi divisorial log terminal pairs* (cf. [F1, Definition 1.1]). For the details of these pairs, see [F1, Section 1]. Note that the notion of semi divisorial log terminal pairs in [Ko3, Definition 5.17] is different from ours.

Definition 2.5 (Slc and sdlt)**.** Let X be a reduced S_2 scheme. We assume that it is pure n-dimensional and normal crossing in codimension one. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We assume that $\Delta = \sum_i a_i \Delta_i$ where $a_i \in \mathbb{Q}$ and Δ_i is an irreducible codimension one closed subvariety of X such that $\mathcal{O}_{X,\Delta_i}$ is a DVR for every i. Let $X = \cup_i X_i$ be the irreducible decomposition and let $\nu : X^\nu := \amalg_i X_i^\nu \to X = \cup_i X_i$ be the normalization. A $\mathbb{Q}$-divisor Θ on X^ν is defined by $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ and a $\mathbb{Q}$-divisor Θ_i on X_i^ν by $\Theta_i := \Theta|_{X_i^\nu}$. We say that (X, Δ) is a *semi log canonical n-fold* (an *slc n-fold*, for short) if (X^ν, Θ) is lc. We say that (X, Δ) is a *semi divisorial log terminal n-fold* (an *sdlt n-fold*, for short) if X_i is normal, that is, X_i^ν is isomorphic to X_i, and (X^ν, Θ) is dlt.

We recall a very important example of slc pairs.

Example 2.6. Let (X, Δ) be a $\mathbb{Q}$-factorial lc pair such that Δ is a $\mathbb{Q}$-divisor. We put $S = \llcorner \Delta \lrcorner$. Assume that $(X, \Delta - \varepsilon S)$ is klt for some $0 < \varepsilon \ll 1$. Then (S, Δ_S) is slc where $K_S + \Delta_S = (K_X + \Delta)|_S$.

Remark 2.7. Let (X, Δ) be a dlt pair such that Δ is a $\mathbb{Q}$-divisor. We put $S = \llcorner \Delta \lrcorner$. Then it is well known that (S, Δ_S) is sdlt where $K_S + \Delta_S = (K_X + \Delta)|_S$.

The following theorem was originally proved by Christopher Hacon (cf. [F7, Theorem 10.4], [KK, Theorem 3.1]). For a simpler proof, see [F6, Section 4].

Theorem 2.8 (Dlt blow-up)**.** *Let X be a normal quasi-projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose that (X, Δ) is lc. Then there exists a projective birational morphism $\varphi : Y \to X$ from a normal quasi-projective variety Y with the following properties*:

(i) *Y is $\mathbb{Q}$-factorial,*

(ii) $a(E, X, \Delta) = -1$ *for every* φ*-exceptional divisor* E *on* Y*, and*
(iii) *for*

$$\Gamma = \varphi_*^{-1}\Delta + \sum_{E:\varphi\text{-}exceptional} E,$$

it holds that (Y, Γ) *is dlt and* $K_Y + \Gamma = \varphi^*(K_X + \Delta)$.

The above theorem is very useful for the study of log canonical singularities (cf. [F3], [F7], [F10], [G1], [G2], [KK], and [FG]). We will repeatedly use it in the subsequent sections.

2.9 (Log pluricanonical representations)**.** Nakamura–Ueno ([NU]) and Deligne proved the following theorem (see [U, Theorem 14.10]).

Theorem 2.10 (Finiteness of pluricanonical representations)**.** *Let* X *be a compact complex Moishezon manifold. Then the image of the group homomorphism*

$$\rho_m : \mathrm{Bim}(X) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, mK_X))$$

is finite, where $\mathrm{Bim}(X)$ *is the group of bimeromorphic maps from* X *to itself.*

For considering the logarithmic version of Theorem 2.10, we need the notion of *B-birational maps* and *B-pluricanonical representations.*

Definition 2.11 ([F1, Definition 3.1])**.** Let (X, Δ) (resp. (Y, Γ)) be a pair such that X (resp. Y) is a normal scheme with a $\mathbb{Q}$-divisor Δ (resp. Γ) such that $K_X + \Delta$ (resp. $K_Y + \Gamma$) is $\mathbb{Q}$-Cartier. We say that a proper birational map $f : (X, \Delta) \dashrightarrow (Y, \Gamma)$ is *B-birational* if there exists a common resolution

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

such that

$$\alpha^*(K_X + \Delta) = \beta^*(K_Y + \Gamma).$$

This means that it holds that $E = F$ when we put $K_W = \alpha^*(K_X + \Delta) + E$ and $K_W = \beta^*(K_Y + \Gamma) + F$.

Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then we define

$$f^*D := \alpha_*\beta^*D.$$

It is easy to see that f^*D is independent of the common resolution $\alpha : W \to X$ and $\beta : W \to Y$.

Finally, we put

$$\mathrm{Bir}(X, \Delta) = \{\sigma \,|\, \sigma : (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-birational}\}.$$

It is obvious that $\mathrm{Bir}(X, \Delta)$ has a natural group structure.

Remark 2.12. In Definition 2.11, let $\psi : X' \to X$ be a proper birational morphism from a normal scheme X' such that $K_{X'} + \Delta' = \psi^*(K_X + \Delta)$. Then we can easily check that $\mathrm{Bir}(X, \Delta) \simeq \mathrm{Bir}(X', \Delta')$ by $g \mapsto \psi^{-1} \circ g \circ \psi$ for $g \in \mathrm{Bir}(X, \Delta)$.

We give a basic example of B-birational maps.

Example 2.13 (Quadratic transformation)**.** Let $X = \mathbb{P}^2$ and let Δ be the union of three general lines on $\mathbb{P}^2$. Let $\alpha : W \to X$ be the blow-up at the three intersection points of Δ and let $\beta : W \to X$ be the blow-down of the strict transform of Δ on W. Then we obtain the *quadratic transformation* φ.

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\varphi"'] & & X \end{tikzcd}$$

For the details, see [H, Chapter V Example 4.2.3]. In this situation, it is easy to see that

$$\alpha^*(K_X + \Delta) = K_W + \Theta = \beta^*(K_X + \Delta).$$

Therefore, φ is a B-birational map of the pair (X, Δ).

Definition 2.14 ([F1, Definition 3.2])**.** Let X be a pure n-dimensional normal scheme and let Δ be a $\mathbb{Q}$-divisor, and let m be a nonnegative integer such that $m(K_X + \Delta)$ is Cartier. A B-birational map $\sigma \in \mathrm{Bir}(X, \Delta)$ defines a linear automorphism of $H^0(X, m(K_X + \Delta))$. Thus we get the group homomorphism

$$\rho_m : \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X + \Delta))).$$

The homomorphism ρ_m is called a *B-pluricanonical representation* or *log pluricanonical representation* for (X, Δ). We sometimes simply denote $\rho_m(g)$ by g^* for $g \in \mathrm{Bir}(X, \Delta)$ if there is no danger of confusion.

In Subsection 3.1, we will introduce and consider *$\widetilde{B}$-birational maps* and *$\widetilde{B}$-pluricanonical representations* for subklt pairs (cf. Definition 3.1). In some sense, they are generalizations of Definitions 2.11 and 2.14. We need them for our proof of Theorem 1.1.

Remark 2.15. Let (X, Δ) be a projective dlt pair. We note that $g \in \mathrm{Bir}(X, \Delta)$ does not necessarily induce a birational map $g|_T : T \dashrightarrow T$, where $T = \llcorner \Delta \lrcorner$ (see Example 2.13). However, $g \in \mathrm{Bir}(X, \Delta)$ induces an automorphism

$$g^* : H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \xrightarrow{\sim} H^0(T, \mathcal{O}_T(m(K_T + \Delta_T)))$$

where $(K_X + \Delta)|_T = K_T + \Delta_T$ and m is a nonnegative integer such that $m(K_X + \Delta)$ is Cartier (see the proof of [F1, Lemma 4.9]). More precisely, let

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "g"'] & & X \end{tikzcd}$$

be a common log resolution such that

$$\alpha^*(K_X + \Delta) = K_W + \Theta = \beta^*(K_X + \Delta).$$

Then we can easily see that

$$\alpha_* \mathcal{O}_S \simeq \mathcal{O}_T \simeq \beta_* \mathcal{O}_S,$$

where $S = \Theta^{=1}$, by the Kawamata–Viehweg vanishing theorem. Thus we obtain an automorphism

$$\begin{aligned} g^* : H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) &\xrightarrow{\beta^*} H^0(S, \mathcal{O}_S(m(K_S + \Theta_S))) \\ &\xrightarrow{\alpha^{*-1}} H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \end{aligned}$$

where $(K_W + \Theta)|_S = K_S + \Theta_S$.

Let us recall an important lemma on B-birational maps, which will be used in the proof of the main theorem (cf. Theorem 3.15).

Lemma 2.16. *Let $f : (X, \Delta) \to (X', \Delta')$ be a B-birational map between projective dlt pairs. Let S be an lc center of (X, Δ) such that $K_S + \Delta_S = (K_X + \Delta)|_S$. We take a suitable common log resolution as in* Definition 2.11.

$$\begin{tikzcd} & (W, \Gamma) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ (X, \Delta) \arrow[rr, dashed, "f"'] & & (X', \Delta') \end{tikzcd}$$

Then we can find an lc center V of (X, Δ) contained in S with $K_V + \Delta_V = (K_X+\Delta)|_V$, an lc center T of (W, Γ) with $K_T+\Gamma_T = (K_X+\Delta)|_T$, and an lc center V' of (X', Δ') with $K_{V'} + \Delta'_{V'} = (K_{X'} + \Delta')|_{V'}$ such that the following conditions hold.

(a) *$\alpha|_T$ and $\beta|_T$ are B-birational morphisms.*

$$\begin{tikzcd} & (T, \Gamma_T) \arrow[dl, "\alpha|_T"'] \arrow[dr, "\beta|_T"] & \\ (V, \Delta_V) & & (V', \Delta'_{V'}) \end{tikzcd}$$

Therefore, $(\beta|_T) \circ (\alpha|_T)^{-1} : (V, \Delta_V) \dashrightarrow (V', \Delta'|_{V'})$ is a B-birational map.

(b) *$H^0(S, m(K_S + \Delta_S)) \simeq H^0(V, m(K_V + \Delta_V))$ by the natural restriction map where m is a nonnegative integer such that $m(K_X + \Delta)$ is Cartier.*

Proof. See Claim (A_n) and Claim (B_n) in the proof of [F1, Lemma 4.9]. □

2.17 (Numerical dimesions)**.** In Section 5, we will use the notion of *Nakayama's numerical Kodaira dimension* for pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on normal projective varieties. For the details, see [N] and [L].

Definition 2.18 (Nakayama's numerical Kodaira dimension (cf. [N, V. 2.5. Definition]))**.** Let D be a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a normal projective variety X and let A be a Carteir divisor on X. If $H^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + A)) \neq 0$ for infinitely many positive integers m, then we put

$$\sigma(D; A) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \,\middle|\, \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + A))}{m^k} > 0 \right\}.$$

If $H^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + A)) \neq 0$ only for finitely many $m \in \mathbb{Z}_{\geq 0}$, then we put $\sigma(D; A) = -\infty$. We define *Nakayama's numerical Kodaira dimension* κ_σ by

$$\kappa_\sigma(X, D) = \max\{\sigma(D; A) \,|\, A \text{ is a Cartier divisor on } X\}.$$

If D is a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a normal projective variety X, then it is well known that D is pseudo-effective and

$$\kappa_\sigma(X, D) = \nu(X, D)$$

where $\nu(X, D)$ is the *numerical Kodaira dimension* of D.

We close this section with a remark on the minimal model program with scaling. For the details, see [BCHM] and [B1].

2.19 (Minimal model program with ample scaling)**.** Let $f : X \to Z$ be a projective morphism between quasi-projective varieties and let (X, B) be a $\mathbb{Q}$-factorial dlt pair. Let H be an effective f-ample $\mathbb{Q}$-divisor on

X such that $(X, B + H)$ is lc and that $K_X + B + H$ is f-nef. Under these assumptions, we can run the minimal model program on $K_X + B$ with scaling of H over Z. We call it *the minimal model program with ample scaling.*

Assume that $K_X + B$ is not pseudo-effective over Z. We note that the above minimal model program always terminates at a Mori fiber space structure over Z. By this observation, the results in [F1, Section 2] hold in every dimension. Therefore, we will freely use the results in [F1, Section 2] for *any* dimensional varieties.

From now on, we assume that K_X+B is pseudo-effective and $\dim X = n$. We further assume that the weak non-vanishing conjecture (cf. Conjecture 5.1) for projective $\mathbb{Q}$-factorial dlt pairs holds in dimension $\leq n$. Then the minimal model program on $K_X + B$ with scaling of H over Z terminates with a minimal model of (X, B) over Z by [B1, Theorems 1.4, 1.5].

3. Finiteness of log pluricanonical representations

In this section, we give a proof of Theorem 1.1. All the divisors in this section are $\mathbb{Q}$-divisors. We do not use $\mathbb{R}$-divisors throughout this section. We divide the proof into the three steps: subklt pairs in 3.1, lc pairs with big log canonical divisor in 3.2, and lc pairs with semi-ample log canonical divisor in 3.3.

3.1. **Klt pairs.** In this subsection, we prove Theorem 1.1 for klt pairs. More precisely, we prove Theorem 1.1 for *$\widetilde{B}$-pluricanonical representations* for projective subklt pairs without assuming the semi-ampleness of log canonical divisors. This formulation is indispensable for the proof of Theorem 1.1 for lc pairs.

First, let us introduce the notion of *$\widetilde{B}$-pluricanonical representations for subklt pairs.*

Definition 3.1 ($\widetilde{B}$-pluricanonical representations for subklt pairs)**.** Let (X, Δ) be an n-dimensional projective subklt pair such that X is smooth and that Δ has a simple normal crossing support. We write $\Delta = \Delta^+ - \Delta^-$ where Δ^+ and Δ^- are effective and have no common irreducible components. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. In this subsection, we always see

$$\omega \in H^0(X, m(K_X + \Delta))$$

as a meromorphic m-ple n-form on X which vanishes along $m\Delta^-$ and has poles at most $m\Delta^+$. By $\mathrm{Bir}(X)$, we mean the group of all the birational mappings of X onto itself. It has a natural group structure induced by the composition of birational maps. We define

$$\widetilde{\mathrm{Bir}}_m(X,\Delta)=\left\{g\in \mathrm{Bir}(X)\ \middle|\ \begin{array}{l} g^*\omega\in H^0(X,m(K_X+\Delta)) \text{ for} \\ \text{every } \omega\in H^0(X,m(K_X+\Delta))\end{array}\right\}.$$

Then it is easy to see that $\widetilde{\mathrm{Bir}}_m(X,\Delta)$ is a subgroup of $\mathrm{Bir}(X)$. An element $g\in\widetilde{\mathrm{Bir}}_m(X,\Delta)$ is called a $\widetilde{B}$*-birational map* of (X,Δ). By the definition of $\widetilde{\mathrm{Bir}}_m(X,\Delta)$, we get the group homomorphism

$$\widetilde{\rho}_m:\widetilde{\mathrm{Bir}}_m(X,\Delta)\to \mathrm{Aut}_{\mathbb{C}}(H^0(X,m(K_X+\Delta))).$$

The homomorphism $\widetilde{\rho}_m$ is called the $\widetilde{B}$*-pluricanonical representation* of $\widetilde{\mathrm{Bir}}_m(X,\Delta)$. We sometimes simply denote $\widetilde{\rho}_m(g)$ by g^* for $g\in\widetilde{\mathrm{Bir}}_m(X,\Delta)$ if there is no danger of confusion. There exists a natural inclusion $\mathrm{Bir}(X,\Delta)\subset\widetilde{\mathrm{Bir}}_m(X,\Delta)$ by the definitions.

Next, let us recall the notion of $L^{2/m}$*-integrable* m*-ple* n*-forms.*

Definition 3.2. Let X be an n-dimensional connected complex manifold and let ω be a meromorphic m-ple n-form. Let $\{U_\alpha\}$ be an open covering of X with holomorphic coordinates

$$(z^1_\alpha, z^2_\alpha,\cdots,z^n_\alpha).$$

We can write

$$\omega|_{U_\alpha}=\varphi_\alpha(dz^1_\alpha\wedge\cdots\wedge dz^n_\alpha)^m,$$

where φ_α is a meromorphic function on U_α. We give $(\omega\wedge\bar\omega)^{1/m}$ by

$$(\omega\wedge\bar\omega)^{1/m}|_{U_\alpha}=\left(\frac{\sqrt{-1}}{2\pi}\right)^n|\varphi_\alpha|^{2/m}dz^1_\alpha\wedge d\bar z^1_\alpha\cdots\wedge dz^n_\alpha\wedge d\bar z^n_\alpha.$$

We say that a meromorphic m-ple n-form ω is $L^{2/m}$*-integrable* if

$$\int_X(\omega\wedge\bar\omega)^{1/m}<\infty.$$

We can easily check the following two lemmas.

Lemma 3.3. *Let X be a compact connected complex manifold and let D be a reduced normal crossing divisor on X. Set $U=X\setminus D$. If ω is an L^2-integrable meromorphic n-form such that $\omega|_U$ is holomorphic, then ω is a holomorphic n-form.*

Proof. See, for example, [S, Theorem 2.1] or [Ka1, Proposition 16]. □

Lemma 3.4 (cf. [G2, Lemma 4.8])**.** *Let (X, Δ) be a projective subklt pair such that X is smooth and Δ has a simple normal crossing support. Let m be a positive integer such that $m\Delta$ is Cartier and let $\omega \in H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ be a meromorphic m-ple n-form. Then ω is $L^{2/m}$-integrable.*

By Lemma 3.4, we obtain the following result. We note that the proof of [G2, Proposition 4.9] works without any changes in our setting.

Proposition 3.5. *Let (X, Δ) be an n-dimensional projective subklt pair such that X is smooth, connected, and Δ has a simple normal crossing support. Let $g \in \widetilde{\mathrm{Bir}}_m(X, \Delta)$ be a $\widetilde{B}$-birational map where m is a positive integer such that $m\Delta$ is Cartier, and let*

$$\omega \in H^0(X, m(K_X + \Delta))$$

be a nonzero meromorphic m-ple n-form on X. Suppose that $g^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{C}$. Then there exists a positive integer $N_{m,\omega}$ such that $\lambda^{N_{m,\omega}} = 1$ and $N_{m,\omega}$ does not depend on g.

Remark 3.6. By the proof of [G2, Proposition 4.9] and [U, Theorem 14.10], we know that $\varphi(N_{m,\omega}) \leq b_n(Y')$, where $b_n(Y')$ is the n-th Betti number of Y' which is in the proof of [G2, Proposition 4.9] and φ is the Euler function.

Proposition 3.7 (cf. [U, Proposition 14.7])**.** *Let (X, Δ) be a projective subklt pair such that X is smooth, connected, and Δ has a simple normal crossing support, and let*

$$\widetilde{\rho}_m : \widetilde{\mathrm{Bir}}_m(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X + \Delta)))$$

be the $\widetilde{B}$-pluricanonical representation of $\widetilde{\mathrm{Bir}}_m(X, \Delta)$ where m is a positive integer such that $m\Delta$ is Cartier. Then $\widetilde{\rho}_m(g)$ is semi-simple for every $g \in \widetilde{\mathrm{Bir}}_m(X, \Delta)$.

Proof. If $\widetilde{\rho}_m(g)$ is not semi-simple, there exist two linearly independent elements $\varphi_1, \ \varphi_2 \in H^0(X, m(K_X + \Delta))$ and nonzero $\alpha \in \mathbb{C}$ such that

$$g^*\varphi_1 = \alpha\varphi_1 + \varphi_2, \ g^*\varphi_2 = \alpha\varphi_2$$

by considering Jordan's decomposition of g^*. Here, we denote $\widetilde{\rho}_m(g)$ by g^* for simplicity. By Proposition 3.5, we see that α is a root of unity. Let l be a positive integer. Then we have

$$(g^l)^*\varphi_1 = \alpha^l\varphi_1 + l\alpha^{l-1}\varphi_2.$$

Since g is a birational map, we have

$$\int_X (\varphi_1 \wedge \bar{\varphi}_1)^{1/m} = \int_X ((g^l)^*\varphi_1 \wedge (g^l)^*\bar{\varphi}_1)^{1/m}.$$

On the other hand, we have

$$\lim_{l\to\infty}\int_X ((g^l)^*\varphi_1\wedge (g^l)^*\bar{\varphi}_1)^{1/m}=\infty.$$

For details, see the proof of [U, Proposition 14.7]. However, we know $\int_X(\varphi_1\wedge\bar{\varphi}_1)^{1/m}<\infty$ by Lemma 3.4. This is a contradiction. $\square$

Proposition 3.8. *The number $N_{m,\omega}$ in* Proposition 3.5 *is uniformly bounded for every $\omega\in H^0(X, m(K_X+\Delta))$. Therefore, we can take a positive integer N_m such that N_m is divisible by $N_{m,\omega}$ for every ω.*

Proof. We consider the projective space bundle

$$\pi: M:=\mathbb{P}_X(\mathcal{O}_X(-K_X)\oplus\mathcal{O}_X)\to X$$

and

$$\begin{aligned} V &:= M\times\mathbb{P}(H^0(X,\mathcal{O}_X(m(K_X+\Delta))))\\ &\to X\times\mathbb{P}(H^0(X,\mathcal{O}_X(m(K_X+\Delta)))).\end{aligned}$$

We fix a basis $\{\omega_0, \omega_1, \ldots, \omega_N\}$ of $H^0(X,\mathcal{O}_X(m(K_X+\Delta)))$. By using this basis, we can identify $\mathbb{P}(H^0(X,\mathcal{O}_X(m(K_X+\Delta))))$ with $\mathbb{P}^N$. We write the coordinate of $\mathbb{P}^N$ as $(a_0:\cdots:a_N)$ under this identification. Set $\Delta=\Delta^+-\Delta^-$, where Δ^+ and Δ^- are effective and have no common irreducible components. Let $\{U_\alpha\}$ be coordinate neighborhoods of X with holomorphic coordinates $(z^1_\alpha, z^2_\alpha,\cdots, z^n_\alpha)$. For any i, we can write ω_i locally as

$$\omega_i|_{U_\alpha}=\frac{\varphi_{i,\alpha}}{\delta_{i,\alpha}}(dz^1_\alpha\wedge\cdots\wedge dz^n_\alpha)^m,$$

where $\varphi_{i,\alpha}$ and $\delta_{i,\alpha}$ are holomorphic with no common factors, and $\frac{\varphi_{i,\alpha}}{\delta_{i,\alpha}}$ has poles at most $m\Delta^+$. We may assume that $\{U_\alpha\}$ gives a local trivialization of M, that is, $M|_{U_\alpha}:=\pi^{-1}U_\alpha\simeq U_\alpha\times\mathbb{P}^1$. We set a coordinate $(z^1_\alpha, z^2_\alpha,\cdots, z^n_\alpha, \xi^0_\alpha:\xi^1_\alpha)$ of $U_\alpha\times\mathbb{P}^1$ with the homogeneous coordinate $(\xi^0_\alpha:\xi^1_\alpha)$ of $\mathbb{P}^1$. Note that

$$\frac{\xi^0_\alpha}{\xi^1_\alpha}=k_{\alpha\beta}\frac{\xi^0_\beta}{\xi^1_\beta}\text{ in } M|_{U_\alpha\bigcap U_\beta},$$

where $k_{\alpha\beta}=\det(\partial z^i_\beta/\partial z^j_\alpha)_{1\leq i,j\leq n}$. Set

$$Y_{U_\alpha}=\{(\xi^0_\alpha)^m\prod_{i=0}^N\delta_{i,\alpha}-(\xi^1_\alpha)^m\sum_{i=0}^N\hat{\delta}_{i,\alpha}a_i\varphi_{i,\alpha}=0\}\subset U_\alpha\times\mathbb{P}^1\times\mathbb{P}^N,$$

where $\hat{\delta}_{i,\alpha}=\delta_{0,\alpha}\cdots\delta_{i-1,\alpha}\cdot\delta_{i+1,\alpha}\cdots\delta_{N,\alpha}$. By easy calculations, we see that $\{Y_{U_\alpha}\}$ can be patched and we obtain Y. We note that Y may have singularities and be reducible. The induced projection $f:Y\to\mathbb{P}^N$

is surjective and equidimensional. Let $q : Y \to X$ be the natural projection. By the same arguments as in the proof of [U, Theorem 14.10], we have a suitable stratification $\mathbb{P}^N = \amalg_i S_i$, where S_i is smooth and locally closed in $\mathbb{P}^N$ for every i, such that $f^{-1}(S_i) \to S_i$ has a simultaneous resolution with good properties for every i. Therefore, we may assume that there is a positive constant b such that for every $p \in \mathbb{P}^N$ we have a resolution $\mu_p : \widetilde{Y}_p \to Y_p := f^{-1}(p)$ with the properties that $b_n(\widetilde{Y}_p) \le b$ and that $\widetilde{\mu}_p^* \Delta \cup \mathrm{Exc}(\widetilde{\mu}_p)$ has a simple normal crossing support, where $\widetilde{\mu}_p : \widetilde{Y}_p \overset{\mu_p}{\to} Y_p \overset{q}{\to} X$. Thus, by Remark 3.6, we obtain Proposition 3.8. □

Now we have the main theorem of this subsection. We will use it in the following subsections.

Theorem 3.9. *Let (X, Δ) be a projective subklt pair such that X is smooth, Δ has a simple normal crossing support, and $m(K_X + \Delta)$ is Cartier where m is a positive integer. Then $\widetilde{\rho}_m(\widetilde{\mathrm{Bir}}_m(X, \Delta))$ is a finite group.*

Proof. By Proposition 3.7, we see that $\widetilde{\rho}_m(g)$ is diagonalizable. Moreover, Proposition 3.8 implies that the order of $\widetilde{\rho}_m(g)$ is bounded by a positive constant N_m which is independent of g. Thus $\widetilde{\rho}_m(\widetilde{\mathrm{Bir}}_m(X, \Delta))$ is a finite group by Burnside's theorem (see, for example, [U, Theorem 14.9]). □

As a corollary, we obtain Theorem 1.1 for klt pairs without assuming the semi-ampleness of log canonical divisors.

Corollary 3.10. *Let (X, Δ) be a projective klt pair such that $m(K_X + \Delta)$ is Cartier where m is a positive integer. Then $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group.*

Proof. Let $f : Y \to X$ be a log resolution of (X, Δ) such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Since

$$\rho_m(\mathrm{Bir}(Y, \Delta_Y)) \subset \widetilde{\rho}_m(\widetilde{\mathrm{Bir}}_m(Y, \Delta_Y)),$$

$\rho_m(\mathrm{Bir}(Y, \Delta_Y))$ is a finite group by Theorem 3.9. Therefore, we obtain that $\rho_m(\mathrm{Bir}(X, \Delta)) \simeq \rho_m(\mathrm{Bir}(Y, \Delta_Y))$ is a finite group. □

3.2. **Lc pairs with big log canonical divisor.** In this subsection, we prove the following theorem. The proof is essentially the same as that of Case 1 in [F1, Theorem 3.5].

Theorem 3.11. *Let (X, Δ) be a projective sublc pair such that $K_X + \Delta$ is big. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. Then $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group.*

Before we start the proof of Theorem 3.11, we give a remark.

Remark 3.12. By Theorem 3.11, when $K_X + \Delta$ is big, Theorem 1.1, the main theorem of this paper, holds true without assuming that $K_X + \Delta$ is semi-ample. Therefore, we state Theorem 3.11 separately for some future usage (cf. Corollary 3.13). In Case 2 in the proof of Theorem 3.15, which is nothing but Theorem 1.1, we will use the arguments in the proof of Theorem 3.11.

Proof. By taking a log resolution, we can assume that X is smooth and Δ has a simple normal crossing support. By Theorem 3.9, we can also assume that $\Delta^{=1} \neq 0$. Since $K_X + \Delta$ is big, for a sufficiently large and divisible positive integer m', we obtain an effective Cartier divisor $D_{m'}$ such that

$$m'(K_X + \Delta) \sim_{\mathbb{Z}} \Delta^{=1} + D_{m'}$$

by Kodaira's lemma. It is easy to see that $\operatorname{Supp} g^*\Delta^{=1} \supset \operatorname{Supp} \Delta^{=1}$ for every $g \in \operatorname{Bir}(X, \Delta)$. This implies that $g^*\Delta^{=1} \geq \Delta^{=1}$. Thus, we have a natural inclusion

$$\operatorname{Bir}(X, \Delta) \subset \widetilde{\operatorname{Bir}}_{m'}\left(X, \Delta - \frac{1}{m'}\Delta^{=1}\right).$$

We consider the $\widetilde{B}$-birational representation

$$\widetilde{\rho}_{m'} : \widetilde{\operatorname{Bir}}_{m'}\left(X, \Delta - \frac{1}{m'}\Delta^{=1}\right) \to \operatorname{Aut}_{\mathbb{C}} H^0(X, m'(K_X + \Delta) - \Delta^{=1}).$$

Then, by Theorem 3.9,

$$\widetilde{\rho}_{m'}\left(\widetilde{\operatorname{Bir}}_{m'}\left(X, \Delta - \frac{1}{m'}\Delta^{=1}\right)\right)$$

is a finite group. Therefore, $\widetilde{\rho}_{m'}(\operatorname{Bir}(X, \Delta))$ is also a finite group. We put $a = |\widetilde{\rho}_{m'}(\operatorname{Bir}(X, \Delta))| < \infty$. In this situation, we can find a $\operatorname{Bir}(X, \Delta)$-invariant non-zero section $s \in H^0(X, a(m'(K_X+\Delta)-\Delta^{=1}))$. By using s, we have a natural inclusion

$$(\spadesuit) \qquad H^0(X, m(K_X + \Delta)) \subseteq H^0(X, (m + m'a)(K_X + \Delta) - a\Delta^{=1}).$$

By the construction, $\operatorname{Bir}(X, \Delta)$ acts on the both vector spaces compatibly. We consider the $\widetilde{B}$-pluricanonical representation

$$\begin{aligned}\widetilde{\rho}_{m+m'a} : \widetilde{\operatorname{Bir}}_{m+m'a}&\left(X, \Delta - \frac{a}{m + m'a}\Delta^{=1}\right) \\ &\to \operatorname{Aut}_{\mathbb{C}} H^0(X, (m + m'a)(K_X + \Delta) - a\Delta^{=1}).\end{aligned}$$

Since

$$\left(X, \Delta - \frac{a}{m + m'a}\Delta^{=1}\right)$$

is subklt, we have that

$$\widetilde{\rho}_{m+m'a}\left(\widetilde{\mathrm{Bir}}_{m+m'a}\left(X, \Delta - \frac{a}{m+m'a}\Delta^{=1}\right)\right)$$

is a finite group by Theorem 3.9. Therefore, $\widetilde{\rho}_{m+m'a}(\mathrm{Bir}(X,\Delta))$ is also a finite group. Thus, we obtain that $\rho_m(\mathrm{Bir}(X,\Delta))$ is a finite group by the $\mathrm{Bir}(X,\Delta)$-equivariant embedding (♠). □

The following corollary is an answer to the question raised by Cacciola and Tasin. It is a generalization of the well-known finiteness of birational automorphisms of varieties of general type (cf. [U, Corollary 14.3]).

Corollary 3.13. *Let (X,Δ) be a projective sublc pair such that $K_X+\Delta$ is big. Then $\mathrm{Bir}(X,\Delta)$ is a finite group.*

Proof. We consider the rational map

$$\Phi_m := \Phi_{|m(K_X+\Delta)|} : X \dashrightarrow \mathbb{P}(H^0(X, m(K_X+\Delta)))$$

associated to the complete linear system $|m(K_X+\Delta)|$, where m is a positive integer such that $m(K_X+\Delta)$ is Cartier. By taking $m \gg 0$, we may assume that $\Phi_m : X \dashrightarrow V$ is birational because $K_X+\Delta$ is big, where V is the image of X by Φ_m. The log pluricanonical representation

$$\rho_m : \mathrm{Bir}(X,\Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X+\Delta)))$$

induces the group homomorphism

$$\bar{\rho}_m : \mathrm{Bir}(X,\Delta) \to \mathrm{Aut}(\mathbb{P}(H^0(X, m(K_X+\Delta)))).$$

Note that $\bar{\rho}_m(g)$ leaves V invariant for every $g \in \mathrm{Bir}(X,\Delta)$ by the construction. Since $\Phi_m : X \dashrightarrow V$ is birational, $\bar{\rho}_m$ is injective. On the other hand, we see that $\bar{\rho}_m(\mathrm{Bir}(X,\Delta))$ is finite by Theorem 3.11. Therefore, $\mathrm{Bir}(X,\Delta)$ is a finite group. □

Remark 3.14. By the proof of Corollary 3.13 and Theorem 3.9, we obtain the following finiteness of $\widetilde{\mathrm{Bir}}_m(X,\Delta)$.

Let (X,Δ) be a projective subklt pair such that X is smooth and that Δ has a simple normal crossing support. Let m be a positive integer such that $m(K_X+\Delta)$ is Cartier and that $|m(K_X+\Delta)|$ defines a birational map. Then $\widetilde{\mathrm{Bir}}_m(X,\Delta)$ is a finite group.

3.3. **Lc pairs with semi-ample log canonical divisor.** Theorem 3.15 is one of the main results of this paper (see Theorem 1.1). We will treat many applications of Theorem 3.15 in Section 4.

Theorem 3.15. *Let (X, Δ) be an n-dimensional projective lc pair such that $K_X + \Delta$ is semi-ample. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. Then $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group.*

Proof. We show the statement by the induction on n. By taking a dlt blow-up (cf. Theorem 2.8), we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. Let $f : X \to Y$ be a projective surjective morphism associated to $k(K_X + \Delta)$ for a sufficiently large and divisible positive integer k. By Corollary 3.10, we may assume that $\llcorner \Delta \lrcorner \neq 0$.

Case 1. $\llcorner \Delta^h \lrcorner \neq 0$, where Δ^h is the horizontal part of Δ with respect to f.

In this case, we put $T = \llcorner \Delta \lrcorner$. Since $m(K_X + \Delta) \sim_{\mathbb{Q},Y} 0$, we see that

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta) - T)) = 0.$$

Thus the restricted map

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(T, \mathcal{O}_T(m(K_T + \Delta_T)))$$

is injective, where $K_T + \Delta_T = (K_X + \Delta)|_T$. Let (V_i, Δ_{V_i}) be the disjoint union of all the i-dimensional lc centers of (X, Δ) for $0 \leq i \leq n-1$. We note that $\rho_m(\mathrm{Bir}(V_i, \Delta_{V_i}))$ is a finite group for every i by the induction on dimension. We put $k_i = |\rho_m(\mathrm{Bir}(V_i, \Delta_{V_i}))| < \infty$ for $0 \leq i \leq n-1$. Let l be the least common multiple of k_i for $0 \leq i \leq n-1$. Let $T = \cup_j T_j$ be the irreducible decomposition. Let g be an element of $\mathrm{Bir}(X, \Delta)$. By repeatedly using Lemma 2.16, for every T_j, we can find lc centers S_j^i of (X, Δ)

$$\begin{array}{ccccccccccc} X & \overset{g}{\dashrightarrow} & X & \overset{g}{\dashrightarrow} & X & \overset{g}{\dashrightarrow} & \cdots & \overset{g}{\dashrightarrow} & X & \dashrightarrow \\ \cup & & \cup & & \cup & & & & \cup & \\ S_j^0 & & S_j^1 & & S_j^2 & & & & S_j^k & \end{array}$$

such that $S_j^0 \subset T_j$, $S_j^i \dashrightarrow S_j^{i+1}$ is a B-birational map for every i, and

$$H^0(T_j, m(K_{T_j} + \Delta_{T_j})) \simeq H^0(S_j^0, m(K_{S_j^0} + \Delta_{S_j^0}))$$

by the natural restriction map, where $K_{T_j} + \Delta_{T_j} = (K_X + \Delta)|_{T_j}$ and $K_{S_j^0} + \Delta_{S_j^0} = (K_X + \Delta)|_{S_j^0}$. Since there are only finitely many lc centers of (X, Δ), we can find $p_j < q_j$ such that $S_j^{p_j} = S_j^{q_j}$ and that $S_j^{p_j} \neq S_j^r$ for $r = p_j + 1, \cdots, q_j - 1$. Therefore, g induces a B-birational map

$$\widetilde{g} : \coprod_{p_j \leq r \leq q_j - 1} S_j^r \dashrightarrow \coprod_{p_j \leq r \leq q_j - 1} S_j^r$$

for every j. We have an embedding

$$H^0(T, \mathcal{O}_T(m(K_T+\Delta_T))) \subset \bigoplus_j H^0(S_j^{p_j}, m(K_{S_j^{p_j}}+\Delta_{S_j^{p_j}})),$$

where $K_{S_j^{p_j}}+\Delta_{S_j^{p_j}} = (K_X+\Delta)|_{S_j^{p_j}}$ for every j. First, by the following commutative diagram (cf. Remark 2.15)

$$\begin{tikzcd} 0 \arrow[r] & H^0(T, \mathcal{O}_T(m(K_T+\Delta_T))) \arrow[r] \arrow[d, "(g^*)^l"'] & \bigoplus_j H^0(S_j^{p_j}, m(K_{S_j^{p_j}}+\Delta_{S_j^{p_j}}))) \arrow[d, "(\widetilde{g}^*)^l=\mathrm{id}"] \\ 0 \arrow[r] & H^0(T, \mathcal{O}_T(m(K_T+\Delta_T))) \arrow[r] & \bigoplus_j H^0(S_j^{p_j}, m(K_{S_j^{p_j}}+\Delta_{S_j^{p_j}}))), \end{tikzcd}$$

we obtain $(g^*)^l = \mathrm{id}$ on $H^0(T, m(K_T+\Delta_T))$. Next, by the following commutative diagram (cf. Remark 2.15)

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{O}_X(m(K_X+\Delta))) \arrow[r] \arrow[d, "(g^*)^l"'] & H^0(T, \mathcal{O}_T(m(K_T+\Delta_T))) \arrow[d, "(g^*)^l=\mathrm{id}"] \\ 0 \arrow[r] & H^0(X, \mathcal{O}_X(m(K_X+\Delta))) \arrow[r] & H^0(T, \mathcal{O}_T(m(K_T+\Delta_T))), \end{tikzcd}$$

we have that $(g^*)^l = \mathrm{id}$ on $H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$. Thus we obtain that $\rho_m(\mathrm{Bir}(X,\Delta))$ is a finite group by Burnside's theorem (cf. [U, Theorem 14.9]).

Case 2. $\llcorner \Delta^h \lrcorner = 0$.

We can construct the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

with the following properties:

(a) $\varphi : X' \to X$ is a log resolution of (X, Δ).
(b) $\psi : Y' \to Y$ is a resolution of Y.
(c) there is a simple normal crossing divisor Σ on Y' such that f' is smooth and $\mathrm{Supp}\,\varphi_*^{-1}\Delta \cup \mathrm{Exc}(\varphi)$ is relatively normal crossing over $Y' \setminus \Sigma$.
(d) $\mathrm{Supp}\, f'^*\Sigma$ and $\mathrm{Supp}\, f'^*\Sigma \cup \mathrm{Exc}(\varphi) \cup \mathrm{Supp}\,\varphi_*^{-1}\Delta$ are simple normal crossing divisors on X'.

Then we have

$$K_{X'} + \Delta_{X'} = f'^*(K_{Y'} + \Delta_{Y'} + M),$$

where $K_{X'} + \Delta_{X'} = \varphi^*(K_X + \Delta)$, $\Delta_{Y'}$ is the discriminant divisor and M is the moduli part of $f' : (X', \Delta_{X'}) \to Y'$. Note that

$$\Delta_{Y'} = \sum (1 - c_Q)Q,$$

where Q runs through all the prime divisors on Y' and

$$c_Q = \sup\{t \in \mathbb{Q} \,|\, K_{X'} + \Delta_{X'} + t f'^*Q \text{ is sublc over the generic point of } Q\}.$$

When we construct $f' : X' \to Y'$, we first make $f : X \to Y$ toroidal by [AK, Theorem 2.1], next make it equi-dimensional by [AK, Proposition 4.4], and finally obtain $f' : X' \to Y'$ by taking a resolution. By the above construction, we can assume that $f' : X' \to Y'$ factors as

$$f' : X' \xrightarrow{\alpha} \widetilde{X} \xrightarrow{\widetilde{f}} Y'$$

with the following properties.

(1) $\widetilde{f} : (\widetilde{U} \subset \widetilde{X}) \to (U_{Y'} \subset Y')$, where $U_{Y'} = Y' \setminus \Sigma$ and $\widetilde{U}$ is some Zariski open set of $\widetilde{X}$, is toroidal and equi-dimensional.
(2) α is a projective biratioinal morphism and is an isomorphism over $U_{Y'}$.
(3) $\widetilde{\varphi} := \varphi \circ \alpha^{-1} : \widetilde{X} \to X$ is a morphism such that $K_{\widetilde{X}} + \Delta_{\widetilde{X}} = \widetilde{\varphi}^*(K_X + \Delta)$ and that $\operatorname{Supp} \Delta_{\widetilde{X}} \subset \widetilde{X} \setminus \widetilde{U}$.
(4) $\widetilde{X}$ has only quotient singularities (cf. [AK, Remark 4.5]).

For the details, see the arguments in [AK]. In this setting, it is easy to see that $\operatorname{Supp} \Delta_{X'}^{=1} \subset \operatorname{Supp} f'^*\Delta_{Y'}^{=1}$. Therefore, $\Delta_{X'}^{=1} \leq f'^*\Delta_{Y'}^{=1}$. We can check that every $g \in \operatorname{Bir}(X', \Delta_{X'}) = \operatorname{Bir}(X, \Delta)$ induces $g_{Y'} \in \operatorname{Bir}(Y', \Delta_{Y'})$ which satisfies the following commutative diagram (see [A, Theorem 0.2] for the subklt case, and [Ko1, Proposition 8.4.9 (3)] for the sublc case).

$$\begin{tikzcd}
X' \arrow[r, dashed, "g"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "f'"] \\
Y' \arrow[r, dashed, "g_{Y'}"'] & Y'
\end{tikzcd}$$

Therefore, we have $\operatorname{Supp} g_{Y'}^*\Delta_{Y'}^{=1} \supset \operatorname{Supp} \Delta_{Y'}^{=1}$. This implies that

$$g_{Y'}^*\Delta_{Y'}^{=1} \geq \Delta_{Y'}^{=1}.$$

Thus there is an effective Cartier divisor E_g on X' such that

$$g^* f'^* \Delta_{Y'}^{=1} + E_g \geq f'^* \Delta_{Y'}^{=1}$$

and that the codimension of $f'(E_g)$ in Y' is ≥ 2. We note the definitions of g^* and $g_{Y'}^*$ (cf. Definition 2.11). Therefore, $g \in \operatorname{Bir}(X', \Delta_{X'})$ induces an automorphism g^* of $H^0(X', m'(K_{X'}+\Delta_{X'})-f'^*\Delta^{=1}_{Y'})$ where m' is a sufficiently large and divisible positive integer m'. It is because

$$\begin{aligned}&H^0(X', m'(K_{X'}+\Delta_{X'})-g^*f'^*\Delta^{=1}_{Y'})\\ &\subset H^0(X', m'(K_{X'}+\Delta_{X'})-f'^*\Delta^{=1}_{Y'}+E_g)\\ &\simeq H^0(X', m'(K_{X'}+\Delta_{X'})-f'^*\Delta^{=1}_{Y'}).\end{aligned}$$

Here, we used the facts that $m'(K_{X'}+\Delta_{X'})=f'^*(m'(K_{Y'}+\Delta_{Y'}+M))$ and that $f'_*\mathcal{O}_{X'}(E_g)\simeq\mathcal{O}_{Y'}$. Thus we have a natural inclusion

$$\mathrm{Bir}(X',\Delta_{X'})\subset\widetilde{\mathrm{Bir}}_{m'}\left(X',\Delta_{X'}-\frac{1}{m'}f'^*\Delta^{=1}_{Y'}\right).$$

Note that

$$\left(X',\Delta_{X'}-\frac{1}{m'}f'^*\Delta^{=1}_{Y'}\right)$$

is subklt because $\Delta^{=1}_{X'}\leq f'^*\Delta^{=1}_{Y'}$. Since $K_{Y'}+\Delta_{Y'}+M$ is (nef and) big, for a sufficiently large and divisible positive integer m', we obtain an effective Cartier divisor $D_{m'}$ such that

$$m'(K_{Y'}+\Delta_{Y'}+M)\sim_{\mathbb{Z}}\Delta^{=1}_{Y'}+D_{m'}.$$

This means that

$$H^0(X', m'(K_{X'}+\Delta_{X'})-f'^*\Delta^{=1}_{Y'})\neq 0.$$

By considering the natural inclusion

$$\mathrm{Bir}(X',\Delta_{X'})\subset\widetilde{\mathrm{Bir}}_{m'}\left(X',\Delta_{X'}-\frac{1}{m'}f'^*\Delta^{=1}_{Y'}\right),$$

we can use the same arguments as in the proof of Theorem 3.11. Thus we obtain the finiteness of B-pluricanonical representations. □

Remark 3.16. Although we did not explicitly state it, in Theorem 3.9, we do not have to assume that X is connected. Similarly, we can prove Theorems 3.11 and 3.15 without assuming that X is connected. For the details, see [G2, Remark 4.4].

We close this section with comments on [F1, Section 3] and [G2, Theorem B]. In [F1, Section 3], we proved Theorem 3.15 for surfaces. There, we do not need the notion of $\widetilde{B}$-birational maps. It is mainly because Y' in Case 2 in the proof of Theorem 3.15 is a curve if (X,Δ) is not klt and $K_X+\Delta$ is not big. Thus, $g_{Y'}$ is an automorphism of Y'. In [G2, Theorem B], we proved Theorem 3.15 under the assumption that $K_X+\Delta\sim_{\mathbb{Q}}0$. In that case, Case 1 in the proof of Theorem 3.15 is sufficient. Therefore, we do not need the notion of $\widetilde{B}$-birational maps in [G2].

4. On abundance conjecture for log canonical pairs

In this section, we treat various applications of Theorem 1.1 on the abundance conjecture for (semi) lc pairs (cf. Conjecture 1.3). We note that we only treat $\mathbb{Q}$-divisors in this section.

Let us introduce the notion of *nef and log abundant $\mathbb{Q}$-divisors*.

Definition 4.1 (Nef and log abundant divisors)**.** Let (X, Δ) be a sublc pair. A closed subvariety W of X is called an *lc center* if there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$ and $f(E) = W$. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is called *nef and log abundant with respect to* (X, Δ) if and only if D is nef and abundant, and $\nu_W^* D|_W$ is nef and abundant for every lc center W of the pair (X, Δ), where $\nu_W : W^\nu \to W$ is the normalization. Let $\pi : X \to S$ be a proper morphism onto a variety S. Then D is *π-nef and π-log abundant with respect to* (X, Δ) if and only if D is π-nef and π-abundant and $(\nu_W^* D|_W)|_{W_\eta^\nu}$ is abundant, where W_η^ν is the generic fiber of $W^\nu \to \pi(W)$. We sometimes simply say that D is nef and log abundant over S.

The following theorem is one of the main theorems of this section (cf. [F2, Theorem 0.1], [F9, Theorem 4.4]). For a relative version of Theorem 4.2, see Theorem 4.12 below. See also Subsection 4.1.

Theorem 4.2. *Let (X, Δ) be a projective lc pair. Assume that $K_X + \Delta$ is nef and log abundant. Then $K_X + \Delta$ is semi-ample.*

Proof. By replacing (X, Δ) with its dlt blow-up (cf. Theorem 2.8), we can assume that (X, Δ) is dlt and that $K_X + \Delta$ is nef and log abundant. We put $S = \llcorner \Delta \lrcorner$. Then (S, Δ_S), where $K_S + \Delta_S = (K_X + \Delta)|_S$, is an sdlt $(n-1)$-fold and $K_S + \Delta_S$ is semi-ample by the induction on dimension and Proposition 4.3 below. By applying Fukuda's theorem (cf. [F5, Theorem 1.1]), we obtain that $K_X + \Delta$ is semi-ample. □

We note that Proposition 4.3 is a key result in this paper. It heavily depends on Theorem 1.1.

Proposition 4.3. *Let (X, Δ) be a projective slc pair. Let $\nu : X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ is semi-ample. Then $K_X + \Delta$ is semi-ample.*

Proof. The arguments in [F1, Section 4] work by Theorem 1.1. As we pointed out in 2.19, we can freely use the results in [F1, Section 2]. The finiteness of B-pluricanonical representations, which was only proved in dimension ≤ 2 in [F1, Section 3], is now Theorem 1.1. Therefore, the results in [F1, Section 4] hold in any dimension. □

By combining Proposition 4.3 with Theorem 4.2, we obtain an obvious corollary (see also Corollary 4.13, Theorem 4.16, and Remark 4.17).

Corollary 4.4. *Let (X, Δ) be a projective slc pair and let $\nu : X^\nu \to X$ be the normalization. If $K_{X^\nu}+\Theta = \nu^*(K_X+\Delta)$ is nef and log abundant, then $K_X + \Delta$ is semi-ample.*

We give one more corollary of Proposition 4.3.

Corollary 4.5. *Let (X, Δ) be a projective slc pair such that $K_X + \Delta$ is nef. Let $\nu : X^\nu \to X$ be the normalization. Assume that X^ν is a toric variety. Then $K_X + \Delta$ is semi-ample.*

Proof. It is well known that every nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a projective toric variety is semi-ample. Therefore, this corollary is obvious by Proposition 4.3. □

Theorem 4.6. *Let (X, Δ) be a projective n-dimensional lc pair. Assume that the abundance conjecture holds for projective dlt pairs in dimension $\leq n-1$. Then $K_X+\Delta$ is semi-ample if and only if $K_X+\Delta$ is nef and abundant.*

Proof. It is obvious that $K_X+\Delta$ is nef and abundant if $K_X+\Delta$ is semi-ample. So, we show that $K_X + \Delta$ is semi-ample under the assumption that $K_X+\Delta$ is nef and abundant. By taking a dlt blow-up (cf. Theorem 2.8), we can assume that (X, Δ) is dlt. By the assumption, it is easy to see that $K_X + \Delta$ is nef and log abundant. Therefore, by Theorem 4.2, we obtain that $K_X + \Delta$ is semi-ample. □

The following theorem is an easy consequence of the arguments in [KMM, Section 7] and Proposition 4.3 by the induction on dimension. We will treat related topics in Section 5 more systematically.

Theorem 4.7. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt n-fold such that $K_X+\Delta$ is nef. Assume that the abundance conjecture for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n$. We further assume that the minimal model program with ample scaling terminates for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n$. Then $K_X + \Delta$ is semi-ample.*

Proof. This follows from the arguments in [KMM, Section 7] by using the minimal model program with ample scaling with the aid of Proposition 4.3. Let H be a general effective sufficiently ample Cartier divisor on X. We run the minimal model program on $K_X + \Delta - \varepsilon\llcorner\Delta\lrcorner$ with scaling of H. We note that $K_X + \Delta$ is numerically trivial on the extremal ray in each step of the above minimal model program if ε is sufficiently small by [B1, Proposition 3.2]. We also note that, by the induction on dimension, $(K_X+\Delta)|_{\llcorner \Delta \lrcorner}$ is semi-ample. For the details, see [KMM, Section 7]. $\square$

Remark 4.8. In the proof of Theorem 4.7, the abundance theorem and the termination of the minimal model program with ample scaling for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n-1$ are sufficient if $K_X+\Delta-\varepsilon \llcorner \Delta \lrcorner$ is not pseudo-effective for every $0<\varepsilon \ll 1$ by [BCHM] (cf. 2.19).

The next theorem is an answer to János Kollár's question for *projective* varieties. He was mainly interested in the case where f is birational.

Theorem 4.9. *Let $f: X \to Y$ be a projective morphism between projective varieties. Let (X, Δ) be an lc pair such that $K_X+\Delta$ is numerically trivial over Y. Then $K_X+\Delta$ is f-semi-ample.*

Proof. By replacing (X, Δ) with its dlt blow-up (cf. Theorem 2.8), we can assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. Let $S=\llcorner \Delta \lrcorner=\cup S_i$ be the irreducible decomposition. If $S=0$, then $K_X+\Delta$ is f-semi-ample by Kawamata's theorem (see [F4, Theorem 1.1]). It is because $(K_X+\Delta)|_{X_\eta} \sim_{\mathbb{Q}} 0$, where X_η is the generic fiber of f, by Nakayama's abundance theorem for klt pairs with numerical trivial log canonical divisor (cf. [N, Chapter V. 4.9. Corollary]). By the induction on dimension, we can assume that $(K_X+\Delta)|_{S_i}$ is semi-ample over Y for every i. Let H be a general effective sufficiently ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $\llcorner H \lrcorner=0$. Then $(X, \Delta+f^*H)$ is dlt, $(K_X+\Delta+f^*H)|_{S_i}$ is semi-ample for every i. By Proposition 4.3, $(K_X+\Delta+f^*H)|_S$ is semi-ample. By applying [F5, Theorem 1.1], we obtain that $K_X+\Delta+f^*H$ is f-semi-ample. We note that $(K_X+\Delta+f^*H)|_{X_\eta} \sim_{\mathbb{Q}} 0$ (see, for example, [G2, Theorem 1.2]). Therefore, $K_X+\Delta$ is f-semi-ample. $\square$

Remark 4.10. In Theorem 4.9, if Δ is an $\mathbb{R}$-divisor, then we also obtain that $K_X+\Delta$ is semi-ample over Y by the same arguments as in [G3, Lemma 6.2] and [FG, Theorem 3.1].

As a corollary, we obtain a relative version of the main theorem of [G2].

Corollary 4.11 (cf. [G2, Theorem 1.2])**.** *Let $f: X \to Y$ be a projective morphism from a projective slc pair (X, Δ) to a (not necessarily irreducible) projective variety Y. Assume that $K_X+\Delta$ is numerically trivial over Y. Then $K_X+\Delta$ is f-semi-ample.*

Proof. Let $\nu: X^\nu \to X$ be the normalization such that $K_{X^\nu}+\Theta=\nu^*(K_X+\Delta)$. By Theorem 4.9, $K_{X^\nu}+\Theta$ is semi-ample over Y. Let H

be a general sufficiently ample $\mathbb{Q}$-divisor on Y such that $K_{X^\nu} + \Theta + \nu^* f^* H$ is semi-ample and that $(X, \Delta + f^*H)$ is slc. By Proposition 4.3, $K_X + \Delta + f^*H$ is semi-ample. In particular, $K_X + \Delta + f^*H$ is f-semi-ample. Then $K_X + \Delta$ is f-semi-ample. □

By the same arguments as in the proof of Theorem 4.9 (resp. Corollary 4.11), we obtain the following theorem (resp. corollary), which is a relative version of Theorem 4.2 (resp. Corollary 4.4).

Theorem 4.12. *Let $f : X \to Y$ be a projective morphism between projective varieties. Let (X, Δ) be an lc pair such that $K_X + \Delta$ is f-nef and f-log abundant. Then $K_X + \Delta$ is f-semi-ample.*

Corollary 4.13. *Let $f : X \to Y$ be a projective morphism from a projective slc pair (X, Δ) to a (not necessarily irreducible) projective variety Y. Let $\nu : X^\nu \to X$ be the normalization such that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$. Assume that $K_{X^\nu} + \Theta$ is nef and log abundant over Y. Then $K_X + \Delta$ is f-semi-ample.*

4.1. **Relative abundance conjecture.** In this subsection, we make some remarks on the relative abundance conjecture.

After we circulated this paper, Hacon and Xu proved the relative version of Theorem 1.4 (cf. Proposition 4.3) in [HX2].

Theorem 4.14 (cf. [HX2, Theorem 1.4])**.** *Let (X, Δ) be an slc pair, let $f : X \to Y$ be a projective morphism onto an algebraic variety Y, and let $\nu : X^\nu \to X$ be the normalization with $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$. If $K_{X^\nu} + \Theta$ is semi-ample over Y, then $K_X + \Delta$ is semi-ample over Y.*

The proof of [HX2, Theorem 1.4] in [HX2, Section 4] depends on Kollár's gluing theory (see, for example, [Ko2], [Ko3], and [HX1]) and the finiteness of the log pluricanonical representation: Theorem 1.1. Note that Hacon and Xu prove a slightly weaker version of Theorem 1.1 in [HX2, Section 3], which is sufficient for the proof of [HX2, Theorem 1.4]. Their arguments in [HX2, Section 3] are more Hodge theoretic than ours and use the finiteness result in the case when the Kodaira dimension is zero established in [G2]. We note that Theorem 4.14 implies the relative versions (or generalizations) of Theorem 4.2, Corollary 4.4, Corollary 4.5, Theorem 4.6, Theorem 4.7, Theorem 4.9, Corollary 4.11, Theorem 4.12, and Corollary 4.13 without assuming the projectivity of varieties. We leave the details as exercises for the reader.

4.2. **Miscellaneous applications.** In this subsection, we collect some miscellaneous applications related to the base point free theorem and the abundance conjecture.

The following theorem is the log canonical version of Fukuda's result.

Theorem 4.15 (cf. [Fk2, Theorem 0.1]). *Let (X, Δ) be a projective lc pair. Assume that $K_X + \Delta$ is numerically equivalent to some semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D. Then $K_X + \Delta$ is semi-ample.*

Proof. By taking a dlt blow-up (cf. Theorem 2.8), we can assume that (X, Δ) is dlt. By the induction on dimension and Proposition 4.3, we have that $(K_X + \Delta)|_{\llcorner \Delta \lrcorner}$ is semi-ample. By [F5, Theorem 1.1], we can prove the semi-ampleness of $K_X + \Delta$. For the details, see the proof of [G2, Theorem 6.3]. □

By using the deep result in [CKP], we have a slight generalization of Theorem 4.15 and [CKP, Corollary 3]. It is also a generalization of Theorem 4.2.

Theorem 4.16 (cf. [CKP, Corollary 3]). *Let (X, Δ) be a projective lc pair and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that D is nef and log abundant with respect to (X, Δ). Assume that $K_X + \Delta \equiv D$. Then $K_X + \Delta$ is semi-ample.*

Proof. By replacing (X, Δ) with its dlt blow-up (cf. Theorem 2.8), we can assume that (X, Δ) is dlt. Let $f : Y \to X$ be a log resolution. We put $K_Y + \Delta_Y = f^*(K_X + \Delta) + F$ with $\Delta_Y = f_*^{-1}\Delta + \sum E$ where E runs through all the f-exceptional prime divisors on Y. We note that F is effective and f-exceptional. By [CKP, Corollary 1],

$$\kappa(X, K_X + \Delta) = \kappa(Y, K_Y + \Delta_Y) \geq \kappa(Y, f^*D + F) = \kappa(X, D).$$

By the assumption, $\kappa(X, D) = \nu(X, D) = \nu(X, K_X + \Delta)$. On the other hand, $\nu(X, K_X + \Delta) \geq \kappa(X, K_X + \Delta)$ always holds. Therefore, $\kappa(X, K_X + \Delta) = \nu(X, K_X + \Delta)$, that is, $K_X + \Delta$ is nef and abundant. By applying the above argument to every lc center of (X, Δ), we obtain that $K_X + \Delta$ is nef and log abundant. Thus, by Theorem 4.2, we obtain that $K_X + \Delta$ is semi-ample. □

Remark 4.17. By the proof of Theorem 4.16, we see that we can weaken the assumption as follows. Let (X, Δ) be a projective lc pair. Assume that $K_X + \Delta$ is numerically equivalent to a nef and abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and that $\nu_W^*((K_X + \Delta)|_W)$ is numerically equivalent to a nef and abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for every lc center W of (X, Δ), where $\nu_W : W^\nu \to W$ is the normalization of W. Then $K_X + \Delta$ is semi-ample.

Theorem 4.18 is a generalization of [G1, Theorem 1.7]. The proof is the same as [G1, Theorem 1.7] once we adopt [F5, Theorem 1.1].

Theorem 4.18 (cf. [G2, Theorems 6.4, 6.5]). *Let (X, Δ) be a projective lc pair such that $-(K_X + \Delta)$ (resp. $K_X + \Delta$) is nef and abundant.*

Assume that $\dim \mathrm{Nklt}(X,\Delta) \le 1$ *where* $\mathrm{Nklt}(X,\Delta)$ *is the non-klt locus of the pair* (X,Δ)*. Then* $-(K_X+\Delta)$ *(resp.* $K_X+\Delta$*) is semi-ample.*

Proof. Let T be the non-klt locus of (X,Δ). By the same argument as in the proof of [G1, Theorem 3.1], we can check that $-(K_X+\Delta)|_T$ (resp. $(K_X+\Delta)|_T$) is semi-ample. Therefore, $-(K_X+\Delta)$ (resp. $K_X+\Delta$) is semi-ample by [F5, Theorem 1.1]. □

Similarly, we can prove Theorem 4.19.

Theorem 4.19. *Let* (X,Δ) *be a projective lc pair. Assume that* $-(K_X+\Delta)$ *is nef and abundant and that* $(K_X+\Delta)|_W \equiv 0$ *for every lc center* W *of* (X,Δ)*. Then* $-(K_X+\Delta)$ *is semi-ample.*

Proof. By taking a dlt blow-up (cf. Theorem 2.8), we can assume that (X,Δ) is dlt. By [G2, Theorem 1.2] (cf. Corollary 4.11), $(K_X+\Delta)|_{\llcorner \Delta \lrcorner}$ is semi-ample. Therefore, $K_X+\Delta$ is semi-ample by [F5, Theorem 1.1]. □

5. Non-vanishing, abundance, and extension conjectures

In this final section, we discuss the relationship among various conjectures in the minimal model program. Roughly speaking, we prove that the abundance conjecture for projective log canonical pairs (cf. Conjecture 1.3) is equivalent to the non-vanishing conjecture (cf. Conjecture 5.7) and the extension conjecture for projective dlt pairs (cf. Conjecture 5.8).

First, let us recall the weak non-vanishing conjecture for projective lc pairs (cf. [B1, Conjecture 1.3]).

Conjecture 5.1 (Weak non-vanishing conjecture)**.** *Let* (X,Δ) *be a projective lc pair such that* Δ *is an* $\mathbb{R}$*-divisor. Assume that* $K_X+\Delta$ *is pseudo-effective. Then there exists an effective* $\mathbb{R}$*-divisor* D *on* X *such that* $K_X+\Delta \equiv D$*.*

Conjecture 5.1 is known to be one of the most important problems in the minimal model theory (cf. [B1]).

Remark 5.2. By [CKP, Theorem 1], $K_X+\Delta \equiv D \ge 0$ in Conjecture 5.1 means that there is an effective $\mathbb{R}$-divisor D' such that $K_X+\Delta \sim_{\mathbb{R}} D'$.

By Remark 5.2 and Lemma 5.3 below, Conjecture 5.1 in dimension $\le n$ is equivalent to Conjecture 1.3 of [B1] in dimension $\le n$ with the aid of dlt blow-ups (cf. Theorem 2.8).

Lemma 5.3. *Assume that* Conjecture 5.1 *holds in dimension* $\leq n$*. Let* $f : X \to Z$ *be a projective morphism between quasi-projective varieties with* $\dim X = n$*. Let* (X, Δ) *be an lc pair such that* $K_X + \Delta$ *is pseudo-effective over* Z*. Then there exists an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* M *on* X *such that* $K_X + \Delta \sim_{\mathbb{R},Z} M$*.*

Proof. Apply Conjecture 5.1 and Remark 5.2 to the generic fiber of f. Then, by [BCHM, Lemma 3.2.1], we obtain M with the required properties. $\square$

We give a small remark on Birkar's paper [B1].

Remark 5.4 (Absolute versus relative)**.** Let $f : X \to Z$ be a projective morphism between *projective* varieties. Let (X, B) be a $\mathbb{Q}$-factorial dlt pair and let $(X, B+C)$ be an lc pair such that $C \geq 0$ and that K_X+B+C is nef over Z. Let H be a very ample Cartier divisor on Z. Let D be a general member of $|2(2\dim X+1)H|$. In this situation, $(X, B+\frac{1}{2}f^*D)$ is dlt, $(X, B+\frac{1}{2}f^*D+C)$ is lc, and $K_X+B+\frac{1}{2}f^*D+C$ is nef by Kawamata's bound on the length of extremal rays. The minimal model program on $K_X+B+\frac{1}{2}f^*D$ with scaling of C is the minimal model program on K_X+B *over* Z with scaling of C. By this observation, the arguments in [B1] work without appealing relative settings if the considered varieties are *projective*.

Theorem 5.5. *The abundance theorem for projective klt pairs in dimension* $\leq n$ *and* Conjecture 5.1 *for projective* $\mathbb{Q}$*-factorial dlt pairs in dimension* $\leq n$ *imply the abundance theorem for projective lc pairs in dimension* $\leq n$*.*

Proof. Let (X, Δ) be an n-dimensional projective lc pair such that $K_X + \Delta$ is nef. As we explained in 2.19, by [B1, Theorems 1.4, 1.5], the minimal model program with ample scaling terminates for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n$. Moreover, we can assume that (X, Δ) is a projective $\mathbb{Q}$-factorial dlt pair by taking a dlt blow-up (cf. Theorem 2.8). Thus, by Theorem 4.7, we obtain the desired result. $\square$

The following corollary is a result on a generalized abundance conjecture formulated by Nakayama's numerical Kodaira dimension κ_σ.

Corollary 5.6 (Generalized abundance conjecture)**.** *Assume that the abundance conjecture for projective klt pairs in dimension* $\leq n$ *and* Conjecture 5.1 *for* $\mathbb{Q}$*-factorial dlt pairs in dimension* $\leq n$*. Let* (X, Δ) *be an* n*-dimensional projective lc pair. Then* $\kappa(X, K_X+\Delta) = \kappa_\sigma(X, K_X+\Delta)$*.*

Proof. We can assume that (X, Δ) is a $\mathbb{Q}$-factorial projective dlt pair by replacing it with its dlt blow-up (cf. Theorem 2.8). Let H be a general effective sufficiently ample Cartier divisor on X. We can run the minimal model program with scaling of H by 2.19. Then we obtain a good minimal model by Theorem 5.5 if $K_X + \Delta$ is pseudo-effective. When $K_X + \Delta$ is not pseudo-effective, we have a Mori fiber space structure. In each step of the minimal model program, κ and κ_σ are preserved. So, we obtain $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$. □

Finally, we explain the importance of Theorem 1.4 toward the abundance conjecture. Let us consider the following two conjectures.

Conjecture 5.7 (Non-vanishing conjecture). *Let (X, Δ) be a projective lc pair such that Δ is an $\mathbb{R}$-divisor. Assume that $K_X + \Delta$ is pseudo-effective. Then there exists an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

As we pointed it out in Remark 5.2, Conjecture 5.7 in dimension n follows from Conjecture 5.1 in dimension n. For related topics on the non-vanishing conjecture, see [DHP, Section 8] and [G4], where Conjecture 5.7 is reduced to the case when X is a smooth projective variety and $\Delta = 0$ by assuming the global ACC conjecture and the ACC for log canonical thresholds (see [DHP, Conjecture 8.2 and Conjecture 8.4]).

Conjecture 5.8 (Extension conjecture for dlt pairs (cf. [DHP, Conjecture 1.3]). *Let $(X, S + B)$ be an n-dimensional projective dlt pair such that B is an effective $\mathbb{Q}$-divisor, $\llcorner S + B \lrcorner = S$, $K_X + S + B$ is nef, and $K_X + S + B \sim_{\mathbb{Q}} D \geq 0$ where $S \subset \operatorname{Supp} D$. Then*

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

In Conjecture 5.8, if $(X, S+B)$ is a plt pair, equivalently, S is normal, then it is true by [DHP, Corollary 1.8].

The following theorem is essentially contained in the proof of [DHP, Theorem 1.4]. However, our proof of Theorem 5.9 is slightly different from the arguments in [DHP] because we directly use Birkar's result [B1, Theorems 1.4 and 1.5] and Theorem 1.4.

Theorem 5.9. *Assume that* Conjecture 5.7 *and* Conjecture 5.8 *hold true in dimension $\leq n$. Then* Conjecture 1.3 *is true in dimension n.*

Proof. By Theorem 1.4, it is sufficient to treat log canonical pairs. This reduction is crucial for our inductive proof. We show the statement by induction on dimension. Note that we can freely use the minimal model program with ample scaling for projective dlt pairs by Conjecture 5.1 and Birkar's results (cf. [B1, Theorems 1.4 and 1.5] and 2.19). By Theorem 4.2 and Corollary 5.6, it is sufficient to show the abundance conjecture for an n-dimensional projective kawamata log terminal pair (X, Δ) such that $K_X + \Delta$ is pseudo-effective. By Conjecture 5.7, we see $\kappa(X, K_X + \Delta) \geq 0$ (cf. [Ch, Corollary 2.1.4]). By Kawamata's well-known inductive argument (cf. [Ka2, Theorem 7.3]), we may assume that $\kappa(X, K_X + \Delta) = 0$. We take an effective divisor D such that $D \sim_{\mathbb{Q}} K_X + \Delta$. We take a resolution $\varphi : Y \to X$ such that $\mathrm{Exc}(\varphi) \cup \operatorname{Supp} f_*^{-1}(\Delta + D)$ is a simple normal crossing divisor on Y. Let B and E be effective $\mathbb{Q}$-divisors satisfying:

$$K_Y + B = \varphi^*(K_X + \Delta) + E,$$

and E and B have no common irreducible components. Now we know that $\kappa(X, K_X + \Delta) = \kappa(Y, K_Y + B) = 0$ and $\kappa_\sigma(X, K_X + \Delta) = \kappa_\sigma(Y, K_Y + B)$. Thus, by replacing X with Y, we may further assume that X is smooth and $\Delta + D$ has a simple normal crossing support. Let

$$\Delta = \sum \delta_i D_i \text{ and } D = \sum d_i D_i$$

be the irreducible decompositions. We put

$$\Delta' = \Delta - \sum_{d_i \neq 0} \delta_i D_i + D_{\mathrm{red}},$$

where $D_{\mathrm{red}} = \sum_{d_i \neq 0} D_i$. Then the effective divisor Δ' satisfies

$$\operatorname{Supp} \Delta' = \operatorname{Supp}(\Delta + D), \ \operatorname{Supp} \llcorner \Delta' \lrcorner = \operatorname{Supp} D,$$

and $\operatorname{Supp} D = \operatorname{Supp}(\Delta' - \Delta)$ since (X, Δ) is klt. Note that

$$\kappa(X, K_X + \Delta) = \kappa(X, K_X + \Delta')$$

and

$$\kappa_\sigma(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta').$$

We take a minimal model

$$f : (X, \Delta') \dashrightarrow (Y, \Gamma')$$

of (X, Δ'). If (Y, Γ') is klt, then $\llcorner \Delta' \lrcorner$ is f-exceptional. Thus we have $K_Y + \Gamma' \sim_{\mathbb{Q}} 0$ since $\operatorname{Supp} \llcorner \Delta' \lrcorner = \operatorname{Supp} D$. Therefore,

$$\kappa_\sigma(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta') = 0.$$

It is a desired result. So, from now on, we assume that $S := \llcorner \Gamma' \lrcorner \neq 0$. Then, by Conjecture 5.8, it holds that

$$H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma'))) \to H^0(S, \mathcal{O}_S(m(K_Y + \Gamma')))$$

is surjective for all sufficiently divisible integers $m \geq 2$. By the hypothesis of the induction, it holds that $K_S + \Gamma_S = (K_Y + \Gamma')|_S$ is semi-ample. Note that the pair (S, Γ_S) is an sdlt pair. In particular, $H^0(S, \mathcal{O}_S(m(K_Y + \Gamma'))) \neq 0$. However, since $\operatorname{Supp} \llcorner \Delta' \lrcorner = \operatorname{Supp} D$ and $\kappa(Y, K_Y + \Gamma') = 0$,

$$H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma'))) \to H^0(S, \mathcal{O}_S(m(K_Y + \Gamma')))$$

is a zero map. It is a contradiction. Thus we see that $S = 0$. Therefore, we obtain $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$. □

We have a generalization of [DHP, Theorem 1.4] as a corollary of Theorem 5.9. For a different approach to the existence of good minimal models, see [B2].

Corollary 5.10 (cf. [DHP, Theorem 1.4]). *Assume that* Conjecture 5.7 *and* Conjecture 5.8 *hold true in dimension* $\leq n$. *Let* $f : X \to Y$ *be a projective morphism between quasi-projective varieties. Assume that* (X, Δ) *is an* n*-dimensional dlt pair such that* $K_X + \Delta$ *is pseudo-effective over* Y. *Then* (X, Δ) *has a good minimal model* (X', Δ') *over* Y.

Proof. By Conjecture 5.7 with Lemma 5.3, we can run the minimal model program with ample scaling (cf. 2.19). Therefore, we can construct a minimal model (X', Δ') over Y. By Theorem 5.9, $K_{X'} + \Delta'$ is semi-ample when Y is a point and Δ' is a $\mathbb{Q}$-divisor. By the relative version of Theorem 4.2 and the induction on dimension, we can check that $K_{X'} + \Delta'$ is semi-ample over Y when Δ' is a $\mathbb{Q}$-divisor. If Δ' is an $\mathbb{R}$-divisor, then we can reduce it to the case when Δ' is a $\mathbb{Q}$-divisor by using Shokurov's polytope (see, for example, the proof of Theorem 3.1 in [FG]) and obtain that $K_{X'} + \Delta'$ is semi-ample over Y. It is a standard argument. □

We close this paper with an easy observation. Conjecture 5.8 follows from Conjecture 1.3 by a cohomology injectivity theorem.

Proposition 5.11. *Assume that* Conjecture 1.3 *is true in dimension* n. *Then* Conjecture 5.8 *holds true in dimension* n. *More precisely, in* Conjecture 5.8, *if* $K_X + S + B$ *is semi-ample, then the restriction map is surjective for every* $m \geq 2$ *such that* $m(K_X + S + B)$ *is Cartier.*

Proof. Let $f : Y \to X$ be a projective birational morphism from a smooth variety Y such that $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}(S + B)$ is a simple normal crossing divisor on Y and that f is an isomorphism over the generic point of every lc center of the pair $(X, S + B)$. Then we can write

$$K_Y + S' + F = f^*(K_X + S + B) + E$$

where S' is the strict transform of S, E is an effective Cartier divisor, F is an effective $\mathbb{Q}$-divisor with $\llcorner F \lrcorner = 0$. Note that E is f-exceptional. We consider the short exact sequence

$$0 \to \mathcal{O}_Y(E - S') \to \mathcal{O}_Y(E) \to \mathcal{O}_{S'}(E) \to 0.$$

By the relative Kawamata–Viehweg vanishing theorem, $R^i f_* \mathcal{O}_Y(E - S') = 0$ for every $i > 0$. Therefore,

$$\mathcal{O}_X \simeq f_* \mathcal{O}_Y(E) \to f_* \mathcal{O}_{S'}(E)$$

is surjective. Thus, we obtain $f_* \mathcal{O}_{S'}(E) \simeq \mathcal{O}_S$. Let m be a positive integer such that $m(K_X + S + B)$ is Cartier with $m \geq 2$. We put $L = m(K_X + S + B)$. It is sufficient to prove that

$$H^0(Y, \mathcal{O}_Y(f^*L + E)) \to H^0(S', \mathcal{O}_{S'}(f^*L + E))$$

is surjective. By Conjecture 1.3, $K_X + S + B$ is semi-ample. Let $g : X \to Z$ be the Iitaka fibration associated to $K_X + S + B$. Then there is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A on Z such that $K_X + S + B \sim_{\mathbb{Q}} g^*A$. We note that

$$\begin{aligned}(f^*L + E - S') - (K_Y + F) &= (m-1)f^*(K_X + S + B) \\ &\sim_{\mathbb{Q}} (m-1)f^*g^*A.\end{aligned}$$

Since $S \subset \operatorname{Supp} D$ and $K_X + S + B \sim_{\mathbb{Q}} D \geq 0$, we have $g \circ f(S') \subsetneq Z$. Note that $g : X \to Z$ is the Iitaka fibration associated to $K_X + S + B$. Therefore, it is easy to see that

$$H^i(Y, \mathcal{O}_Y(f^*L + E - S')) \to H^i(Y, \mathcal{O}_Y(f^*L + E))$$

is injective for every i because $|kf^*g^*A - S'| \neq \emptyset$ for $k \gg 0$ (see, for example, [F7, Theorem 6.1]). In particular,

$$H^1(Y, \mathcal{O}_Y(f^*L + E - S')) \to H^1(Y, \mathcal{O}_Y(f^*L + E))$$

is injective. Thus we obtain that

$$H^0(Y, \mathcal{O}_Y(f^*L + E)) \to H^0(S', \mathcal{O}_{S'}(f^*L + E))$$

is surjective. It implies the desired surjection

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B))).$$

We finish the proof. $\square$

Proposition 5.11 shows that Conjecture 5.8 is a reasonable conjecture in the minimal model theory.

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